ASYMPTOTIC METHODS IN INVERSE SCATTERING FOR INHOMOGENEOUS MEDIA

by

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ABSTRACT

In this thesis we study three different problems associated to the detection of two types of material defects: interfacial cracks and delaminations. In Chapter 2 we address the problem of interfacial crack detection in layered isotropic elastic media. In the first part, a well-posedness result is established, and we use this result in the second part of the chapter to adapt the Factorization Method (FM) in order to propose a reconstruction algorithm. In Chapter 3 we consider the problem of detecting if two materials that should be in contact have separated or delaminated. The goal is to find an acoustic technique to detect the delamination. We model the delamination as a thin opening between two materials of different acoustic properties, and using asymptotic techniques we derive an asymptotic model where the delaminated region is replaced by jump conditions on the acoustic field and flux. The asymptotic model has potential singularities due to the edges of the delaminated region, and we show that the forward problem is well posed for a large class of possible delaminations. We then design a special Linear Sampling Method (LSM) for detecting the shape of the delamination assuming that the background, undamaged, state is known. In Chapter 4 we consider the problem of detecting planar delaminated regions of constant thickness. Here we aim to develop an electromagnetic technique to detect the delamination. Again, we derive a asymptotic model where the delaminated region is replaced by jump conditions on the electric and magnetic fields. We show that the forward problem is well posed under some assumptions on the material properties, and finally adapt again a LSM to detect the shape of the delamination assuming that the background state is known. In all three chapters we show, by numerical experiments, that our nondestructive testing (NDT) methods can indeed be used to determine the shape of the corresponding defect.

Chapter 1

INTRODUCTION

Inverse scattering is a field of science concerned with the development of techniques to infer physical properties of a system, from measurements of how incoming waves are scattered in the presence of the system. Among examples of such problems, are the identification of the shape of an obstacle and the determination of relevant material properties of an object to which no direct access is available. These kind of problems have important applications in medical imaging (e.g. [4, 16]), nondestructive testing of materials (e.g. [44, 2, 68]), geological exploration (see [42, 64, 88]), etc. The mathematical methods that have been developed to solve inverse scattering problems are diverse, and can be divided into three main categories (see [55]): weak scattering methods, optimization methods and qualitative methods. Weak scattering methods are based on approximations that neglect multiple scattering effects, and which are valid only in the sub-resonant region (where either the wave number k of the interrogating waves or the material contrasts are very small). In contrast, both optimization methods and qualitative methods have been successful in tackling inverse scattering problems in more general regimes. These two families of methods are based on substantially different approaches, each of them with their own advantages and disadvantages. Optimization methods lead to final reconstruction algorithms involving iterative schemes, where in each iteration a forward problem has to be solved. They are numerically expensive and require substantial *a priori* information, although have the advantage of providing information on both geometrical and physical parameters of the object. Qualitative methods, on the contrary, typically require little *a priori* information and result in cheap final reconstruction or identification algorithms, but at the expense of recovering only partial information: either the geometry or the physical parameters of the object. For a general overview of inverse scattering methods we refer the reader to [32].

In this thesis we will develop nondestructive testing (NDT) methods for the identification of material defects (delamination and cracks), based on two of such qualitative methods: the linear sampling method (LSM) and the factorization method (FM). The LSM was first developed by D. Colton and A. Kirsch in 1996 [31], and the FM was invented by A. Kirsch in 1998 [56, 54].

The general idea of both the LSM and FM is to find the location of an object (in our case a defect) by the construction of an indicator function whose support is precisely the object. The algorithm to construct such indicator function consists in sampling points on a grid where the object is expected to be, and then use a criterion provided by the LSM or the FM to decide whether the point belongs or not to the support of the object.

The LSM criterion to construct the indicator function is based on solving a linear (ill-posed) equation for each sampling point, with the implementation of a regularization scheme, and then to use this regularized solution to approximate the indicator function.

In contrast, the FM criteria is based on the characterization of the range of an operator that can be constructed from measurable scattering data. Based on the exact knowledge of this range, the indicator function can be found directly. Although the FM is mathematically more satisfactory, the assumptions required by this method are much more restrictive than those for the LSM.

In both cases, the LSM and FM require multistatic data for a single wave number $k = 2\pi/\lambda$, meaning that the information is collected from an array of sources and receivers.

Note that one can increase the resolution of the LSM and FM by simply increasing the wave number k of the interrogating waves [45].

We refer the reader to [56], [21] and [23] for a variety examples of applications of LSM and FM.

We start the thesis with Chapter 2, where we address the problem of interfacial crack detection. The general problem of crack detection has been an active research topic in both engineering and mathematics due to its broad range of applications, including hydrology, geothermal sources, and environmental protection. It has been experimentally confirmed that both the geometry and the interfacial conditions of the fracture are equally important to its response to a given "activation" [76]. In our approach, we present a method that captures the geometrical distribution of interfacial cracks, in an elastic isotropic layered medium, assuming that a linear contact law is satisfied on the crack. This means that the relation between the traction and the jump of the displacements across the discontinuity is linear, given by the so-called stiffness matrix. The method developed is based on an adaptation of the FM, and it is partially a generalization of previous work [77] to the case of inhomogeneous media. The results of this chapter will appear soon in [34].

In Chapters 3 and 4, we turn our attention to the problem of the identification of delamination. Delamination is a defect that occurs when two materials that should be bonded together, partially separate. This is a common failure occuring, for example, in composite or stratified media that are subject to repeated stresses (see [72, 19, 82]).

Delamination is considered one of the most critical defects in engineering, because it changes the structural stability of the system (see [72]), and the development of nondestructive testing (NDT) of delamination is therefore an important area of research (e.g. see [83, 44]).

In order to obtain a model that provides a unified picture of scattering by a thin crack of arbitrary shape, the first parts of Chapters 3 and 4 are devoted to a presentation of *reduced* models for the scattering of acoustic and electromagnetic waves. The full derivation of the reduced models can be found in Appendix B (for acoustic scattering) and in Appendix C (for electromagnetic scattering).

These reduced models are derived from asymptotic expressions of the fields, that exploit the fact that the maximum thickness of the delamination δ is a small parameter (with respect to both the size of the obstacle and to the wave-length λ).

The origin of these type of methods goes back to the work of M.A. Leontovich and S.M. Rytov [62, 78]. In his work, Leontovich derived for the first time the now well-known surface impedance boundary condition (SIBC) to approximate the scattering of an electromagnetic wave by a highly conducting obstacle. Rytov, however, seems to be the first one to have expressed the fields, in the vicinity of a thin film as a power series with respect to the thickness of the film δ (see [86]). Using this approach Rytov provided a general method to derive high order models with better accuracy.

In the context of highly conducting obstacles, subsequent work has resulted in the derivation of approximate boundary conditions, the so-called Generalized Impedance Boundary Conditions (GIBCs) [49, 81]. The ideas have been applied beyond this setting to the case of perfect conductors (e.g. [48, 14, 66]), to the study of thin periodic films and rough surfaces (e.g. [73, 74, 15, 35, 36, 37, 38]), and to thin transmitting coatings (e.g. [71, 30, 40]). In these two last settings, the concept of approximate transmission conditions (ATCs) has been developed. The general idea is to approximate the effect of the thin layer by suitable jump -or transmission- conditions for the fields.

For a more exhaustive review on the development of these asymptotic methods, we refer the reader to the introductory chapter in [35].

For our delamination problem, we derived ATCs models by formally considering an asymptotic expression of the field in a vicinity of the thin delamination, as a power series with respect to δ. These ATCs models are new with respect to previous approaches in the sense that we allow the thin domain to be supported in a surface that has a non-empty boundary (in previous works the thin films were supported in a closed surface). This brings into play potential singularities in the asymptotic expansion, and extra difficulties in the analysis of the well-posedness of the ATCs models.

In Chapter 3, the ATCs model for acoustic scattering considers jumps of the fields on a single surface, and thus is considered to be a *crack-type* model. However, in the electromagnetic case a *crack-type* ATCs (whose full derivation can be found in Appendix C, model I) leads to a model where the signs of the coefficients involved in the ATCs are not compatible with a classical splitting into coercive plus compact terms, that would lead to a well-posedness result. This is not a surprise since it has been shown in [30] that the corresponding model in the time-domain is unstable. Therefore, in Chapter 4, instead of considering *crack-type* ATCs with jumps of traces of the fields *on the same surface*, we consider a different model where the traces of the fields are taken on the *two different surfaces*, sometimes called *Chun's-type* ATCs, that constitute the outer and inner boundaries of the delamination. The full derivation of a general setting for this model can be found in Appendix C, model II.

After having established the ATCs models, a subsequent part of Chapters 3 and 4 is devoted to the numerical validation of these new ATCs models as "good" approximations to the original full model. In both cases we perform numerical experiments and study the numerical convergence of the ATCs models to the standard one, as the small parameter δ approaches to zero.

Later on, in Chapter 3, we study the reduced acoustic ATCs model and prove that, in certain geometrical and material regimes, the model is well-posed. Moreover, we finish the chapter presenting a NDT algorithm for detecting delamination using acoustic waves, based on a LSM adapted precisely to the ATCs model. This NDT can be regarded as a first step to develop methods to identify delaminations using elastic waves (e.g. ultrasound), and it is also a preliminary step for NDT using electromagnetic waves (e.g. microwaves), which is the topic of discussion of Chapter 4.

Due to technical difficulties associated with the reduced ATCs model for electromagnetic scattering, in Chapter 4 we restrict our discussion to the case of planar delaminations of constant thickness. For this case, a well-posedness result based in a Helmholtz decomposition is provided. The last part of Chapter 4 gives the development of a NDT of planar delaminations by adapting the LSM to this model. Preliminary numerical results are also presented.

The results in Chapter 3 have been already published in [24], while the results of Chapter 4 will soon appear in [25] and [26].

Chapter 2

ACTIVE ULTRASONIC SENSING OF INTERFACIAL CRACKS IN LAYERED ELASTIC MEDIA

2.1 The problem

In this first part of the thesis, the aim is to develop a nondestructive testing algorithm to detect the location and size of fractures at the interface of two linear isotropic elastic materials that, in the undamaged or *background* state, were bonded together.

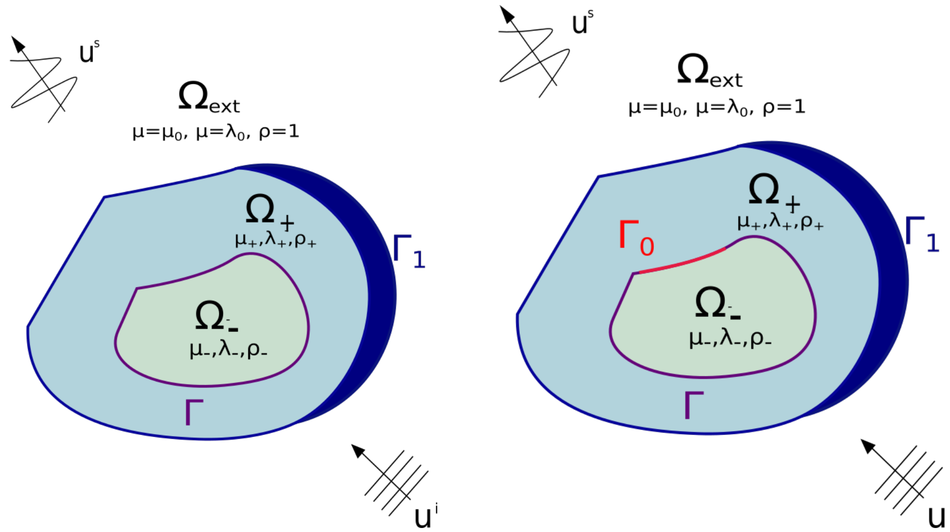
To this end, we study the scattering of a linear elastic wave by a bounded, isotropic penetrable obstacle, $\Omega \subset \mathbb{R}^3$, with Lipschitz continuous boundary Γ_1, which is embedded in a homogeneous medium $\Omega_{ext} := \mathbb{R}^3 \setminus \overline{\Omega}$, see Figure 2.1. In the scenario that we consider, Ω is composed by two layers of different materials, Ω_- and Ω_+, which have also a closed, Lipschitz continuous, common interface Γ.

In the damaged configuration, one part of Γ has been damaged and a fracture $\Gamma_0 \subset \Gamma$ has appeared (Figure 2.1, Panel b)). In general, Γ_0 is an open surface, and its relative boundary on Γ will be denoted by $\partial\Gamma_0$.

All three domains Ω_+, Ω_-, Ω_{ext} have different material properties, characterized by their relative density ρ and the fourth-order elasticity tensor $\mathbf{C}$ that defines the linear constitutive relation between strain and stress. In the context of isotropic elastic material, is given by [80]:

$$\mathbf{C}_{ijk\ell} = \lambda\delta_{ij}\delta_{\ell k} + \mu(\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}), \tag{2.1}$$

where λ and μ are the so-called Lamé parameters. It can be shown that $\mu > 0$ is the shear modulus, while λ does not have a direct physical interpretation but satisfies $K = \lambda + \frac{2}{3}\mu$, where K is the bulk modulus of the material. This imposes the natural



(a) (b)

Figure 2.1: Panel (a) Transversal cut of the undamaged, also called *background* state. The stratified media consists of two materials Ω_- and Ω_+. Panel (b) Transversal cut the damaged configuration. The crack Γ_0 occurs at the interface of the two layers.

condition that $3\lambda + 2\mu > 0$. However, in this work we will restrict ourselves to the case $\lambda > 0$, which is reasonable in most common situations [63].

In the setting we consider, both ρ, λ and μ are constant in the exterior domain Ω_{ext} with values $\rho = 1$, $\lambda = \lambda_0$, and $\mu = \mu_0$, while in Ω_- and Ω_+ they are known piece-wise smooth functions, with possible jumps at the interfaces Γ, Γ_1, satisfying:

$$(\lambda_+|_\Gamma - \lambda_-|_\Gamma)(\mu_+|_\Gamma - \mu_-|_\Gamma) \geq 0 \quad \text{and} \quad (\lambda_+|_{\Gamma_1} - \lambda_0)(\mu_+|_{\Gamma_1} - \mu_0) \geq 0, \tag{2.2}$$

where $\lambda_\pm, \mu_\pm$ are the Lamé coefficients λ, μ in $\Omega_\pm$, respectively.

Remark 2.1.1. *The monotonicity condition (2.2) for the Lamé parameters in transmission problems is necessary in order to have a unique representation of the fields in terms of single- and double-layer potentials in the context of Lipschitz domains (see [41]).*

Let $\mathbf{v}$ be a function defined in Ω_+ and Ω_-. In the sequel, we denote the jump of $\mathbf{v}$ across Γ_0 by $[\mathbf{v}] := \mathbf{v}^+|_{\Gamma_0} - \mathbf{v}^-|_{\Gamma_0}$. Then if we assume a linear contact law on the fracture Γ_0 ([77]), the displacement $\mathbf{u}$ satisfies:

$$\partial_{\boldsymbol{\nu}}^* \mathbf{u} = \mathbf{K}[\mathbf{u}], \text{ on } \Gamma_0, \tag{2.3}$$

where $\mathbf{K} \in L^\infty(\Gamma_0)^{3\times 3}$ is the so-called stiffness matrix, which is assumed to be symmetric in order to be consistent with the reciprocity principle, and

$$\partial_{\boldsymbol{\nu}}^* \mathbf{u} := \boldsymbol{\nu} \cdot \mathbf{C} : \nabla \mathbf{u}$$

is the co-normal derivative or "traction", where $\boldsymbol{\nu}$ is the unit normal vector on Γ, pointing into Ω_+.

Denote by Δ^* the elastic differential operator, defined by

$$\Delta^* \mathbf{u} := \nabla \cdot (\mathbf{C} : \nabla \mathbf{u}) = \nabla \cdot (2\mu\, \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}_{3\times 3}), \tag{2.4}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the strain tensor corresponding to $\mathbf{u}$.

Then the scattering of the total elastic field $\mathbf{u} \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Gamma_0})^3$ in the damaged configuration is, in the frequency domain, given by

$$\Delta^* \mathbf{u} + \rho\omega^2 \mathbf{u} = \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Gamma_0}, \tag{2.5}$$

$$\partial_{\boldsymbol{\nu}}^* \mathbf{u} = \mathbf{K}[\mathbf{u}] \text{ on } \Gamma_0, \tag{2.6}$$

and where in the unbounded domain Ω_{ext} the total field can be decomposed as $\mathbf{u} = \mathbf{u}^{sc} + \mathbf{u}^i$, where $\mathbf{u}^i$ denotes the incident field, and $\mathbf{u}^{sc}$ the radiating field that satisfies the Kupradze radiation conditions (see [59]):

$$\frac{\partial \mathbf{u}^p}{\partial r} - ik_p \mathbf{u}^p = O\left(\frac{1}{r}\right) \quad \text{and} \quad \frac{\partial \mathbf{u}^s}{\partial r} - ik_p \mathbf{u}^s = O\left(\frac{1}{r}\right), \quad \text{as } r \to \infty, \tag{2.7}$$

where $r = |\mathbf{x}|$, the limits are uniform with respect to $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and where

$$\mathbf{u}^p \;:=\; \frac{1}{k_s^2 - k_p^2}(\Delta + k_s^2)\mathbf{u}^{sc} \text{ in } \Omega, \tag{2.8}$$

$$\mathbf{u}^s \;:=\; \frac{1}{k_p^2 - k_s^2}(\Delta + k_p^2)\mathbf{u}^{sc} \text{ in } \Omega, \tag{2.9}$$

for

$$k_s^2 = \frac{\omega^2}{\mu_0} \quad \text{and} \quad k_p^2 = \frac{\omega^2}{\lambda_0 + 2\mu_0}. \tag{2.10}$$

2.2 The well-posedness of the direct problem

The well-posedness of problem (2.5)-(2.7), is proven using the same arguments given in the case where the background is homogeneous, and already studied in [77]. For the reader's convenience we provide here the details.

In order to study our problem in a Fredholm operator framework, it is convenient to consider an equivalent formulation of the problem (2.5)-(2.7) involving only bounded domains. To this end, let $B_R \subset \mathbb{R}^3$ be an arbitrary ball of radius $R > 0$, such that that $\overline{\Omega} \subset B_R$. As usual, multiplying equation (2.5) by a test vector field $\mathbf{v}$ and integrating by parts in B_R, we get the following equivalent variational formulation of our problem (2.5)-(2.6): Seek $\mathbf{u} \in H^1(B_R \setminus \overline{\Gamma}_0)^3$ such that

$$A(\mathbf{u},\mathbf{v}) \;=\; \mathcal{L}(\mathbf{v}) \qquad \forall \mathbf{v},\mathbf{u} \in H^1(B_R \setminus \overline{\Gamma}_0)^3, \tag{2.11}$$

where

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) \;&=\; \int_{B_R} \overline{\nabla \mathbf{v}} : \mathbf{C} : \nabla \mathbf{u} \;\,\mathrm{d}\mathbf{y} - \omega^2 \int_{B_R} \rho\, \overline{\mathbf{v}} \cdot \mathbf{u} \;\,\mathrm{d}\mathbf{y} \\ &\quad + \int_{\Gamma_0} [\overline{\mathbf{v}}] \cdot \mathbf{K}\,[\mathbf{u}] \;\,\mathrm{d}s(\mathbf{y}) - \int_{S_R} \overline{\mathbf{v}} \cdot \mathcal{T}_R \mathbf{u} \;\,\mathrm{d}s(\mathbf{y}), \qquad (2.12) \\ \mathcal{L}(\mathbf{v}) \;&=\; + \int_{S_R} \left\{ \overline{\mathbf{v}} \cdot \partial^*_{\boldsymbol{\nu}} \mathbf{u}^i - \overline{\mathbf{v}} \cdot \mathcal{T}_R \mathbf{u}^i \right\} \,\mathrm{d}s(\mathbf{y}), \end{aligned}$$

and where the Dirichlet-to-Neumann operator $\mathcal{T}_R : H^{1/2}(S_R)^3 \to H^{-1/2}(S_R)^3$ is defined by

$$\mathcal{T}_R \phi = \partial^*_{\boldsymbol{\nu}} \mathbf{u}_\phi |_{S_R}, \tag{2.13}$$

where $\mathbf{u}_\phi$ satsfies

$$\Delta^* \mathbf{u}_\phi + \omega^2 \mathbf{u}_\phi \;=\; \mathbf{0} \text{ in } \mathbb{R}^3 \setminus \overline{B_R}, \tag{2.14}$$

$$\mathbf{u}_\phi|_{S_R} \;=\; \phi \text{ on } S_R, \tag{2.15}$$

and the Kupradze conditions (2.7). It is well known that the operator $\mathcal{T}_R$ is well defined and bounded [21].

Define the following Sobolev spaces:

$$H^{1/2}(\Gamma_0) \;:=\; \left\{ u|_{\Gamma_0} \,|\, u \in H^{1/2}(\Gamma) \right\}, \tag{2.16}$$

$$\widetilde{H}^{1/2}(\Gamma_0) \;:=\; \left\{ u \in H^{1/2}(\Gamma_0) \,|\, \mathrm{supp}\ u \subset \Gamma_0 \right\}, \tag{2.17}$$

where supp u is the essential support of u defined as the largest relatively closed subset of Γ such that $u = 0$ almost everywhere in $\Gamma \setminus \text{supp } u$.

Moreover, it is well known (see [63, 21]) that:

$$\widetilde{H}^{1/2}(\Gamma_0) \;\; = \;\; \Big\{ u \in H^{1/2}(\Gamma_0) \, | \, \widetilde{u} \in H^{1/2}(\Gamma) \subset \Gamma_0 \Big\}, \tag{2.18}$$

where $\widetilde{u}$ is the extension by zero of u to Γ.

Both spaces $H^{1/2}(\Gamma_0)$ and $\widetilde{H}^{1/2}(\Gamma_0)$ can be endowed with the restricted $H^{1/2}(\Gamma)-$inner product, and in such a case they are Hilbert spaces.

And finally, the associated dual spaces of $\widetilde{H}^{1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$ are, respectively:

$$H^{-1/2}(\Gamma_0) \;\; = \;\; \Big(\widetilde{H}^{1/2}(\Gamma_0) \Big)^*, \tag{2.19}$$

$$\widetilde{H}^{-1/2}(\Gamma_0) \;\; = \;\; \Big(H^{1/2}(\Gamma_0) \Big)^*, \tag{2.20}$$

where

$$H^{-1/2}(\Gamma_0) \;\; := \;\; \Big\{ v|_{\Gamma_0} \, | \, v \in H^{-1/2}(\Gamma) \Big\}, \tag{2.21}$$

$$\widetilde{H}^{-1/2}(\Gamma_0) \;\; := \;\; \Big\{ v \in H^{-1/2}(\Gamma_0) \, | \, \text{supp } v \subset \Gamma_0 \Big\}. \tag{2.22}$$

The space duality (2.19)-(2.20) are defined precisely in terms of the following natural duality pairings:

$$\langle v, u \rangle_{H^{-1/2}(\Gamma_0), \widetilde{H}^{1/2}(\Gamma_0)} \;\; := \;\; \langle v, \widetilde{u} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}, \tag{2.23}$$

where on the right-hand-side of (2.23) $\widetilde{u}$ is the extension by zero of u to Γ, and

$$\langle v, u \rangle_{\widetilde{H}^{-1/2}(\Gamma_0), H^{1/2}(\Gamma_0)} \;\; := \;\; \langle \widetilde{v}, u \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}, \tag{2.24}$$

where $\widetilde{v} \in H^{-1/2}(\Gamma)$ is the extension by zero of v.

It is then true that:

$$\widetilde{H}^{1/2}(\Gamma_0) \subset H^{1/2}(\Gamma_0) \subset L^2(\Gamma_0) \subset \widetilde{H}^{-1/2}(\Gamma_0) \subset H^{-1/2}(\Gamma_0), \tag{2.25}$$

and all the embeddings are bounded.

Remark 2.2.1. Observe that for $\mathbf{u} \in H^1(B_R \setminus \overline{\Gamma}_0)^3$, the jump $[\mathbf{u}] \in \widetilde{H}^{1/2}(\Gamma_0)^3$. Moreover, if the stiffness matrix $\mathbf{K} \in L^\infty(\Gamma_0)^{3\times 3}$, then $\mathbf{K}\,[\mathbf{u}] \in H^{-1/2}(\Gamma_0)^3$ (see corollary 8.8 in [58]).

Therefore, on one hand the interfacial condition (2.6) makes mathematical sense, and the term:

$$\int_{\Gamma_0} [\overline{\mathbf{v}}] \cdot \mathbf{K}\,[\mathbf{u}]\ \ \mathrm{d}s(\mathbf{y})$$

appearing in the expression (2.12) is considered in the sense of the duality pairing $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma_0)^3, \widetilde{H}^{1/2}(\Gamma_0)^3}$, and then pivoting with respect to the $L^2(\Gamma_0)^3$ inner product.

Lemma 2.2.1. *The Dirichlet-to-Neumann operator $\mathcal{T}_R$ can be decomposed as $\mathcal{T}_R = \mathcal{T}_R^c + \mathcal{T}_R^0$, where $\mathcal{T}_R^c : H^{1/2}(S_R)^3 \to H^{-1/2}(S_R)^3$ is compact and $-\mathcal{T}_R^0 : H^{1/2}(S_R)^3 \to H^{-1/2}(S_R)^3$ is non-negative and self-adjoint. Moreover,*

$$\Im\langle \mathcal{T}_R \boldsymbol{\phi},\, \boldsymbol{\phi}\rangle_{S_R} > 0 \text{ for all } \boldsymbol{\phi} \in H^{1/2}(S_R)^3 \setminus \{0\}. \tag{2.26}$$

For the proof of this result see Lemma 1 in [77].

Theorem 2.2.1. *(Well-posedness of the crack problem) Let $\mathbf{K} \in L^\infty(\Gamma_0)^{3\times 3}$ such that $\Im(\overline{\boldsymbol{v}} \cdot \mathbf{K}\boldsymbol{v}) \le 0$ for all $\boldsymbol{v} \in H^{1/2}(\Gamma_0)^3$. Then the variational problem (2.11) is well-posed.*

Proof. Define

$$\begin{aligned} A_0(\mathbf{u},\mathbf{v}) &= \int_{B_R} \overline{\nabla \mathbf{v}} : \mathbf{C} : \nabla \mathbf{u}\ \ \mathrm{d}\,\mathbf{y} + \omega^2 \int_{B_R} \overline{\mathbf{v}} \cdot \mathbf{u}\ \ \mathrm{d}\,\mathbf{y} \\ &- \int_{S_R} \mathcal{T}_R^0 \mathbf{u} \cdot \overline{\mathbf{v}}\ \ \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{2.27}$$

$$\begin{aligned} B(\mathbf{u},\mathbf{v}) &= -\omega^2 \int_{B_R} (1+\rho)\overline{\mathbf{v}} \cdot \mathbf{u}\ \ \mathrm{d}\,\mathbf{y} \\ &+ \int_{\Gamma_0} [\overline{\mathbf{v}}] \cdot \mathbf{K}\,[\mathbf{u}]\ \ \mathrm{d}s(\mathbf{y}) - \int_{S_R} \overline{\mathbf{v}} \cdot \mathcal{T}_R^c \mathbf{u}\ \ \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{2.28}$$

so that $A = A_0 + B$.

Notice that from (2.4) and from Korn's inequality [63],

$$
\begin{aligned}
|A_0(\mathbf{u},\mathbf{u})| &= \int_{B_R} \left\{2\mu|\boldsymbol{\varepsilon}(\mathbf{u})|^2 + \lambda|\nabla\cdot\mathbf{u}|^2 + \omega^2|\mathbf{u}|^2\right\} \,\mathrm{d}\,\mathbf{y} \\
&- \int_{S_R} \mathcal{T}_R^0\mathbf{u}\cdot\overline{\mathbf{u}}\ \mathrm{d}s(\mathbf{y}) \qquad (2.29)\\
&\geq \int_{B_R} \left\{2\mu|\boldsymbol{\varepsilon}(\mathbf{u})|^2 + \omega^2|\mathbf{u}|^2\right\} \,\mathrm{d}\,\mathbf{y} \qquad (2.30)\\
&\geq C\,\|\mathbf{u}\|^2_{H^1(B_R\setminus\overline{\Gamma}_0)}\,, \qquad (2.31)
\end{aligned}
$$

where $C > 0$ is a constant independent of $\mathbf{u}$, implying that $A_0 : H^1(B_R\setminus\overline{\Gamma}_0)^3\times H^1(B_R\setminus\overline{\Gamma}_0)^3 \to \mathbb{C}$ is coercive.

On the other hand, for all $\mathbf{u},\mathbf{v}\in H^1(B_R\setminus\overline{\Gamma}_0)^3$,

$$
\begin{aligned}
|B(\mathbf{u},\mathbf{v})| &\leq \omega^2(1+\|\rho\|_\infty)\,\|\mathbf{u}\|_{L^2(B_R\setminus\overline{\Gamma}_0)^3}\,\|\mathbf{v}\|_{L^2(B_R\setminus\overline{\Gamma}_0)^3} \qquad (2.32)\\
&+ \|\mathbf{K}\|_\infty\,\|[\mathbf{u}]\|_{L^2(\Gamma_0)^3}\,\|[\mathbf{v}]\|_{L^2(\Gamma_0)^3} + \|\mathcal{T}_R^c\mathbf{u}\|_{H^{-1/2}(S_R)^3}\,\|\mathbf{v}\|_{H^{1/2}(S_R)^3}\,, \qquad (2.33)
\end{aligned}
$$

and then for all $\|\mathbf{v}\|_{H^1(B_R\setminus\overline{\Gamma}_0)^3} = 1$,

$$
|B(\mathbf{u},\mathbf{v})| \leq C_0\,\|\mathbf{u}\|_{L^2(B_R\setminus\overline{\Gamma}_0)^3} + C_1\,\|\mathbf{K}\|_\infty\,\|[\mathbf{u}]\|_{L^2(\Gamma_0)^3} + C_2\,\|\mathcal{T}_R^c\mathbf{u}\|_{H^{-1/2}(S_R)^3}, \qquad (2.34)
$$

where $C_0 = \omega^2(1+\|\rho\|_\infty)$, C_1 is twice the norm of the trace operator from $H^1(B_R\setminus\overline{\Gamma}_0)^3 \to L^2(\Gamma_0)^3$, and $C_2 > 0$ is the norm of the compact trace operator from $H^1(B_R\setminus\overline{\Gamma}_0)^3 \to H^{1/2}(S_R)^3$.

Let $\{\mathbf{u}_n\}$ be a sequence that converges weakly to $\mathbf{0}$ in $H^1(B_R\setminus\overline{\Gamma}_0)^3$, then, from the compactness of $H^1(B_R\setminus\overline{\Gamma}_0)^3 \subset L^2(B_R\setminus\overline{\Gamma}_0)^3$, the boundedness of the trace operator $H^1(B_R\setminus\overline{\Gamma}_0)^3 \to H^{1/2}(B_R\setminus\overline{\Gamma}_0)^3$, together with the compactness of the embedding $\widetilde{H}^{1/2}(\Gamma_0)^3 \subset L^2(\Gamma_0)^3$, and the compactness of $\mathcal{T}_R^c : H^{1/2}(S_R)^3 \to H^{-1/2}(S_R)^3$, we conclude that $B(\mathbf{u}_n,\mathbf{v}) \to 0$ for all $\mathbf{v}$ in $H^1(B_R\setminus\overline{\Gamma}_0)^3$. Hence, $B(\cdot,\cdot)$ is a compact sesquilinear form. Therefore, A is the sum of a coercive and a compact sequilinear form, and thus the uniqueness of the solution to the problem is equivalent to its well posedness.

To prove uniqueness, suppose that $A(\mathbf{u},\mathbf{u})=0$, and take the imaginary part. Then:

$$0 \;=\; \int_{\Gamma_0} \Im(\mathbf{K}\,[\mathbf{u}]\cdot[\overline{\mathbf{u}}])\;\mathrm{d}s(\mathbf{y}) - \Im\Big\{\int_{S_R} \mathcal{T}_R\mathbf{u}\cdot\overline{\mathbf{u}}\;\mathrm{d}s(\mathbf{y})\Big\}. \tag{2.35}$$

By the properties of $\mathcal{T}_R$ stated in Lemma 2.2.1 (and pivoting with $L^2(S_R)^3$),

$$Im\Big\{\int_{S_R} \mathcal{T}_R\mathbf{u}\cdot\overline{\mathbf{u}}\;\mathrm{d}s(\mathbf{y})\Big\} \;\geq\; 0. \tag{2.36}$$

Since $\Im([\overline{\mathbf{u}}]\cdot\mathbf{K}\,[\mathbf{u}])\leq 0$, then necessarily:

$$\Im\Big\{\int_{S_R} \mathcal{T}_R\mathbf{u}\cdot\overline{\mathbf{u}}\;\mathrm{d}s(\mathbf{y})\Big\} \;=\; 0. \tag{2.37}$$

Therefore, $\mathbf{u}=\mathbf{0}$ on S_R, and therefore, by the unique continuation principle, $\mathbf{u}=\mathbf{0}$ in B_R, which finishes the proof. □

2.3 The inverse problem

2.3.1 Preliminary notation and concepts

For any $\widehat{\mathbf{d}}\in\mathbb{S}^2$ and $\mathbf{p}\in\mathbb{R}^3$, we define the pressure plane waves $\mathbf{u}^{i,p}(\cdot,\widehat{\mathbf{d}},\mathbf{p})$ and the shear plane waves $\mathbf{u}^{i,s}(\cdot,\widehat{\mathbf{d}},\mathbf{p})$ with incidence direction $\widehat{\mathbf{d}}$ and polarization vector $\mathbf{p}$, by

$$\mathbf{u}^{i,s}(\mathbf{x},\widehat{\mathbf{d}},\mathbf{p}) = \widehat{\mathbf{d}}\times(\mathbf{p}\times\widehat{\mathbf{d}})\;e^{ik_s\mathbf{x}\cdot\widehat{\mathbf{d}}}, \quad \text{and} \quad \mathbf{u}^{i,p}(\mathbf{x},\widehat{\mathbf{d}},\mathbf{p}) = (\mathbf{p}\cdot\widehat{\mathbf{d}})\widehat{\mathbf{d}}\;e^{ik_p\mathbf{x}\cdot\widehat{\mathbf{d}}}, \tag{2.38}$$

where k_s and k_p are defined by (2.10) These two orthogonal families of plane waves constitute important analytic solutions to the homogeneous problem $\Delta_0^*\mathbf{u}+\omega^2\mathbf{u}=\mathbf{0}$ in $\mathbb{R}^3$, where Δ_0^* refers to the elastic differential operator (2.4) corresponding to constant Lamé coeafficients λ_0 and μ_0 in $\mathbb{R}^3$.

The incident plane-wave tensor $\mathbf{W}^i(\cdot,\widehat{\mathbf{d}})\in\mathcal{C}^\infty(\mathbb{R}^3)^{3\times3}$ defined by

$$\mathbf{W}^i(\mathbf{x},\widehat{\mathbf{d}}) := e^{ik_s\mathbf{x}\cdot\widehat{\mathbf{d}}}(\mathbf{I}_{3\times3}-\widehat{\mathbf{d}}\otimes\widehat{\mathbf{d}}) + e^{ik_p\mathbf{x}\cdot\widehat{\mathbf{d}}}\widehat{\mathbf{d}}\otimes\widehat{\mathbf{d}},$$

then satisfies

$$\mathbf{W}^i(\cdot,\widehat{\mathbf{d}})\mathbf{p} = \mathbf{u}^{i,s}(\cdot,\widehat{\mathbf{d}},\mathbf{p}) + \mathbf{u}^{i,p}(\cdot,\widehat{\mathbf{d}},\mathbf{p}) \quad \text{in } \mathbb{R}^3, \tag{2.39}$$

for all $\widehat{\mathbf{d}} \in \mathbb{S}^2$ and $\mathbf{p} \in \mathbb{R}^3$.

Next we introduce in this section some necessary concepts and notation for our analysis. In the sequel, $\mathbf{u}_b$ will be called a total field solution to the background problem due to the incident field $\mathbf{u}^i$, if it satisfies:

$$\mathbf{u}_b \in H^1_{loc}(\mathbb{R}^3)^3 \tag{2.40}$$

$$\Delta^* \mathbf{u}_b + \omega^2 \rho \mathbf{u}_b = \mathbf{0} \quad \text{in } \mathbb{R}^3, \tag{2.41}$$

$$\mathbf{u}_b = \mathbf{u}_b^{sc} + \mathbf{u}^i \qquad \text{in } \Omega_{ext}, \tag{2.42}$$

and where $\mathbf{u}_b^{sc}$ satisfies the Kupradze radiation conditions (2.7).

Let $\widehat{\mathbf{d}} \in \mathbb{S}^2$ and $\mathbf{p} \in \mathbb{R}^3$, then denote by $\mathbf{u}_b(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$ the total field solution to the background problem (2.40)-(2.42) when the incident field is precisely the pressure and shear plane wave combination $\mathbf{u}^i(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) := \mathbf{W}^i(\cdot, \widehat{\mathbf{d}})\mathbf{p}$. Since the mapping $\mathbf{p} \mapsto \mathbf{u}_b(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$ is linear, then there is a second order response tensor $\mathbf{W}_b(\cdot, \widehat{\mathbf{d}})$ such that $\mathbf{W}_b(\cdot, \widehat{\mathbf{d}})\mathbf{p} = u_b(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$.

Given a scattered wave $\mathbf{u}^{sc}$ that solves $\Delta^* \mathbf{u}^{sc} + \omega^2 \mathbf{u}^{sc} = \mathbf{0}$ in Ω_{ext}, it can be shown that it has the asymptotic behavior ([59]):

$$\mathbf{u}^{sc}(\mathbf{x}) = \alpha_p \frac{e^{ik_p r}}{r} \mathbf{u}^{p,\infty}(\widehat{\mathbf{x}}) + \alpha_s \frac{e^{ik_s r}}{r} \mathbf{u}^{s,\infty}(\widehat{\mathbf{x}}) + O\left(\frac{1}{r^2}\right), \tag{2.43}$$

where, in turn, $\mathbf{u}^{p,\infty}$, $\mathbf{u}^{s,\infty}$ are the corresponding acoustic far-field patterns of the waves $\mathbf{u}^s$ and $\mathbf{u}^p$ defined by (2.8)-(2.9) (see e.g. [21],[32]), and

$$\alpha_p = \frac{1}{4\pi(\lambda_0 + 2\mu_0)} \quad \text{and} \quad \alpha_s = \frac{1}{4\pi\mu_0}. \tag{2.44}$$

The elastic far-field pattern of $\mathbf{u}^{sc}$ is defined by

$$\mathbf{u}^\infty := \mathbf{u}^{p,\infty} \oplus \mathbf{u}^{s,\infty}. \tag{2.45}$$

Denote by $\mathbf{\Gamma}_0(\cdot, \mathbf{z})$ the fundamental solution, also called the Kupradze matrix associated with the Lamé coefficients of λ_0 and μ_0, that satisfies:

$$\Delta_0^* \mathbf{\Gamma}_0 + w^2 \mathbf{\Gamma}_0 = -\delta_{\mathbf{z}} \, \mathbf{I}_{3\times 3} \text{ in } \mathbb{R}^3, \tag{2.46}$$

where $\delta_{\mathbf{z}}$ is the Dirac distribution with support in $\mathbf{z}$.

It is well known that $\mathbf{\Gamma}_0$ is given ([59]) by

$$\mathbf{\Gamma}_0(\mathbf{x},\mathbf{z}) = -\alpha_s\phi_{k_s}(\mathbf{x},\mathbf{z})\mathbf{I}_{3\times 3} - \beta_s(\nabla_x\otimes\nabla_x)\phi_{k_s}(\mathbf{x},\mathbf{z}) - \beta_p(\nabla_x\otimes\nabla_x)\phi_{k_p}(\mathbf{x},\mathbf{z}),$$

where k_s and k_p are defined by (2.10), α_s is defined by (2.44), $\beta_s = \frac{1}{4\pi\omega^2}$, $\beta_p = -\frac{1}{4\pi\omega^2}$, and where for any $k\in\mathbb{R}$, $\phi_k(\mathbf{x},\mathbf{z}) = \frac{e^{ik|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|}$, which is the well-known fundamental solution of the Helmholtz operator $\Delta u + k^2 u$ in $\mathbb{R}^3$.

The far-field pattern of the Kupradze matrix is given by

$$\mathbf{\Gamma}_0^{\infty}(\widehat{\mathbf{x}},\mathbf{z}) = \mathbf{\Gamma}_0^{p,\infty}(\widehat{\mathbf{x}},\mathbf{z}) \oplus \mathbf{\Gamma}_0^{s,\infty}(\widehat{\mathbf{x}},\mathbf{z}), \tag{2.47}$$

where

$$\mathbf{\Gamma}_0^{p,\infty}(\widehat{\mathbf{x}},\mathbf{z}) = \gamma\, e^{-ik_s\widehat{\mathbf{x}}\cdot\mathbf{z}}(\mathbf{I}_{3\times 3} - \widehat{\mathbf{x}}\otimes\widehat{\mathbf{x}}) \text{ and } \mathbf{\Gamma}_0^{p,\infty}(\widehat{\mathbf{x}},\mathbf{z}) = \gamma\, e^{-ik_p\widehat{\mathbf{x}}\cdot\mathbf{z}}\,\widehat{\mathbf{x}}\otimes\widehat{\mathbf{x}}, \tag{2.48}$$

and $\gamma = \frac{1}{4\pi}$. Then, for every fixed $\widehat{\mathbf{x}}\in\mathbb{S}^2$, $\mathbf{\Gamma}_0^{\infty}(\widehat{\mathbf{x}},\cdot) = \gamma\mathbf{W}^i(\cdot,-\widehat{\mathbf{x}})$.

2.3.2 The mixed reciprocity principle and the scattering operator

The inverse problem we aim to tackle is to identify the part Γ_0 of the interface Γ where the crack is located, from far field measurements corresponding to incident plane-waves in all possible directions $\widehat{\mathbf{d}}\in\mathbb{S}^2$ and polarization vectors $\mathbf{p}\in\mathbb{R}^3$.

It will be proven in the following section that this can indeed be done, using a version of the Factorization Method, first introduced by A. Kirsch [56].

To this end, we assume that we know the background configuration, that is, the configuration in absence of the defect Γ_0.

For every $\mathbf{z}\in\mathbb{R}^3$, let $\mathbb{G}_b(\cdot,\mathbf{z})$ in $H^1_{loc}(\mathbb{R}^3\setminus\{\mathbf{z}\})^{3\times 3}$ be the (radiating) Green's matrix associated with the background problem, that is,

$$\Delta^*\mathbb{G}_b(\cdot,\mathbf{z}) + \omega^2\rho\mathbb{G}_b(\cdot,\mathbf{z}) = -\delta_{\mathbf{z}}\,\mathbf{I}_{3\times 3} \text{ in } \mathbb{R}^3, \tag{2.49}$$

and such that $\mathbb{G}_b(\cdot,\mathbf{z})\mathbf{p}$ satisfies the elastic radiation condition (2.7) for every $\mathbf{p}\in\mathbb{R}^3$. Then we have the following:

Theorem 2.3.1 (Mixed reciprocity principle). *Let $\mathbb{G}_b^\infty(\cdot,\mathbf{z})$ denote the far-field pattern of $\mathbb{G}_b(\cdot,\mathbf{z})$, then the following relation holds*

$$\mathbb{G}_b^\infty(\widehat{\mathbf{d}},\mathbf{z}) = \gamma\mathbf{W}_b(\mathbf{z},-\widehat{\mathbf{d}}) \textit{ for all } \mathbf{z}\in\mathbb{R}^3 \textit{ and } \widehat{\mathbf{d}}\in\mathbb{S}.$$

Proof. Case 1. Let $\mathbf{z}\in\Omega_{ext}$.

By definition, all the columns of $\mathbb{G}_b(\cdot,\mathbf{z})-\mathbf{\Gamma}_0(\cdot,\mathbf{z})$ are non-singular radiating solutions of $\Delta_0^*\mathbf{u}+\rho\omega^2\mathbf{u}=\mathbf{0}$ in Ω_{ext}. Therefore, after integrating by parts twice using Betti's formulas and recalling that both $\mathbb{G}_b(\cdot,\mathbf{z})-\mathbf{\Gamma}_0(\cdot,\mathbf{z})$ and $\mathbf{\Gamma}_0(\cdot,\mathbf{y})$ satisfy the radiation conditions (2.7):

$$\begin{aligned}(\mathbb{G}_b-\mathbf{\Gamma}_0)(\mathbf{x},\mathbf{z}) &= \int_{\Gamma_1}\Big\{\partial^*_{\boldsymbol{\nu}(y)}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})(\mathbb{G}_b-\mathbf{\Gamma}_0)(\mathbf{y},\mathbf{z}) \\ &- \mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}_y}(\mathbb{G}_b-\mathbf{\Gamma}_0)(\mathbf{y},\mathbf{z})\Big\}\,\mathrm{d}s(\mathbf{y}) \\ &= \int_{\Gamma_1}\big\{\mathbb{G}_b(\mathbf{y},\mathbf{z})\partial^*_{\boldsymbol{\nu}(y)}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y}) \\ &- \mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}(y)}\mathbb{G}_b(\mathbf{y},\mathbf{z})\big\}\,\mathrm{d}s(\mathbf{y}),\end{aligned} \tag{2.50}$$

for all $\mathbf{x}\in\Omega_{ext}$, and where in the second line we have used Lemma A.0.2, i.e.,

$$\int_{\Gamma_1}\big\{\mathbf{\Gamma}_0(\mathbf{y},\mathbf{z})\partial^*_{\boldsymbol{\nu}_y}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})-\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}_y}\mathbf{\Gamma}_0(\mathbf{y},\mathbf{z})\big\}\,\mathrm{d}s(\mathbf{y})=\mathbf{0}.$$

Then, from (2.50), and from the fact that $\mathbf{\Gamma}_0^\infty(\widehat{\mathbf{x}},\mathbf{z})=\gamma\mathbf{W}^i(\mathbf{z},-\widehat{\mathbf{x}})$,

$$\begin{aligned}&\mathbb{G}_b^\infty(\widehat{\mathbf{x}},\mathbf{z})-\gamma\mathbf{W}^i(\mathbf{z},-\widehat{\mathbf{x}}) \\ &= \gamma\textstyle\int_{\Gamma_1}\Big\{\mathbb{G}_b(\mathbf{y},\mathbf{z})\partial^*_{\boldsymbol{\nu}_y}\mathbf{W}^i(\mathbf{y},-\widehat{\mathbf{x}})-\partial^*_{\boldsymbol{\nu}_y}\mathbb{G}_b^i(\mathbf{y},\mathbf{z})\mathbf{W}^i(\mathbf{y},-\widehat{\mathbf{x}})\Big\}\,\mathrm{d}s(\mathbf{y})\end{aligned} \tag{2.51}$$

for all $\mathbf{x}\in\Omega_{ext}$.

On the other hand, the columns of the scattered field associated with the background problem, $\mathbf{W}_b^{sc}(\cdot,-\widehat{\mathbf{x}})$, are also radiating solutions of $\Delta^*\mathbf{u}+\omega^2\rho\mathbf{u}=\mathbf{0}$ in Ω_{ext}, so

$$\begin{aligned}&\int_{\Gamma_1}\Big\{(\mathbf{\Gamma}_0-\mathbb{G}_b)(\mathbf{y},\mathbf{z})\partial^*_{\boldsymbol{\nu}_y}\mathbf{W}^{sc}_{b,ext}(\mathbf{y},-\widehat{\mathbf{x}}) \quad - \\ &\partial^*_{\boldsymbol{\nu}_y}(\mathbf{\Gamma}_0-\mathbb{G}_b)(\mathbf{y},\mathbf{z})\mathbf{W}^{sc}_{b,ext}(\mathbf{y},-\widehat{\mathbf{x}})\Big\}\,\mathrm{d}s(\mathbf{y}) \quad = \quad \mathbf{0},\end{aligned}$$

but by the integral representation of $\mathbf{W}_b^{sc}(\cdot, -\widehat{\mathbf{x}})$ in Ω_{ext} (see [59]), we have that

$$\begin{aligned}
\mathbf{W}_b^{sc}(\mathbf{z}, -\widehat{\mathbf{x}}) &= \int_{\Gamma_1} \Big\{ \partial^*_{\boldsymbol{\nu}(y)} \mathbf{\Gamma}_0(\mathbf{y}, \mathbf{z}) \mathbf{W}^{sc}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) \\
&- \mathbf{\Gamma}_0(\mathbf{y}, \mathbf{z}) \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}^{sc}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}) \\
&= \int_{\Gamma_1} \Big\{ \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \mathbf{W}^{sc}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) \\
&- \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}^{sc}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}). \qquad (2.52)
\end{aligned}$$

Additionally, from the transmission conditions of the background problem,

$$\begin{aligned}
& \int_{\Gamma_1} \Big\{ \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \mathbf{W}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) - \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}) \\
= & \int_{\Gamma_1} \Big\{ \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \mathbf{W}_{b,+}(\mathbf{y}, -\widehat{\mathbf{x}}) - \mathbb{G}_{b,+}(\mathbf{z}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}_{b,+}(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}) \\
= & \int_{\Omega_+ \cup \Omega_-} \Big\{ \Delta^*_y \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \mathbf{W}_b(\mathbf{y}, -\widehat{\mathbf{x}}) - \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \Delta^*_y \mathbf{W}_b(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Gamma} \Big\{ \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \left[\mathbf{W}_b\right](\mathbf{y}, -\widehat{\mathbf{x}}) - \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \left[\partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}_b\right](\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}) \\
= & \ \mathbf{0}. \qquad (2.53)
\end{aligned}$$

So from (2.52) and (2.53), since $\mathbf{u}_b = \mathbf{u}_b^{sc} + \mathbf{u}^i$, we have that

$$\mathbf{W}_b^{sc}(\mathbf{z}, -\widehat{\mathbf{x}}) = \int_{\Gamma_1} \Big\{ \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}^i(\mathbf{y}, -\widehat{\mathbf{x}}) - \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{z}, \mathbf{y}) \mathbf{W}^i(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}). \quad (2.54)$$

And then, from (2.51),

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) = \gamma \mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{x}}).$$

Case 2. Let $\mathbf{z} \in \Omega_+ \cup \Omega_-$. Then $\mathbb{G}_b(\cdot, \mathbf{z})$ is a smooth radiating solution of $\Delta^* \mathbf{u} + \rho\omega^2 \mathbf{u} = \mathbf{0}$ in Ω_{ext}, so integrating by parts twice using Betti's formulas, and using the fact that both $\mathbf{\Gamma}_0(\cdot, \mathbf{y})$ and $\mathbb{G}_b(\cdot, \mathbf{z})$ radiation condition:

$$\begin{aligned}
\mathbb{G}_b(\mathbf{x}, \mathbf{z}) &= \int_{\Gamma_1} \Big\{ \partial^*_{\boldsymbol{\nu}} \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y}) \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \\
&- \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \Big\} \mathrm{d}s(\mathbf{y}), \qquad (2.55)
\end{aligned}$$

for every $\mathbf{x} \in \Omega_{ext}$, and then

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) = \int_{\Gamma_1} \left\{ \partial^*_{\boldsymbol{\nu}(y)} \boldsymbol{\Gamma}_0^\infty(\widehat{\mathbf{x}}, \mathbf{y}) \mathbb{G}_b(\mathbf{y}, \mathbf{z}) - \boldsymbol{\Gamma}_0^\infty(\widehat{\mathbf{x}}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \right\} \, \mathrm{d}s(\mathbf{y}). \tag{2.56}$$

Moreover, since the rows of $\mathbf{W}_b^{sc}(\cdot, -\widehat{\mathbf{x}})$ are all radiating solutions of $\Delta^* \mathbf{u} + \rho\omega^2 \mathbf{u} = 0$ in Ω_{ext},

$$\int_{\Gamma_1} \left\{ \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \mathbf{W}^{sc}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) - \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}^{sc}_{b,ext}(\mathbf{y}, -\widehat{\mathbf{x}}) \right\} \, \mathrm{d}s(\mathbf{y}) = \mathbf{0}, \tag{2.57}$$

and then adding (2.56) and γ(2.57),

$$\begin{aligned}
\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) &= \gamma \int_{\Gamma_1} \Big\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \partial^*_{\boldsymbol{\nu}_y} \mathbf{W}_b(\widehat{\mathbf{x}}, \mathbf{y}) \\
&- \partial^*_{\boldsymbol{\nu}_y} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \mathbf{W}_b(\widehat{\mathbf{x}}, \mathbf{y}) \Big\} \, \mathrm{d}s(\mathbf{y}) \\
&= \gamma \int_{\Omega_+ \cup \Omega_-} \Big\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \Delta^*_{\boldsymbol{\nu}_y} \mathbf{W}_b(\mathbf{y}, -\widehat{\mathbf{x}}) \\
&- \Delta^*_{\boldsymbol{\nu}_y} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \mathbf{W}_b(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}\,\mathbf{y} \\
&+ \gamma \int_{\Gamma} \Big\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \left[\partial^*_{\boldsymbol{\nu}_y} \mathbf{W}_b \right] (\mathbf{y}, -\widehat{\mathbf{x}}) \\
&- \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \left[\mathbf{W}_b \right] (\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}s(\mathbf{y}) \\
&= \gamma \int_{\Omega_+ \cup \Omega_-} \Big\{ -\rho\omega^2 \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \mathbf{W}_b(\mathbf{y}, -\widehat{\mathbf{x}}) \\
&+ \left(\delta_{\mathbf{z}}(\mathbf{y}) \mathbf{I}_{3\times 3} + \rho\omega^2 \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \right) \mathbf{W}_b(\mathbf{y}, -\widehat{\mathbf{x}}) \Big\} \, \mathrm{d}\,\mathbf{y} \\
&= \gamma \, \mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{x}}).
\end{aligned}$$

Finally, by continuity of $\mathbb{G}_b$ and $\mathbf{W}_b$, we know that the identity $\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \cdot) = \gamma \mathbf{W}_b(\cdot, -\widehat{\mathbf{x}})$ holds everywhere in $\mathbb{R}^3$. $\square$

Definition 2.3.1. *The scattering operator $\mathcal{S}_b : L^2(\mathbb{S}^2)^3 \to L^2(\mathbb{S}^2)^3$ is defined by*

$$\mathcal{S}_b = I + 2ik\gamma|\alpha|^2 \mathcal{F}_b,$$

where I denotes the identity map, $k = k_p \oplus k_s$, and $\gamma = -\frac{1}{4\pi}$ and $\alpha = \alpha_p \oplus \alpha_s$.

Remark 2.3.1. By similar arguments to those given in [56], it can be proven that the scattering operator is unitary, that is $\mathcal{S}_b \mathcal{S}_b^* = \mathcal{S}_b^* \mathcal{S}_b = I$.

The following result describes the role of the scattering operator. The proof given below follows the proof of Theorem 2.3 in [17], but we include it here for the reader's convenience.

Proposition 2.3.1. *For every* $\mathbf{z} \in \Omega$ *and* $\widehat{\mathbf{x}} \in \mathbb{S}^2$*,*

$$\mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{x}}) = \mathcal{S}_b(\overline{\mathbf{W}_b(\mathbf{z}, \cdot)})(\widehat{\mathbf{x}}). \tag{2.58}$$

Proof. Let $\mathbf{y} \in \Omega_{ext}$ and $\mathbf{z} \in \Omega$. Choose $R > 0$ large enough so that the open ball B_R satisfies $\{\mathbf{y}\} \cup \overline{\Omega} \subset B_R$. Then, the columns of both $\mathbb{G}_b(\cdot, \mathbf{z}) - \overline{\mathbb{G}_b(\cdot, \mathbf{z})}$ and $\mathbb{G}_b(\cdot, \mathbf{z}) - \mathbf{\Gamma}_0(\cdot, \mathbf{z})$ are regular solutions to $\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = \mathbf{0}$ in Ω_{ext}.
Using Betti's formulae in $\Omega_R := \Omega_{ext} \cap B_R$,

$$\begin{aligned}
& \mathbb{G}_b(\mathbf{y}, \mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{y}, \mathbf{z})} \\
= & \int_{S_R} \Big\{ \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x}, \mathbf{z})}) \\
- & \partial^*_{\boldsymbol{\nu}(x)} \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y}) (\mathbb{G}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x}, \mathbf{z})}) \Big\} \, \mathrm{d}s(\mathbf{x}) \\
- & \int_{\Gamma_1} \Big\{ \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z})}) \\
- & \partial^*_{\boldsymbol{\nu}(x)} \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y}) (\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z})}) \Big\} \, \mathrm{d}s(\mathbf{x}).
\end{aligned} \tag{2.59}$$

On the other hand, again as a consequence of Betti's formulae,

$$\begin{aligned}
0 = & \int_{S_R} \Big\{ (\mathbb{G}_b(\mathbf{x}, \mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y})) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x}, \mathbf{z})}) \\
- & \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}_b(\mathbf{x}, \mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y})) (\mathbb{G}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x}, \mathbf{z})}) \Big\} \, \mathrm{d}s(\mathbf{x}) \\
- & \int_{\Gamma_1} \Big\{ (\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y})) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z})}) \\
- & \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x}, \mathbf{y})) (\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z}) - \overline{\mathbb{G}^{ext}_b(\mathbf{x}, \mathbf{z})}) \Big\} \, \mathrm{d}s(\mathbf{x}).
\end{aligned} \tag{2.60}$$

Thus combining identities (2.60) and (2.60),

$$\begin{aligned}
&\mathbb{G}_b(\mathbf{y},\mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{y},\mathbf{z})} \\
= &\int_{S_R} \Big\{ \mathbb{G}_b(\mathbf{x},\mathbf{y}) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}_b(\mathbf{x},\mathbf{z}) \\
- &\overline{\mathbb{G}_b(\mathbf{x},\mathbf{z})}) - \partial^*_{\boldsymbol{\nu}(x)} \mathbb{G}_b(\mathbf{x},\mathbf{y}) (\mathbb{G}_b(\mathbf{x},\mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x},\mathbf{z})}) \Big\} \,\mathrm{d}s(\mathbf{x}) \\
- &\int_{\Gamma_1} \Big\{ \mathbb{G}_b^{ext}(\mathbf{x},\mathbf{y}) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}_b^{ext}(\mathbf{x},\mathbf{z}) - \overline{\mathbb{G}_b^{ext}(\mathbf{x},\mathbf{z})}) \\
- &\partial^*_{\boldsymbol{\nu}(x)} \mathbb{G}^{ext}(\mathbf{x},\mathbf{y}) (\mathbb{G}_b^{ext}(\mathbf{x},\mathbf{z}) - \overline{\mathbb{G}_b^{ext}(\mathbf{x},\mathbf{z})}) \Big\} \,\mathrm{d}s(\mathbf{x}),
\end{aligned} \tag{2.61}$$

and applying Betti's formulae in Ω for the integral on Γ_1 in (2.61),

$$\begin{aligned}
&\mathbb{G}_b(\mathbf{y},\mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{y},\mathbf{z})} \\
= &\int_{S_R} \Big\{ \mathbb{G}_b(\mathbf{x},\mathbf{y}) \partial^*_{\boldsymbol{\nu}(x)} (\mathbb{G}_b(\mathbf{x},\mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x},\mathbf{z})}) \\
- &\partial^*_{\boldsymbol{\nu}(x)} \mathbb{G}_b(\mathbf{x},\mathbf{y}) (\mathbb{G}_b(\mathbf{x},\mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{x},\mathbf{z})}) \Big\} \,\mathrm{d}s(\mathbf{x}) \\
= &-\int_{S_R} \Big\{ \mathbb{G}_b(\mathbf{x},\mathbf{y}) \partial^*_{\boldsymbol{\nu}(x)} \overline{\mathbb{G}_b(\mathbf{x},\mathbf{z})} \\
+ &\partial^*_{\boldsymbol{\nu}(x)} \mathbb{G}_b(\mathbf{x},\mathbf{y}) \overline{\mathbb{G}_b(\mathbf{x},\mathbf{z})} \Big\} \,\mathrm{d}s(\mathbf{x}),
\end{aligned} \tag{2.62}$$

where in the last step we used that both $\mathbb{G}_b(\cdot,\mathbf{z})$ and $\mathbb{G}_b(\cdot,\mathbf{y})$ are smooth in $\mathbb{R}^3 \setminus \overline{B_R}$ and radiating. Therefore, taking the limit when $R \to \infty$,

$$\begin{aligned}
&\mathbb{G}_b(\mathbf{y},\mathbf{z}) - \overline{\mathbb{G}_b(\mathbf{y},\mathbf{z})} \\
= &\int_{\mathbb{S}^2} \Big\{ \alpha_p \mathbb{G}_b^{p,\infty}(\mathbf{x},\mathbf{y}) \overline{ik_p \alpha_p \mathbb{G}_b^{p,\infty}(\widehat{\mathbf{x}},\mathbf{z})} + \alpha_s \mathbb{G}_b^{s,\infty}(\widehat{\mathbf{x}},\mathbf{y}) \overline{ik_s \alpha_s \mathbb{G}_b^{s,\infty}(\widehat{\mathbf{x}},\mathbf{z})} \\
- &\alpha_p ik_p \mathbb{G}_b^{p,\infty}(\mathbf{x},\mathbf{y}) \overline{\alpha_p \mathbb{G}_b^{p,\infty}(\widehat{\mathbf{x}},\mathbf{z})} - \alpha_s ik_s \mathbb{G}_b^{s,\infty}(\widehat{\mathbf{x}},\mathbf{y}) \overline{\alpha_s \mathbb{G}_b^{s,\infty}(\widehat{\mathbf{x}},\mathbf{z})} \Big\} \,\mathrm{d}s(\widehat{\mathbf{x}}) \\
= &\, 2i \int_{\mathbb{S}^2} \Big\{ k_p |\alpha_p|^2 \mathbb{G}_b^{p,\infty}(\mathbf{x},\mathbf{y}) \overline{\mathbb{G}_b^{p,\infty}(\widehat{\mathbf{x}},\mathbf{z})} + k_s |\alpha_s|^2 \mathbb{G}_b^{s,\infty}(\widehat{\mathbf{x}},\mathbf{y}) \overline{\mathbb{G}_b^{s,\infty}(\widehat{\mathbf{x}},\mathbf{z})} \Big\} \,\mathrm{d}s(\widehat{\mathbf{x}}) \\
= &\, 2ik|\alpha|^2 \int_{\mathbb{S}^2} \Big\{ \mathbb{G}_b^{\infty}(\mathbf{x},\mathbf{y}) \overline{\mathbb{G}_b^{\infty}(\widehat{\mathbf{x}},\mathbf{z})} \Big\} \,\mathrm{d}s(\widehat{\mathbf{x}}),
\end{aligned} \tag{2.63}$$

where $k = k_p \oplus k_s$ and $\alpha = \alpha_p \oplus \alpha_s$.

On the other hand from the mixed reciprocity principle Theorem 2.3.1 and corollary A.0.1,

$$\begin{aligned}
&\gamma \mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{x}}) - \gamma \overline{\mathbf{W}_b(\mathbf{z}, \widehat{\mathbf{x}})} \\
= &\int_{\Gamma_1} \partial^*_{\boldsymbol{\nu}(y)} \boldsymbol{\Gamma}^\infty_0(\widehat{\mathbf{x}}, \mathbf{y})(\mathbb{G}^{ext}_b(\mathbf{y}, \mathbf{z}) - \overline{\mathbb{G}^{ext}_b(\mathbf{y}, \mathbf{z})}) \\
- &\int_{\Gamma_1} \boldsymbol{\Gamma}^\infty_0(\widehat{\mathbf{x}}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)}(\mathbb{G}^{ext}_b(\mathbf{y}, \mathbf{z}) - \overline{\mathbb{G}^{ext}_b(\mathbf{y}, \mathbf{z})}) \\
= &\; 2ik|\alpha|^2 \int_{\mathbb{S}^2} \int_{\Gamma_1} \left\{ \partial^*_{\boldsymbol{\nu}(y)} \boldsymbol{\Gamma}^\infty_0(\widehat{\mathbf{x}}, \mathbf{y}) \left(\mathbb{G}^\infty_b(\widehat{\mathbf{d}}, \mathbf{y}) \overline{\mathbb{G}^\infty_b(\widehat{\mathbf{d}}, \mathbf{z})} \right) \right\} \, \mathrm{d}s(\mathbf{y}) \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
- &\; 2ik|\alpha|^2 \int_{\mathbb{S}^2} \int_{\Gamma_1} \left\{ \boldsymbol{\Gamma}^\infty_0(\widehat{\mathbf{x}}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \left(\mathbb{G}^\infty_b(\widehat{\mathbf{d}}, \mathbf{y}) \overline{\mathbb{G}^\infty_b(\widehat{\mathbf{d}}, \mathbf{z})} \right) \right\} \, \mathrm{d}s(\mathbf{y}) \, \mathrm{d}s(\widehat{\mathbf{d}}),
\end{aligned} \tag{2.64}$$

and again using the mixed reciprocity relation Theorem 2.3.1,

$$\begin{aligned}
&\gamma \mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{x}}) - \gamma \overline{\mathbf{W}_b(\mathbf{z}, \widehat{\mathbf{x}})} \\
= &\; 2ik\gamma|\alpha|^2 \int_{\mathbb{S}^2} \overline{\mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{d}})} \Bigg\{ \int_{\Gamma_1} \partial^*_{\boldsymbol{\nu}(y)} \boldsymbol{\Gamma}^\infty_0(\widehat{\mathbf{x}}, \mathbf{y}) \mathbb{G}^\infty_b(\widehat{\mathbf{d}}, \mathbf{y}) \\
- &\; \boldsymbol{\Gamma}^\infty_0(\widehat{\mathbf{x}}, \mathbf{y}) \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}^\infty_b(\widehat{\mathbf{d}}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \Bigg\} \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
= &\; 2ik\gamma^2|\alpha|^2 \int_{\mathbb{S}^2} \overline{\mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{d}})} \mathbf{W}^\infty_b(\widehat{\mathbf{d}}, -\widehat{\mathbf{x}}) \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
= &\; 2ik\gamma^2|\alpha|^2 \int_{\mathbb{S}^2} \overline{\mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{d}})} \mathbf{W}^\infty_b(\widehat{\mathbf{d}}, -\widehat{\mathbf{x}}) \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
= &\; 2ik\gamma^2|\alpha|^2 \int_{\mathbb{S}^2} \overline{\mathbf{W}_b(\mathbf{z}, -\widehat{\mathbf{d}})} \mathbf{W}^\infty_b(\widehat{\mathbf{x}}, -\widehat{\mathbf{d}}) \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
= &\; 2ik\gamma^2|\alpha|^2 \int_{\mathbb{S}^2} \overline{\mathbf{W}_b(\mathbf{z}, \widehat{\mathbf{d}})} \mathbf{W}^\infty_b(\widehat{\mathbf{x}}, \widehat{\mathbf{d}}) \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
= &\; 2ik\gamma^2|\alpha|^2 \mathcal{F}_b(\overline{\mathbf{W}_b(\mathbf{z}, \cdot)})(\widehat{\mathbf{x}}),
\end{aligned} \tag{2.65}$$

which finishes the proof. □

2.3.3 The factorization method

For later use we define the *defective problem* as follows. Given $\mathbf{h} \in H^{-1/2}(\Gamma_0)^3$, we seek $\mathbf{u} \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Gamma}_0)^3$ such that

$$\Delta^* \mathbf{u} + \rho\omega^2 \mathbf{u} = \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Gamma \cup \Gamma_1} \tag{2.66}$$

$$\partial^*_{\boldsymbol{\nu}} \mathbf{u} = \mathbf{K}[\mathbf{u}] - \mathbf{h}, \text{ on } \Gamma_0, \tag{2.67}$$

and such that $\mathbf{u}$ satisfies the radiation condition (2.7).

Now let $\mathbf{g} \in L^2(\mathbb{S}^2)^3$. Then $\mathbf{g} = \mathbf{g}^s \oplus \mathbf{g}^p$, where

$$\mathbf{g}^p(\widehat{\mathbf{d}}) := (\widehat{\mathbf{d}} \otimes \widehat{\mathbf{d}})\mathbf{g} \text{ and } \mathbf{g}^s(\widehat{\mathbf{d}}) := (\mathbf{I}_{3\times 3} - \widehat{\mathbf{d}} \otimes \widehat{\mathbf{d}})\mathbf{g}. \tag{2.68}$$

The elastic Herglotz wave associated with $\mathbf{g}$ is defined by ([33])

$$\mathbf{v}_{\mathbf{g}}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{g}^s e^{ik_s \widehat{\mathbf{d}}\cdot\mathbf{x}} \ \mathrm{d}s(\widehat{\mathbf{d}}) + \int_{\mathbb{S}^2} \mathbf{g}^p e^{ik_p \widehat{\mathbf{d}}\cdot\mathbf{x}} \ \mathrm{d}s(\widehat{\mathbf{d}}), \text{ for } \mathbf{x} \in \mathbb{R}^3. \tag{2.69}$$

Let $\mathcal{F} : L^2(\mathbb{S}^2)^3 \to L^2(\mathbb{S}^2)^3$ denote the far-field operator associated with the crack problem, defined by

$$\mathcal{F}\mathbf{g} = \mathbf{u}^\infty_g,$$

where $\mathbf{u}^\infty_g$ is the far field pattern of the scattered field associated with the crack problem (2.5)-(2.6) when the incident field is precisely the Hergoltz wave $\mathbf{v}_{\mathbf{g}}$.

In a similar manner, we denote by $\mathcal{F}_b : L^2(\mathbb{S}^2)^3 \to L^2(\mathbb{S}^2)^3$ the far-field operator associated with the background, defined by

$$\mathcal{F}_b\mathbf{g} = \mathbf{u}^\infty_{g,b},$$

where $\mathbf{u}^\infty_{g,b}$ is the far field pattern of the scattered field associated with the background problem (2.40)-(2.42) when the incident field is again the Hergoltz wave $\mathbf{v}_{\mathbf{g}}$.

Finally, we define the far-field operator associated with the defect, $\mathcal{F}_D := \mathcal{F} - \mathcal{F}_b$. Notice that $\mathcal{F}_D\mathbf{g}$ is the far field pattern of the scattered field due to (2.66)-(2.67) when the source term correspnds to $\mathbf{h} = \partial^*_{\boldsymbol{\nu}} \mathbf{u}_{\mathbf{g},b}$.

The main result of this paper consists of a characterization of the range of $\mathcal{F}_D$, stated in Theorem 2.3.3, and which is the basis of the reconstruction algorithm of Γ_0. This theorem is a consequence of the abstract result Theorem 3.2 in [18], that for the reader's convenience we state as Theorem 2.3.2 below. This abstract result is in turn an adaptation of Theorem 2.15 in [56] and a particular case of Theorem 2.1 in [61]. Given a Hilbert space Z and a bounded linear operator $L : Z \to Z$, we define

$$Re(L) := \frac{L + L^*}{2} \quad \text{and} \quad Im(L) := \frac{L - L^*}{2i}. \tag{2.70}$$

Theorem 2.3.2. *Let $H \subset U \subset H^*$ be a Gelfand triple with a Hilbert space U and a reflexive Banach space H such that the embedding is dense. Moreover, let Y be a second Hilbert space and let $F : Y \to Y$, $\mathcal{H} : Y \to H$, and $T : H \to H^*$ be linear bounded operators such that we make the following assumptions:*

(A1) $\mathcal{H}^*$ is compact with dense range.

(A2) $Re(T) = C + K$ with some compact operator K and some self-adjoint and coercive operator $C : H \to H^*$, i.e, there exists $c > 0$ with $\langle \phi, C\phi \rangle \geq |\phi|^2$ for all $\phi \in H$, and $Im(T)$ is positive on the closure of the range of $\mathcal{H}$.

Then the operator $F^\# = |Re(F)| + Im(F)$ is positive, and the ranges of $\mathcal{H}^* : H^* \to Y$ and $(F^\#)^{1/2} : Y \to Y$ coincide.

In what follows we will prove that a factorization with the properties stated in Theorem 2.3.2 is satisfied in our case.

Define the Hergoltz operator by $\mathscr{H} : L^2(\mathbb{S}^2)^3 \to H^{-1/2}(\Gamma_0)^3$ such that $\mathscr{H}\mathbf{g} = \partial^*_{\boldsymbol{\nu}}\mathbf{u}_{\mathbf{g},b}$. Then we have the following:

Lemma 2.3.1. *Assume that there are no non-trivial background fields $\mathbf{u}_{\mathbf{g},b}$ such that $\partial^*_{\boldsymbol{\nu}}\mathbf{u}_{\mathbf{g},b}|_{\Gamma_0} = \mathbf{0}$. Then the Hergoltz operator $\mathscr{H}$ is injective with dense range and its conjugate transpose operator $\mathscr{H}^* : \widetilde{H}^{1/2}(\Gamma_0)^3 \to L^2(\mathbb{S}^2)^3$ satisfies:*

$$\mathcal{S}_b\mathscr{H}^*\boldsymbol{\eta} = \int_{\Gamma_0} \partial^*_{\boldsymbol{\nu}(y)}\mathbb{G}^\infty(\cdot, \mathbf{y})\boldsymbol{\eta}(\mathbf{y}) \;\, ds(\mathbf{y}). \tag{2.71}$$

Proof. Let $\eta \in \widetilde{H}^{1/2}(\Gamma_0)^3$, then

$$\begin{aligned}
(\mathscr{H}\mathbf{g}, \boldsymbol{\eta})_{L^2(\Gamma_0)^3} &= \int_{\Gamma_0} \partial^*_{\boldsymbol{\nu}} \mathbf{u}_{\mathbf{g},b}(\mathbf{y}) \cdot \overline{\boldsymbol{\eta}}(\mathbf{y})\, ds(\mathbf{y}) \\
&= \int_{\Gamma_0} \overline{\boldsymbol{\eta}}(\mathbf{y}) \cdot \int_{\mathbb{S}^2} \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}_b(\mathbf{y}, \widehat{\mathbf{d}}) \mathbf{g}(\widehat{\mathbf{d}})\, ds(\widehat{\mathbf{d}})\, ds(\mathbf{y}) \\
&= \int_{\mathbb{S}^2} \mathbf{g}(\widehat{\mathbf{d}}) \cdot \int_{\Gamma_0} \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}^\infty_b(-\widehat{\mathbf{d}}, \mathbf{y}) \overline{\boldsymbol{\eta}}(\mathbf{y})\, ds(\mathbf{y})\, ds(\widehat{\mathbf{d}}) \\
&= (\mathbf{g}, \mathscr{H}^*\boldsymbol{\eta})_{L^2(\mathbb{S}^2)^3},
\end{aligned} \tag{2.72}$$

thus,

$$\mathscr{H}^*\boldsymbol{\eta} = \int_{\Gamma_0} \overline{\partial^*_{\boldsymbol{\nu}} \mathbb{G}^\infty_b(-\cdot, \mathbf{y})} \boldsymbol{\eta}(\mathbf{y})\, ds(\mathbf{y}), \tag{2.73}$$

and from Proposition 2.3.1, it follows immediately identity (2.71). To prove the injectivity of $\mathscr{H}$, observe that if $\mathscr{H}\mathbf{g} = \mathbf{0}$, then from the assumption, $\mathbf{u}_{\mathbf{g},b} = \mathbf{0}$, and by the well posedness of the background problem (2.40)-(2.42), this means that $\mathbf{v}_{\mathbf{g}} = \mathbf{0}$, and hence $\mathbf{g} = \mathbf{0}$.

To prove the denseness of the range of $\mathscr{H}$, we will show that $\mathscr{H}^*$ is injective. From (2.73),

$$(\mathscr{H}^*\boldsymbol{\eta})(\widehat{\mathbf{d}}) = \overline{\mathbf{u}^\infty(-\widehat{\mathbf{d}})},$$

where $\mathbf{u}^\infty$ is the far field pattern of

$$\mathbf{u} = \int_{\Gamma_0} \partial^*_{\boldsymbol{\nu}} \mathbb{G}_b(\cdot, \mathbf{y}) \overline{\boldsymbol{\eta}(\mathbf{y})}\, ds(\mathbf{y}),$$

which is a generalized double-layer potential. Therefore, we know that $\mathbf{u} \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Gamma_0})^3$ and solves

$$\begin{aligned}
\Delta^*\mathbf{u} + \omega^2 \rho \mathbf{u} &= \mathbf{0} \text{ in } \mathbb{R}^3 \setminus (\Gamma_1 \cup \Gamma), && (2.74) \\
[\mathbf{u}] &= \overline{\boldsymbol{\eta}} \text{ on } \Gamma_0. && (2.75)
\end{aligned}$$

Thus if $\mathscr{H}^*\boldsymbol{\eta} = \mathbf{0}$, then $\mathbf{u}^\infty = \mathbf{0}$, and by Rellich's lemma (Lemma 2.11 in [32]) $\mathbf{u} = \mathbf{0}$ in Ω_{ext}, and by continuity of both the displacement and traction on Γ_1, Holmgren's theorem implies that $\mathbf{u} = \mathbf{0}$ in an open neighborhood of Γ_1. By the unique continuation principle, $\mathbf{u} = \mathbf{0}$ in Ω_+. Again, by Holmgren's theorem $\mathbf{u} = \mathbf{0}$ in an open neighborhood of $\Gamma \setminus \Gamma_0$, and thus by the unique continuation principle, $\mathbf{u} = \mathbf{0}$ in Ω_-. Therefore $[\mathbf{u}] = \mathbf{0}$ and then $\boldsymbol{\eta} = \mathbf{0}$, which finishes the proof. $\square$

Remark 2.3.2. Therefore, defining $T : H^{1/2}(\Gamma_0)^3 \to \widetilde{H}^{1/2}(\Gamma_0)^3$ by $T\mathbf{h} = [\mathbf{w_h}]$, where $\mathbf{w_h} \in H^1(\mathbb{R}^3 \setminus \overline{\Gamma_0})^3$ solves (2.66)-(2.67), we get automatically the factorization $\mathcal{F}_D = \mathcal{S}_b \mathscr{H}^* T \mathscr{H}$.

Lemma 2.3.2. *The operator T defined in Remark 2.3.2, is linear, bounded and there exist two bounded linear operators $T_0, T_c : H^{1/2}(\Gamma_0)^3 \to \widetilde{H}^{1/2}(\Gamma_0)^3$ such that $T = T_0 + T_c$, where T_0 is coercive and self-adjoint, and T_c is compact.*
Therefore, $Re(T) = T_0 + Re(T_c)$, and $Re(T_c)$ is compact.

Proof. The boundedness of operator T is an immediate consequence of the well-posedness of the problem (2.66)-(2.67).

Define $T_0 : H^{1/2}(\Gamma_0)^3 \to \widetilde{H}^{1/2}(\Gamma_0)^3$ by $T_0\mathbf{h} = [\mathbf{u}_0]$, where $\mathbf{u}_0 \in H^1(B_R \setminus \overline{\Gamma_0})^3$ satisfies

$$A_0(\mathbf{u}_0, \mathbf{v}) = \int_{\Gamma_0} \mathbf{h} \cdot [\overline{\mathbf{v}}] \; \mathrm{d}s(\mathbf{y}), \quad \text{for all } \mathbf{v} \in H^1(B_R \setminus \overline{\Gamma_0})^3. \tag{2.76}$$

Notice that $\mathbf{u}_0$ satisfies:

$$\begin{aligned} \Delta^* \mathbf{u}_0 - \mathbf{u}_0 &= \mathbf{0} \quad \text{in } B_R \setminus (\Gamma \cup \Gamma_1), & (2.77)\\ \partial^*_{\boldsymbol{\nu}} \mathbf{u}_0 &= -\mathbf{h} \quad \text{on } \Gamma_0. & (2.78) \end{aligned}$$

Since A_0 is coercive and self-adjoint, T_0 is well defined, bounded and self-adjoint too. Moreover, from the properties of the trace theorem,

$$\begin{aligned} ||\mathbf{h}||^2_{\widetilde{H}^{1/2}(\Gamma_0)^3} &= ||\partial^*_{\boldsymbol{\nu}} \mathbf{u}_0||^2_{\widetilde{H}^{1/2}(\Gamma_0)^3} \leq C \left(||\Delta^* \mathbf{u}_0||^2_{L^2(B_R)^3} + ||\mathbf{C} : \nabla \mathbf{u}_0||^2_{L^2(B_R)^3} \right) & (2.79)\\ &\leq \widetilde{C} \left(||\mathbf{u}_0||^2_{L^2(B_R)^3} + ||\nabla \mathbf{u}_0||^2_{L^2(B_R)^3} \right) & (2.80)\\ &\leq \widetilde{C}_1 |A_0(\mathbf{u}_0, \mathbf{u}_0))| = \widetilde{C}_1 \left| \int_{\Gamma_0} \mathbf{h} \cdot \overline{T_0 \phi} \; \mathrm{d}s(\mathbf{y}) \right|, & (2.81) \end{aligned}$$

and then

$$||\mathbf{h}||_{\widetilde{H}^{1/2}(\Gamma_0)^3} \leq \widetilde{C}_1 |\langle T_0 \mathbf{h}, \mathbf{h} \rangle|, \tag{2.82}$$

thus T_0 is coercive.

On the other hand, notice that if $T_c := T - T_0$, then, by definition, $T_c\mathbf{h} = [\mathbf{u}_c]$, where $\mathbf{u}_c = \mathbf{u} - \mathbf{u}_0$ satisfies the variational problem:

$$A_0(\mathbf{u}_c, \mathbf{v}) = -B(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H^1(B_R \setminus \overline{\Gamma_0})^3. \tag{2.83}$$

Since A_0 is coercive and B is compact, then the mapping $\mathbf{u} \mapsto \mathbf{u}_c$ is compact, and from the trace theorem together with the well posedness of the problem (2.66)-(2.67) that $\mathbf{u}$ satisfies, the mapping $\mathbf{h} \mapsto [\mathbf{u}_c]$, i.e. the operator T_c, is also compact. The fact that $Re(T_c)$ is compact is therefore an immediate consequence, and then the proof is complete. □

Lemma 2.3.3. *Assume that $\mathbf{K} \in L^\infty(\Gamma_0)^3$ is such that $\Im(\overline{\boldsymbol{\eta}} \cdot \mathbf{K}\boldsymbol{\eta}) \leq 0$ for all $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(\Gamma_0)^3$. Then the operator $Im(T) = \frac{T-T^*}{2i}$ is positive definite, i.e.,*

$$\langle Im(T)\mathbf{h}, \mathbf{h}\rangle > 0 \quad \textit{for all } \mathbf{h} \in H^{-1/2}(\Gamma_0)^3. \tag{2.84}$$

Proof. From the definition of T, we know that given $\mathbf{h} \in H^{-1/2}(\Gamma_0)^3$,

$$\int_{\Gamma_0} \mathbf{h} \cdot \overline{[\mathbf{w_h}]} \; \mathrm{d}s(\mathbf{y}) \;\; = \;\; A(\mathbf{w_h}, \mathbf{w_h}), \tag{2.85}$$

where $A(\cdot,\cdot)$ is defined by (2.12). Therefore for any given $\mathbf{h} \in H^{-1/2}(\Gamma_0)^3$,

$$\begin{aligned} \frac{1}{2i}\left(\langle T\mathbf{h}, \mathbf{h}\rangle - \langle \mathbf{h}, T\mathbf{h}\rangle\right) &= \frac{1}{2i}\left(\int_{\Gamma_0} \overline{\mathbf{h}} \cdot [\mathbf{w_h}] \; \mathrm{d}s(\mathbf{y}) - \int_{\Gamma_0} \mathbf{h} \cdot \overline{[\mathbf{w_h}]} \; \mathrm{d}s(\mathbf{y})\right) \\ &= -\Im(A(\mathbf{w_h}, \mathbf{w_h})) \\ &= -\int_{\Gamma_0} \Im(\overline{[\mathbf{w_h}]} \cdot \mathbf{K}[\mathbf{w_h}]) + \Im\left(\int_{S_R} \overline{\mathbf{w_h}} \cdot \mathcal{T}_R \mathbf{w_h} \; \mathrm{d}s(\mathbf{y})\right) \\ &> 0, \end{aligned} \tag{2.86}$$

where in the last line we used the properties of $\mathbf{K}$, together with those of the Dirichlet-to-Neumann map stated in Lemma 2.2.1. □

For a given open surface $L \subset \Gamma$ and a density $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$, we define the test function $\boldsymbol{\phi}_L^\infty \in L^2(\mathbb{S}^2)^3$ by

$$\boldsymbol{\phi}_L^\infty = \int_L \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}^\infty(\cdot, \mathbf{y})\boldsymbol{\eta}(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) = \gamma \int_L \partial^*_{\boldsymbol{\nu}(y)} \mathbf{W}_b(\mathbf{y}, -\cdot)\boldsymbol{\eta}(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}). \tag{2.87}$$

Lemma 2.3.4. *The operator* $\mathscr{G} := \mathcal{S}_b\mathscr{H}^*T$ *is such that,* $\boldsymbol{\phi}_L^\infty \in$ *Range*$(\mathscr{G})$ *for all* $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$ *not vanishing identically on any subset of* L *of positive Lebesgue measure, if and only if* $L \subset \Gamma_0$.

Proof. Observe that, by definition, $\mathscr{G} : H^{-1/2}(\Gamma_0)^3 \to L^2(\mathbb{S}^2)^3$ is such that $\mathscr{G}\mathbf{h} = \mathbf{w}^\infty$, where $\mathbf{w}^\infty$ is the far field pattern of the solution $\mathbf{w}$ to the defective problem (2.66)-(2.67).

Assume first that $L \subset \Gamma_0$. Given $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$, then $\widetilde{\boldsymbol{\eta}}$, the extension by zero of $\boldsymbol{\eta}$ to Γ_0, is in $\widetilde{H}^{1/2}(\Gamma_0)^3$. Thus we know that $\boldsymbol{\phi}_{\boldsymbol{L}}^{\boldsymbol{\infty}}$ is the far field pattern of the generalized double-layer potential:

$$\mathbf{w} = \int_{\Gamma_0} \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\cdot, \mathbf{y})\widetilde{\boldsymbol{\eta}}(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}), \tag{2.88}$$

which is in $H^1_{loc}(\mathbb{R}^3 \setminus \overline{L})^3$ and satisfies:

$$\Delta^*\mathbf{w} + \omega^2\rho\mathbf{w} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus (\Gamma_1 \cup \Gamma), \tag{2.89}$$

$$[\mathbf{w}] = \boldsymbol{\eta} \quad \text{on } \Gamma_0, \tag{2.90}$$

and $\mathbf{w}$ satisfies the Kupradze radiation conditions. Thus, if we define $\mathbf{h} := \mathbf{K}\widetilde{\boldsymbol{\eta}} - \partial^*_{\boldsymbol{\nu}}\mathbf{w}$, then $\mathbf{h} \in H^{-1/2}(\Gamma_0)^3$ and $\mathscr{G}\mathbf{h} = \boldsymbol{\phi}_{\boldsymbol{L}}^{\boldsymbol{\infty}}$.

Conversely, assume there is $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$ such that $\boldsymbol{\eta}$ does not vanish in any subset of L of positive Lebesgue measure, and such that $\boldsymbol{\phi}_{\boldsymbol{L}}^{\boldsymbol{\infty}}$ in *Range*$(\mathscr{G})$, but $L \not\subset \Gamma_0$. By definition of $\mathscr{G}$, $\boldsymbol{\phi}_{\boldsymbol{L}}^{\boldsymbol{\infty}}$ is the far-field pattern of $\mathbf{w} \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Gamma_0})^3$ that satisfies the defective problem (2.66)-(2.67) for some $\mathbf{h} \in H^{-1/2}(\Gamma_0)^3$.

Therefore , $\boldsymbol{\phi}_{\boldsymbol{L}}^{\boldsymbol{\infty}}$ is the far field pattern of the two potentials:

$$P_L\boldsymbol{\eta} = \int_L \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\cdot, \mathbf{y})\boldsymbol{\eta}(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}), \tag{2.91}$$

and

$$\mathbf{w} = \int_{\Gamma_0} \partial^*_{\boldsymbol{\nu}(y)} \mathbb{G}_b(\cdot, \mathbf{y})[\mathbf{w}](\mathbf{y}) \; \mathrm{d}s(\mathbf{y}). \tag{2.92}$$

By Rellich's lemma and the unique continuation principle, we know that both potentials are identical in $\mathbb{R}^3 \setminus (L \cup \Gamma_0)$. However, by assumption, there exists $\mathbf{x} \in L$ and an open neighborhood V_δ of $\mathbf{x}$ such that on $V_\delta \cap L \subset (L \setminus \overline{\Gamma_0})$ where the density $\boldsymbol{\eta}$ does not vanish. Hence, the potential P_L has a discontinuity on $\mathbf{x}$ along the normal direction to L, whereas $\mathbf{w}$ is continuous at the same point, and this is a contradiction. $\square$

The following corollary follows immediately from Lemma 2.3.4, and considering the properties of the scattering operator.

Corollary 2.3.1. *The operator $\mathscr{H}^*$ is such that, $\mathcal{S}_b^* \boldsymbol{\phi}_L^\infty \in Range(\mathscr{H}^*)$ for all $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$ not vanishing identically on any subset of L of positive Lebesgue measure, if and only if $L \subset \Gamma_0$.*

In summary, we have proved the following theorem, which is the basis for the NDT algorithm to detect interfacial cracks.

Theorem 2.3.3. *Under the assumptions of Lemmas 2.3.1 and 2.3.2, $\widetilde{F}_D := \mathcal{S}_b^* F_D$ satisfies that:*

i) The operator $\widetilde{F}_D^\# := |Re(\widetilde{F}_D)| + Im(\widetilde{F}_D)$ is positive, and the ranges of $\mathscr{H}^* : \widetilde{H}^{1/2}(\Gamma_0)^3 \to L^2(\mathbb{S}^2)^3$ and $(\widetilde{F}_D^\#)^{1/2} : L^2(\mathbb{S}^2)^3 \to L^2(\mathbb{S}^2)^3$ coincide.

ii) $\mathcal{S}_b^* \boldsymbol{\phi}_L^\infty \in Range((\widetilde{F}_D^\#)^{1/2})$ for all $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$ such that $\boldsymbol{\eta}$ does not vanish identically in any subset of L of positive Lebesgue measure, if and only if $L \subset \Gamma_0$.

Therefore, the reconstruction of Γ_0 can in principle be done by solving the so-called far-field equation:

$$(\widetilde{F}_D^\#)^{1/2} \mathbf{g} = \widetilde{\boldsymbol{\phi}_L^\infty}, \tag{2.93}$$

where $\widetilde{\boldsymbol{\phi}_L^\infty} := \mathcal{S}_b^* \boldsymbol{\phi}_L^\infty$, for all possible open surfaces $L \subset \Gamma$.
From Picard's criterion, Theorem 2.7 in [21], the following result is an immediate consequence.

Corollary 2.3.2. *Let $\{\mu_\ell, \boldsymbol{\psi}_\ell\}_{\ell=1}^\infty$ be the eigensystem of $\widetilde{F}_D^\#$, then: $L \subset \Gamma_0$ if and only if*

$$\sum_{\ell=1}^\infty \frac{|(\widetilde{\boldsymbol{\phi}_L^\infty}, \boldsymbol{\psi}_\ell)_{L^2(\mathbb{S}^2)^3}|^2}{|\mu_\ell|} \;<\; \infty, \tag{2.94}$$

where $\widetilde{\boldsymbol{\phi}_L^\infty} := \mathcal{S}_b^* \boldsymbol{\phi}_L^\infty$ and the density $\boldsymbol{\eta} \in \widetilde{H}^{1/2}(L)^3$ in the definition (2.87) of $\boldsymbol{\phi}_L^\infty$ is such that $\boldsymbol{\eta}$ does not vanish identically in any subset of L of positive Lebesgue measure.

2.4 Numerical examples

The numerical simulations presented in this section were carried on using a boundary elements method by Fatemeh Pourahmadian, and will published in [34].

For our numerical simulations we let L shrink to a point $\mathbf{z} \in \Gamma$. More precisely, let $\boldsymbol{\eta}$ in the definition of our test function (2.87) be such that $\boldsymbol{\eta} \sim \delta_{\mathbf{z}} \mathbf{e}_\ell$, where $\{\mathbf{e}_\ell\}_{\ell=1}^3$. Therefore, the three corresponding test functions are:

$$\phi^\infty_{\mathbf{z},\ell} := \boldsymbol{\nu}_z \cdot \mathbf{C} : \nabla_z \mathbf{W}_b(\mathbf{z}, -\cdot)\mathbf{e}_\ell, \quad \text{for all } \ell = 1, 2, 3. \tag{2.95}$$

Define

$$\mathbf{f}_{\mathbf{z}} := ({\phi^\infty_{\mathbf{z},1}}^T, {\phi^\infty_{\mathbf{z},2}}^T, {\phi^\infty_{\mathbf{z},3}}^T)^T, \tag{2.96}$$

and let M be the discrete far-field operator $\mathcal{F}_D$, and S^* the discrete $\mathcal{S}_b^*$. Then $A = |Re(S^*M)| + Im(S^*M)$ is the discretized $\widetilde{F}_D^\#$. Let $\{\mu_\ell, \boldsymbol{\psi}_\ell\}_{\ell=1}^{3M}$ be an eigensystem of A, then by Corollary 2.3.2, we expect that $\mathbf{z} \in \Gamma_0$ if and only if the discrete Picard criterion holds:

$$G(\mathbf{z}) := \sum_{\ell=0}^{N_T} \frac{|\tilde{\mathbf{f}}_{\mathbf{z}} \cdot \overline{\boldsymbol{\phi}_\ell}|^2}{|\mu_\ell|} < \infty, \tag{2.97}$$

where $\mathbf{f}^*_{\mathbf{z}} = S^* \mathbf{f}_{\mathbf{z}}$ and the natural number $N_T < 3N$ is a heuristic truncation level (see Theorem 2.11 in [21]). This can therefore be used to construct an approximate indicator function of Γ_0, by:

$$\mathbb{1}_{\Gamma_0} = \begin{cases} 1 & \text{if } G(\mathbf{z})^{-1} > \tau_{tol}, \\ 0 & \text{otherwise,} \end{cases}$$

for a suitable threshold parameter τ_{tol}.

2.4.1 Numerical examples

The numerical experiments that we consider are the examples corresponding to Fig. 2.2, panels (a) and (b).

Motivated by engineering applications where the aim is the detection of fractures at interfaces of concrete and the external homogeneous domain, the cracks in our numerical examples are placed at the exterior interface Γ_1 (as opposed to previous sections where $\Gamma_0 \subset \Gamma$). In this setting, however, the techniques are the same and thus the examples presented below serve as a proof of principle for our method.

The first example, shown in Fig. 2.2 panel (a), is a configuration where the inhomogeneity Ω is the ellipsoid in $\mathbb{R}^3$ whose boundary satisfies the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{2.98}$$

where $a = 4.5$, $b = 4$, $c = 6$. The interface Γ between Ω_- and Ω_+, is the ellipsoid with cannonical equation (2.98) for $a = 3$, $b = 2.5$ and $c = 2$. The material properties in the exterior domain Ω_{ext} are $\rho = 1$, $\lambda_0 = 1$, and $\mu_0 = 1$, whereas $\rho_+ = 0.75$, $\lambda_+ = 8/60$, $\mu_+ = 0.2$, $\rho_- = 1.5$, $\lambda_+ = 6/10$, and $\mu_+ = 0.4$. The interfacial crack $\Gamma_0 \subset \Gamma_1$ is the shaded surface.

Define $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ to be an orthonormal set of vectors that span the tangent plane at a point on Γ_0, then the vectors $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\nu}\}$ are a local basis that is well defined everywhere on Γ_0. In this local surface basis the stiffness matrix is given by $\mathbf{K} = diag\{1, 1, 1\}$.

The second example, shown in Fig. 2.2 panel (b), corresponds to an inhomogeneity that has three connected components: a cube Ω_c, an ellipsoid Ω_e and a sphere Ω_s. The domain Ω_c has side length of $l = 1.8$, and it is centered at $(0, 3, 3)$, the domain Ω_e obeys the cannonical equation (2.98) for $a = 3$, $b = 2$, $c = 4$, and finally Ω_s is a sphere of radius $r = 2$ and center $(0, -4, -2)$.

In this case, $\Gamma = \emptyset$, $\Omega_- = \emptyset$, i.e., the inhomogeneity has a single layer $\Omega_+ = \Omega_c \cup \Omega_e \cup \Omega_s$. The three connected components of Ω_+ have the same constant material properties given by $\rho_+ = 0.5$, $\lambda_+ = 2/15$, $\mu_+ = 0.2$, whereas the material properties in the exterior domain $\Omega_{ext} = \mathbb{R}^3 \setminus \overline{\Omega_+}$ are $\rho = 1$, $\lambda_0 = 1$, and $\mu_0 = 1$. Finally, the stiffness matrix with respect to the local surface coordinates $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\nu}\}$ is given by

$$\mathbf{K} = \begin{cases} diag\{0, 0, 0\} & \text{in } \Gamma_0 \cap (\Omega_c \cup \Omega_s) \\ diag\{2, 2, 2\} & \text{in } \Gamma_0 \cap \Omega_e. \end{cases}$$

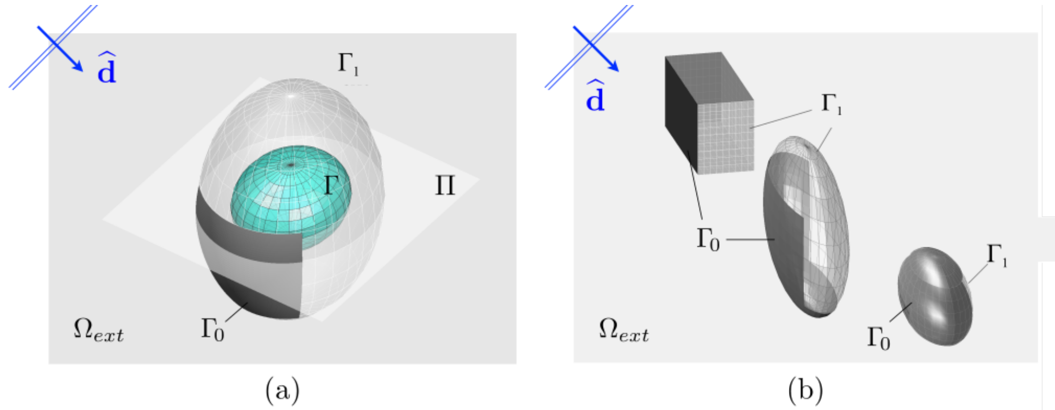


Figure 2.2: Panel (a) shows a connectd ellipsoidal inhomogeneity that consists of two layers, and where $\Gamma_0 \subset \Gamma_1$. Panel (b) shows an inhomogeneity with three connected components of the same material, a cube Ω_c, an ellipsoid Ω_e and a sphere Ω_s. The shaded regions on their surfaces are the three connected components of $\Gamma_0 \subset \Gamma_1$.

In this second example $\Gamma_0 \subset \Gamma_1$ has three different connected components, each of them is the shaded surface at the boundaries of Ω_c, Ω_e and Ω_s, respectively.

On both examples, the excitation frequency was chosen as $\omega = 4$, and to construct M as explained in the previous subsection, the far-field pattern of the radiating solutons associated with 450 incident plane waves (corresponding to 150 incident directions $\widehat{\mathbf{d}}$ and their 3 possible linearly independent polarization vectors $\mathbf{p}$) were computed. The results of the reconstruction of Γ_0 for the first example are shown in Fig. 2.3, and the results of the reconstruction of Γ_0 for the second example are shown in Fig. 2.4. In each case, on panel (a) it is shown the computation of the function G defined by (2.97), whereas on panel (b), the indicator function $\mathbb{1}_{\Gamma_0}$ as defined in the previous subsection is plotted.

As we can see, the numerical examples presented in Figs 2.3 and 2.4, show that the NDT algorithm for interfacial crack detection that derives from Theorem 2.3.3, is suitable for finding the geometry of the fractures using differential measurements, i.e. by comparison to a known healthy (background) configuration.

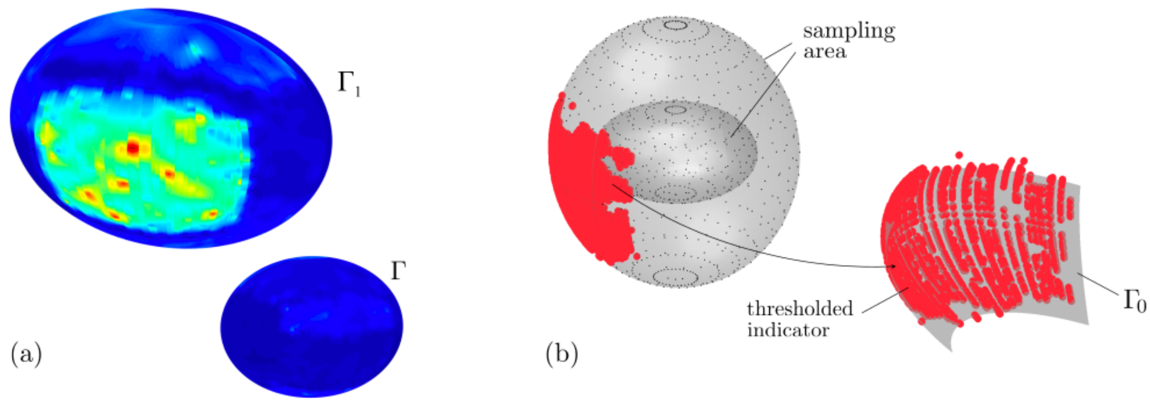


Figure 2.3: Interfacial crack reconstruction for example 1. Panel (a) shows function G defined by (2.97) on each of the interfaces Γ_1 and Γ. Panel (b) shows the plot of the indicator function $\mathbb{1}_{\Gamma_0}$.

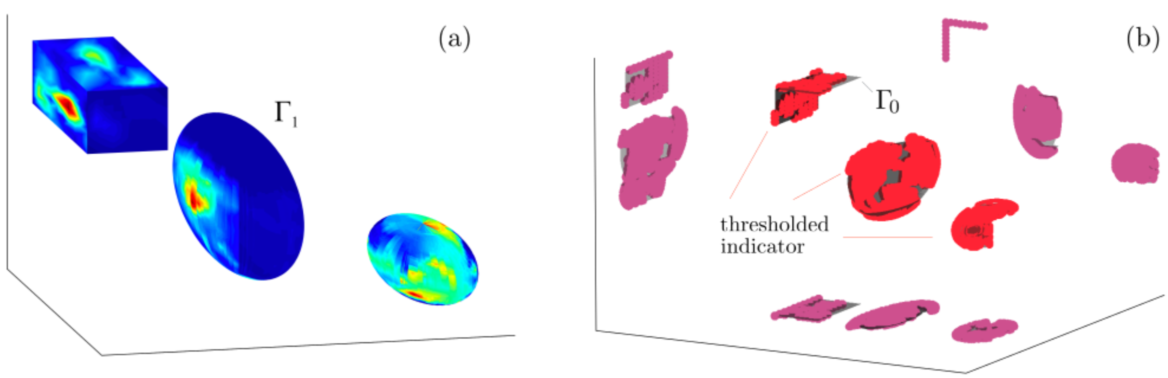


Figure 2.4: Interfacial crack reconstruction for example 2. Panel (a) shows function G defined by (2.97) on the interface Γ_1. Panel (b) shows the plot of the indicator function $\mathbb{1}_{\Gamma_0}$.

Remark 2.4.1. The results presented here are true not only in the case of interfacial cracks; they hold for cracks embedded in an inhomogeneous media Ω with the following properties:

• Ω is the interior of $\overline{\cup_{\ell=1}^{N}\Omega_\ell}$, where $\{\Omega_\ell\}_{\ell=1}^{N}$ is a collection of connected domains with Lipschitz continuous boundary.

• The restricted material properties $\mu_\ell := \mu\big|_{\Omega_\ell}$, $\lambda_\ell := \lambda\big|_{\Omega_\ell}$, and $\rho_\ell := \rho\big|_{\Omega_\ell}$, are continuous.

• Adjacent domains Ω_ℓ and Ω_j, such that $\Gamma_{ell,j} := \partial\Omega_\ell \cap \partial\Omega_j \neq \emptyset$, satisfy the monotonicity condition (2.2), which in this context reads:

$$(\lambda_\ell\big|_{\Gamma_{\ell,j}} - \lambda_j\big|_{\Gamma_{\ell,j}})(\mu_\ell\big|_{\Gamma_{\ell,j}} - \mu_j\big|_{\Gamma_{\ell,j}}) \geq 0. \tag{2.99}$$

Conclusion

The problem of elastic wave scattering in isotropic layered media was shown to be well-posed. The NDT of interfacial cracks based on an adaptation of the FM and proposed in this chapter, was successfully proved analitycally, with the auxiliary mixed reciprocity principle and the definition of the elastic scattering operator. Numerical experiments show that the method can be used in practice and that it works for the detection and geometrical reconstruction of multiple disconnected cracks at the interface of two layers of different materials.

The method proposed in this chapter can easily be adapted to a more general regime, including not only interfacial cracks, but the case when the crack crosses transversally the interface between two internal layers, as long as the material properties satisfy the conditions stated in Remark 2.4.1.

The results presented throughout this chapter will appear in [34].

Chapter 3

NONDESTRUCTIVE TESTING OF THE DELAMINATED INTERFACE BETWEEN TWO MATERIALS: THE ACOUSTIC CASE

3.1 The problem

As has already been mentioned in the general introduction of the thesis, the delamination of two materials occurs when one material becomes partially detached from the other. This process is common in composite structures [19], concrete [82] and many other engineering applications (e.g. [83, 44]). In this chapter, we will develop an inverse scattering approach to the detection of delamination using acoustic waves.
The material presented here, including the figures, has been already published as [24].

In this problem, we consider two materials that should have a coincident boundary (in the undamaged or background state) and we wish to detect if there is a part of the common boundary where the two materials have separated. In particular we want to determine the size and position of the delamination.

More precisely, we denote by $\Omega \subset \mathbb{R}^m$, $m = 2, 3$ the support of the inhomogeneity to be tested which in absence of delamination is composed of two different materials adjacent to one another. For short, denote μ_+, n_+ and μ_-, n_- the material properties in Ω_+ and Ω_-, respectively. We denote their bounded support by Ω_- and Ω_+, respectively, and the shared interface by $\Gamma := \partial\Omega_-$ (i.e. $\Omega = \Omega_- \cup \Omega_+$). Both the outer boundary $\partial\Omega_+$ of the domain Ω_+ and the boundary $\partial\Omega_-$ of the simply connected domain Ω_- are assumed to be piece-wise smooth, unless mentioned otherwise, and $\boldsymbol{\nu}$ denotes the unit normal always oriented outwards to the region bounded by the curve. For simplicity we let $\Omega_{\text{ext}} := \mathbb{R}^m \setminus \overline{\Omega}$. Furthermore, we assume that along a part of the interface, denoted here by $\Gamma_0 \subset \Gamma$, these two materials have detached (delaminated) and we model this fact with the appearance of an opening with support Ω_δ and material properties μ_δ,

n_δ (see Fig. 3.1). Note that $\Gamma_0 = \Omega_\delta \cap \Gamma$. The material properties (possibly complex valued) in each of the domains are assumed to be smooth, i.e. $\mu_+, n_+ \in \mathcal{C}^1(\Omega_+)$, $\mu_-, n_- \in \mathcal{C}^1(\Omega_-)$ and $\mu_\delta, n_\delta \in \mathcal{C}^1(\Omega_\delta)$ (however note that across the interfaces there are discontinuities in the material properties).

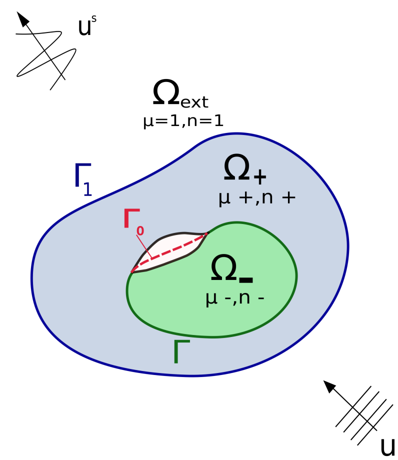


Figure 3.1: Layered media with a thin delamination at the interface of two layers Ω_- and Ω_+. The opening Ω_δ, with coefficients μ_δ, n_δ is shown as the white region.

Assuming now that the incident field and the other fields in the problem are time harmonic (i.e. the time dependent incident field is of the form $\Re\left(u^i(\mathbf{x})e^{i\omega t}\right)$ where ω is the angular frequency), then the total field $u^{ext} = u^s + u^i$ in Ω_{ext}, where u^s is the scattered field, and the fields u^+, u^- and U inside Ω_+, Ω_- and Ω_δ, respectively, satisfy

$$\Delta u^{ext} + k^2 u^{ext} = 0 \qquad \text{in} \qquad \Omega_{\text{ext}}, \tag{3.1}$$

$$\nabla \cdot \left(\frac{1}{\mu_+} \nabla u^+ \right) + k^2 n_+ u^+ = 0 \qquad \text{in} \qquad \Omega_+, \tag{3.2}$$

$$\nabla \cdot \left(\frac{1}{\mu_-} \nabla u^- \right) + k^2 n_- u^- = 0 \qquad \text{in} \qquad \Omega_-, \tag{3.3}$$

$$\nabla \cdot \left(\frac{1}{\mu_\delta} \nabla U \right) + k^2 n_\delta U = 0 \qquad \text{in} \qquad \Omega_\delta. \tag{3.4}$$

Here the wave number $k = \omega / c_{\text{ext}}$ with c_{ext} denoting the sound speed of the homogeneous background. Across the interfaces the fields on either side and their conormal derivatives are continuous, i.e.

$$u^{ext} = u^+ \qquad \text{and} \qquad \frac{\partial u^{ext}}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_+} \frac{\partial u^+}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma_1, \tag{3.5}$$

$$u^+ = u^- \qquad \text{and} \qquad \frac{1}{\mu_+} \frac{\partial u^+}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_-} \frac{\partial u^-}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma \backslash \overline{\Gamma}_0, \tag{3.6}$$

$$U = u^+ \qquad \text{and} \qquad \frac{1}{\mu_\delta} \frac{\partial U}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_+} \frac{\partial u^+}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma_+, \tag{3.7}$$

$$U = u^- \qquad \text{and} \qquad \frac{1}{\mu_\delta} \frac{\partial U}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_-} \frac{\partial u^-}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma_-. \tag{3.8}$$

Of course the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{\frac{m-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{3.9}$$

uniformly in $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, where $\mathbf{x} \in \mathbb{R}^m$ and $r = |\mathbf{x}|$. In this chapter we consider plane waves as incident fields which are given by $u^i(\mathbf{x}) := e^{ik\mathbf{x}\cdot\widehat{\mathbf{d}}}$ where the unit vector $\widehat{\mathbf{d}}$ is the incident direction. Instead of plane waves, it is also possible to consider incident waves due to point sources located outside Ω, in which case the obvious modifications need to be made in the formulation of the problem.

The goal of the this study is to propose and analyze a Linear Sampling Method (LSM) type scheme for detecting the delaminated region using remote measurements of acoustic waves scattered by the structure. In practice, the thickness of the opening is much smaller than both the interrogating wave length in free space $\lambda = \frac{2\pi}{k}$ and the thickness of the layers of background material. This introduces an essential computational difficulty in the numerical solution of the forward problem. In the following section we take advantage of the small scale of the thickness and, using an asymptotic method from [10, 71], we derive an approximate model of the delaminated structure where the opening Ω_δ is replaced by new jump relations for u^+ and u^- across the delaminated part Γ_0 that account for the presence of the opening. This is undertaken in Section 3.2 using formal asymptotic methods. Before analyzing the model further, we then demonstrate numerically that the asymptotic model predicts correctly the acoustic field and far field pattern of the scattered field for a particular model scatterer incorporating a delamination of small positive maximum width.

Remark 3.1.1. In the acoustic scattering case in $\mathbb{R}^m$, $m = 2, 3$, each material is characterized by two parameters, both related to the mass density function, ρ. In this case, if ρ_{ext} denotes the constant density of the homogeneous exterior domain Ω_{ext}, then the relative constitutive material properties are the scalar fields $\mu = \rho/\rho_{ext}$ and $n = \frac{c\rho}{c_{ext}\rho_{ext}}$, where c is the speed of sound field, and c_{ext} is the speed of sound in the exterior domain Ω_{ext}.
In the case $m = 2$, the model that we present here corresponds also to the scattering of spatially polarized electromagnetic waves when the obstacle is an infinite cylinder (see [32]). In such a case, μ and n correspond to the so called relative magnetic permeability and relative electric permitivity parameters. In this setting, n is a complex scalar field, where the imaginary part is related to the conductivity of the material.

Although there has been considerable work on the asymptotics of scattering from thin films (see for example [8, 5, 6, 7, 9, 49, 35, 10, 71]), the novelty of our reduced problem is that the delamination covers only a portion of the interface. The thickness of the delamination vanishes at its boundary and this introduces potential singularities into the asymptotic model. Therefore in Section 3.3 we analyze the forward reduced problem using an appropriate variational formulation and show that under reasonable

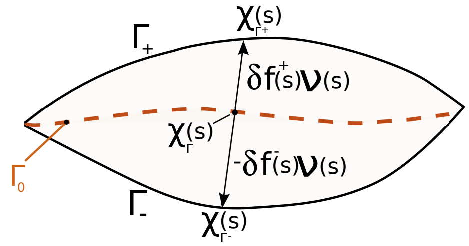


Figure 3.2: Zoom of the thin delamination Ω_δ, and the parametrization of the boundaries Γ_- and Γ_+. Here δ scales the width of the delamination and is assumed small compared to other characteristic dimensions of the problem.

conditions on the constitutive parameters and on the shape of the delamination the forward asymptotic model has a solution (indeed it is this variational scheme that was used to generate the finite element solution used in Section 3.2). Of course a thorough understanding of the forward model is also needed in our analysis of the inverse problem.

The inverse problem under study is precisely formulated in Section 3.4. We assume that the background or undamaged state is known, and then seek to determine the delaminated region Γ_0 using remote (far field) acoustic measurements. In preparation for the analysis of our scheme and to allow a simple calculation of the right hand side of the far field equation we then prove a new mixed reciprocity result for layered media. Next in Section 3.4.2 we give details of the LSM: in particular we seek to determine whether small artificial test arcs on the interface are within the delamination or in the undamaged region. This requires a suitable testing function for the LSM adapted to the delamination problem. We then prove the usual theorem for the LSM suggesting that an approximate solution of the far field equation can be used as an indicator function for the delamination.

Finally in Section 3.5 we test the inversion scheme on synthetic data for a special choice of the testing function from Section 3.4.2. In particular we show that our LSM can detect delamination even in the presence of noise on the data, and that multiple delaminated regions can be detected.

3.2 An approximate asymptotic model

In this section we assume $m = 2$ and, focusing our attention on a neighborhood of the opening Ω_δ, use formal asymptotic analysis to derive an approximate model that takes into account the thin opening Ω_δ. To this end, we start by assuming that the portion Γ_0 of the boundary can be written in the form

$$\Gamma_0 := \{\mathbf{x}_\Gamma(s), \quad s \in [0, L]\},$$

where $\mathbf{x}_\Gamma \in \mathcal{C}^1[0, L]$ is the counter-clockwise arc-length parametrization of Γ_0. If the curve Γ_0 is regular and $c(s)$ denotes its curvature at $\mathbf{x}_\Gamma(s)$, then $0 \leq c_m := \max\{|c(s)| : s \in [0, L]\}$ is finite. Hence, in the neighborhood of Γ_0, one can define the curvilinear coordinates $(s, \eta) \in [0, L] \times (-\frac{1}{c_m}, \frac{1}{c_m})$ by

$$x = \mathbf{x}_\Gamma(s) + \eta \boldsymbol{\nu}(s),$$

where we recall that $\boldsymbol{\nu}$ is the unit normal vector on Γ_0 oriented outward to Ω_- (and taking $\frac{1}{c_m} = \infty$ if $c_m = 0$). Therefore, if the curvature of Γ_0 is small enough, both the outer and inner boundaries of Ω_δ, denoted here by Γ_+ and Γ_-, can be written in this coordinate system as

$$\Gamma_+ = \left\{\mathbf{x}_{\Gamma_+}(s) := \mathbf{x}_\Gamma(s) + \delta f^+(s)\boldsymbol{\nu}(s), \quad s \in [0, L]\right\}$$

and

$$\Gamma_- = \left\{\mathbf{x}_{\Gamma_-}(s) := \mathbf{x}_\Gamma(s) - \delta f^-(s)\boldsymbol{\nu}(s), \quad s \in [0, L]\right\}.$$

Note that the function $\delta(f^+ + f^-)(s)$ defined on Γ_0 describes the thickness of the delamination. Here δ is a small parameter (compared to both the wave length and the size of the domains involved), and $\max_{s\in[0,L]} f^\pm(s) = 1$ (see Figure 3.2).

In an open neighborhood of Ω_δ, we can now express the fields U, u^-, and u^+ in terms of the curvilinear variables (s, η). Ignoring small neighborhoods of the tip points $s = 0$ and $s = L$, since Ω_δ plays here the role of a boundary layer, in order to transfer the small parameter δ from the geometry to the expression of the fields we make a stretching change of variables inside Ω_δ defined by $\zeta = \frac{\eta}{\delta}$. Hence, $\zeta = \frac{\eta}{\delta}$ and s

are now the new coordinates inside Ω_δ. Next, following [10] and [71], we formally make the following ansatz for the fields U and $u^\pm$ in an open neighborhood of Ω_δ

$$U(s,\zeta)=\sum_{j=0}^{\infty}\delta^j U_j(s,\zeta) \tag{3.10}$$

and

$$u^\pm(s,\eta)=\sum_{j=0}^{\infty}\delta^j u_j^\pm(s,\eta), \tag{3.11}$$

where neither $u_j^\pm$ nor U_j depend on δ any longer. Furthermore, we expand each of the terms $u_j^\pm(s,\eta)$ in a power series with respect to the normal direction coordinate η around zero, i.e.

$$u_j^\pm(s,\eta)=u_j^\pm(s,0)+\eta\frac{\partial}{\partial\eta}u_j^\pm(s,0)+\frac{\eta^2}{2}\frac{\partial^2}{\partial\eta^2}u_j^\pm(s,0)+...$$

and after plugging in (3.11) we finally obtain the following expression for $u^\pm(s,\eta)$,

$$u^\pm(s,\eta)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\delta^j\frac{\eta^k}{k!}\frac{\partial^k}{\partial\eta^k}u_j^\pm(s,0). \tag{3.12}$$

Now based on the ansatz (3.10) and (3.12), and using the equations along with the transmission conditions, we can formally obtain an approximate model for the field in the opening Ω_δ. For detailed calculations we refer the reader to Appendix B (see also [71]) and in the following we simply sketch the steps that lead to our approximate model.

3.2.1 The approximate transmission conditions

First we consider the expressions (3.10) and (3.12) which we substitute in (3.6), (3.7) and (3.8). To this end, starting with the *Dirichlet part of the transmission conditions* on $\Gamma_\pm$, we can write

$$U(s,\pm f^\pm)=\sum_{j}^{\infty}\delta^j U_j(s,\pm f^\pm),$$

and

$$u^\pm(s,\pm\delta f^\pm)=\sum_{j=0}^{\infty}\delta^j\sum_{k=0}^{j}\frac{(\pm1)^{j-k}(f^\pm)^{j-k}}{(j-k)!}\frac{\partial^{j-k}}{\partial\eta^{j-k}}u_k^\pm(s,0).$$

Then the Dirichlet part of the transmission condition can be directly computed by equating terms with the same powers of δ. Doing so leads to

$$U_j(s,\pm f^\pm) = \sum_{k=0}^{j} \frac{(\pm 1)^{j-k}(f^\pm)^{j-k}}{j-k!}\frac{\partial^{j-k}}{\partial\eta^{j-k}}u_k^\pm(s,0) \qquad \text{for all } j=0,1,2,... \tag{3.13}$$

Next we deal with the *Neumann part of the transmission conditions* on $\Gamma_\pm$. Unlike the Dirichlet part, the Neumann part of the transmission conditions is more delicate, because in order to compute the co-normal derivatives at $\Gamma_\pm$, one has to take into account the expression in curvilinear coordinates of the normal vectors to those curves. To this end, as discussed in [10], the normal vectors $\boldsymbol{\nu}^\pm$ on $\Gamma^\pm$ have the following expressions

$$\boldsymbol{\nu}^\pm = \frac{1}{|\boldsymbol{\tau}^\pm|}\left((1\pm\delta f^\pm)\boldsymbol{\nu} \mp \delta\frac{df^\pm}{ds}\boldsymbol{\tau}\right),$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the outer unit normal vector and the unit tangential vector defined on Γ_0, respectively, whereas the tangent vectors $\boldsymbol{\tau}^\pm(s) := \frac{d}{ds}\mathbf{x}_{\Gamma^\pm}(s)$ to $\Gamma_\pm$ are not unit vectors. Next, in curvilinear coordinates the gradient operator takes the form

$$\nabla u(\mathbf{x}) = \frac{1}{1+\eta c}\frac{\partial u}{\partial s}\boldsymbol{\tau} + \frac{\partial u}{\partial \eta}\boldsymbol{\nu}$$

where $c := c(s)$ denotes the curvature function of Γ_0. Thus we now have all the ingredients to compute the Neumann part of the transmission conditions, and after straightforward but long calculations (see Appendix B), the Neumann transmission conditions

$$\boldsymbol{\nu}^\pm \cdot \nabla u^\pm|_{\Gamma^\pm} = \boldsymbol{\nu}^\pm \cdot \nabla U|_{\Gamma^\pm},$$

imply the following expression:

$$\begin{aligned}
\pm\tfrac{df^\pm}{ds}&\left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_{j-1}}{\partial s}(s,\pm f^\pm) - \tfrac{1}{\mu_\pm}\textstyle\sum_{k=0}^{j-1}\tfrac{(\pm1)^{j-k-1}(f^\pm)^{j-k-1}}{(j-k-1)!}\tfrac{\partial^{j-k}u_k^\pm}{\partial\eta^{j-k-1}\partial s}(s,0)\right)\\
&= \left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_{j+1}}{\partial \zeta}(s,\pm f^\pm) - \tfrac{1}{\mu_\pm}\textstyle\sum_{k=0}^{j}\tfrac{(\pm1)^{j-k}(f^\pm)^{j-k}}{(j-k)!}\tfrac{\partial^{j-k+1}u_k^\pm}{\partial\eta^{j-k+1}}(s,0)\right)\\
&\pm 2f^\pm c\left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_{j}}{\partial \zeta}(s,\pm f^\pm) - \tfrac{1}{\mu_\pm}\textstyle\sum_{k=0}^{j-1}\tfrac{(\pm1)^{j-k-1}(f^\pm)^{j-k-1}}{(j-k-1)!}\tfrac{\partial^{j-k}u_k^\pm}{\partial\eta^{j-k}}(s,0)\right)\\
&+c^2(f^\pm)^2\left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_{j-1}}{\partial \zeta}(s,\pm f^\pm) - \tfrac{1}{\mu_\pm}\textstyle\sum_{k=0}^{j-2}\tfrac{(\pm1)^{j-k-2}(f^\pm)^{j-k-2}}{(j-k-2)!}\tfrac{\partial^{j-k-1}u_k^\pm}{\partial\eta^{j-k-1}}(s,0)\right),
\end{aligned} \tag{3.14}$$

for $j = -1, 0, 1, 2, ...$, for all $s \in [0, L]$ and with the convention that $U_l = 0$ and $u_l = 0$ for $l < 0$.

Next, we consider the *partial differential equation satisfied by* U_j. To this end, we write the differential operators in curvilinear coordinates and obtain

$$\nabla \cdot \left(\frac{1}{\mu} \nabla w \right) = \frac{1}{(1+\eta c)} \frac{\partial}{\partial s} \left(\frac{1}{\mu} \frac{1}{(1+\eta c)} \frac{\partial w}{\partial s} \right) + \frac{1}{(1+\eta c)} \frac{\partial}{\partial \eta} \left(\frac{(1+\eta c)}{\mu} \frac{\partial w}{\partial \eta} \right).$$

Therefore, the equation satisfied by the field U inside Ω_δ in the new curvilinear coordinates is given by

$$\frac{1}{(1+\delta\zeta c)} \frac{\partial}{\partial s} \left(\frac{1}{\mu} \frac{1}{(1+\delta\zeta c)} \frac{\partial U}{\partial s} \right) + \frac{1}{\delta} \frac{1}{(1+\delta\zeta c)} \frac{\partial}{\partial \zeta} \left(\frac{(1+\delta\zeta c)}{\delta\mu} \frac{\partial U}{\partial \zeta} \right) + k^2 n_\delta U = 0.$$

Now substituting the ansatz (3.10) and collecting the terms corresponding to same powers of δ, we obtain

$$\begin{gathered} \tfrac{\partial}{\partial \zeta} \left(\tfrac{1}{\mu_\delta} \tfrac{\partial}{\partial \zeta} \right) U_j + \left(3\zeta c \tfrac{\partial}{\partial \zeta} \left(\tfrac{1}{\mu_\delta} \tfrac{\partial}{\partial \zeta} \right) + \tfrac{c}{\mu_\delta} \tfrac{\partial}{\partial \zeta} \right) U_{j-1} + \\ \left(\tfrac{\partial}{\partial s} \left(\tfrac{1}{\mu_\delta} \tfrac{\partial}{\partial s} \right) + 3\zeta^2 c^2 \tfrac{\partial}{\partial \zeta} \left(\tfrac{1}{\mu_\delta} \tfrac{\partial}{\partial \zeta} \right) + \tfrac{2c^2\zeta}{\mu_o} \tfrac{\partial}{\partial \zeta} + k^2 n_\delta \right) U_{j-2} + \\ \left(\zeta c \tfrac{\partial}{\partial s} \left(\tfrac{1}{\mu_\delta} \tfrac{\partial}{\partial s} \right) + \zeta^3 c^3 \tfrac{\partial}{\partial \zeta} \left(\tfrac{1}{\mu_\delta} \tfrac{\partial}{\partial \zeta} \right) + \tfrac{c^3\zeta^2}{\mu_o} \tfrac{\partial}{\partial \zeta} - \tfrac{\zeta c'}{\mu_\delta} \tfrac{\partial}{\partial s} + 3\zeta c k^2 n_\delta \right) U_{j-3} + \\ + 3\zeta^2 c^2 k^2 n_\delta U_{j-4} + \zeta^3 c^3 k^2 n_\delta U_{j-5} = 0, \end{gathered} \tag{3.15}$$

for $j = 0, 1, 2...$, and where again $c := c(s)$ is the curvature of Γ_0 and conveying that $U_l = 0$ for negative l.

The recursive relations for the transmission conditions (3.13) and (3.14), and the partial differential equation (3.15) of the three lowest order terms U_0, U_1, U_2 allow us to derive relations between the jumps and mean values of the outer fields u_0 and u_1 and their co-normal derivatives across Γ_0. In the following we summarize these relations (we refer the reader to Appendix B and [71] for details):

$$\begin{gathered} [u_0] = 0, \\ \left[\tfrac{1}{\mu} \tfrac{\partial u_0}{\partial \boldsymbol{\nu}} \right] = 0, \\ [u_1] = 2 \left\langle f(\mu_\delta - \mu) \right\rangle \left\langle \tfrac{1}{\mu} \tfrac{\partial u_0}{\partial \boldsymbol{\nu}} \right\rangle, \\ \left[\tfrac{1}{\mu} \tfrac{\partial u_1}{\partial \boldsymbol{\nu}} \right] = 2 \left(\tfrac{\partial}{\partial s} \left(\left\langle f \left(\tfrac{1}{\mu} - \tfrac{1}{\mu_\delta} \right) \right\rangle \tfrac{\partial}{\partial s} \right) + k^2 \left\langle f \left(n - n_\delta \right) \right\rangle \right) \left\langle u_0 \right\rangle. \end{gathered} \tag{3.16}$$

Here $[u_i] := u_i^+(s,0) - u_i^-(s,0)$ and $\langle u_i \rangle := (u_i^+(s,0) + u_i^-(s,0))/2$, $i = 0,1$, are the point wise jump and average values of the outer fields on Γ_0. Analogously we use the symbols $\left[\frac{1}{\mu}\frac{\partial u_i}{\partial \boldsymbol{\nu}}\right]$ and $\left\langle \frac{1}{\mu}\frac{\partial u_i}{\partial \boldsymbol{\nu}} \right\rangle$ for the jump and average values of the co-normal derivative on Γ_0, and similar definitions for the average values $\langle f\,(n - n_\delta)\rangle$, $\left\langle f\left(\frac{1}{\mu} - \frac{1}{\mu_\delta}\right)\right\rangle$, and $\langle f(\mu_\delta - \mu)\rangle$. Therefore, noting that $u^\pm = u_0^\pm + \delta u_1^\pm + O(\delta^2)$, after dropping the $O(\delta^2)-$terms, we finally obtain the *Approximate Transmission Conditions* (ATCs) of the second order

$$
\begin{aligned}
[u] &= \alpha \left\langle \frac{1}{\mu}\frac{\partial u}{\partial \boldsymbol{\nu}} \right\rangle \text{ on } \Gamma_0, && (3.17)\\
\left[\frac{1}{\mu}\frac{\partial u}{\partial \boldsymbol{\nu}}\right] &= \left(-\frac{\partial}{\partial s}\langle \beta f\rangle \frac{\partial}{\partial s} + \gamma\right)\langle u\rangle \text{ on } \Gamma_0, && (3.18)
\end{aligned}
$$

where

$$
\alpha = 2\delta\,\langle f(\mu_\delta - \mu)\rangle\,, \quad \beta^\pm = 2\delta\left(\frac{1}{\mu_\delta} - \frac{1}{\mu^\pm}\right), \quad \gamma = 2\delta k^2\,\langle f\,(n - n_\delta)\rangle\,. \tag{3.19}
$$

It is worthwhile noticing that all the three coefficients involved in the expression of the ATCs depend on the thickness and the shape of the defect Ω_δ, as well as on the contrasts between material properties of the two delaminated layers $\Omega_\pm$ and the original thin delamination Ω_δ.

Remark 3.2.1. *We remark that our asymptotic expressions along with the derivation of the ATCs are merely formal. Although not needed to write down the final asymptotic model, in our derivation process we have used the assumption that the functions $f^\pm$ are regular at the end points of Γ_0 meaning in particular that $f^\pm(0) = f^\pm(L) = 0$. In the case of regular $f^\pm$, a rigorous justification of the asymptotic model can be done following the approach in [37, 35] for periodic interfaces with constant width.*

3.2.2 Formulation of the approximate model

We can now replace the original problem (3.1)-(3.4), (3.5)-(3.8) and (3.9) by an approximate problem, here referred to as the crack problem, where the opening Ω_δ is replaced by the portion Γ_0 of Γ where the fields satisfy the jump conditions derived above. In an abuse of notation, from now on $u^{\pm}$ will refer to the solution of the approximate problem. We define then the forward approximate scattering problem (i.e. the crack problem): given the plane wave incident field $u^i(\mathbf{x}) := e^{ik\mathbf{x}\cdot\widehat{\mathbf{d}}}$ find the total fields $u^{ext} = u^s + u^i$, u^+ and u^- satisfying

$$\Delta u^{ext} + k^2 u^{ext} = 0 \qquad \text{in} \qquad \Omega_{\text{ext}}, \tag{3.20}$$

$$\nabla \cdot \left(\frac{1}{\mu_+} \nabla u^+ \right) + k^2 n_+ u^+ = 0 \qquad \text{in} \qquad \Omega_+, \tag{3.21}$$

$$\nabla \cdot \left(\frac{1}{\mu_-} \nabla u^- \right) + k^2 n_- u^- = 0 \qquad \text{in} \qquad \Omega_-, \tag{3.22}$$

and

$$u^{ext} = u^+ \quad \text{and} \quad \tfrac{\partial u^{ext}}{\partial \boldsymbol{\nu}} = \tfrac{1}{\mu_+} \tfrac{\partial u^+}{\partial \boldsymbol{\nu}} \qquad \text{on } \Gamma_1, \tag{3.23}$$

$$[u] = 0 \quad \text{and} \quad \left[\tfrac{1}{\mu} \tfrac{\partial u}{\partial \boldsymbol{\nu}} \right] = 0 \qquad \text{on } \Gamma \backslash \overline{\Gamma}_0, \tag{3.24}$$

$$[u] = \alpha \left\langle \tfrac{1}{\mu} \tfrac{\partial u}{\partial \boldsymbol{\nu}} \right\rangle \quad \text{and} \quad \left[\tfrac{1}{\mu} \tfrac{\partial u}{\partial \boldsymbol{\nu}} \right] = \left(-\tfrac{\partial}{\partial s} \langle \beta f \rangle \tfrac{\partial}{\partial s} + \gamma \right) \langle u \rangle \qquad \text{on } \Gamma_0, \tag{3.25}$$

along with the Sommerfeld radiation condition (3.9) for the scattered field u^s (see Figure 3.3), where we recall $[w] = w^+ - w-$ and $\langle w \rangle = (w^+ + w^-)/2$, and α, $\beta^{\pm}$, and γ are given by (3.19). We remark that although our formal asymptotic calculations are performed only in the two-dimensional case, for the analysis in the following will assume that the approximate model (3.20)-(3.22), (3.23)-(3.25) and (3.9) is valid in the three-dimensional case also. Of course in the three-dimensional case the boundary differential operator $\partial/\partial s \langle \beta f \rangle \partial/\partial s$ is replaced by the Laplace-Beltrami operator in the divergence form $\nabla_\Gamma \cdot \langle \beta f \rangle \nabla_\Gamma$, i.e. (3.25) is replaced by

$$[u] = \alpha \left\langle \frac{1}{\mu} \frac{\partial u}{\partial \boldsymbol{\nu}} \right\rangle \quad \text{and} \quad \left[\frac{1}{\mu} \frac{\partial u}{\partial \boldsymbol{\nu}} \right] = \left(-\nabla_\Gamma \cdot \langle \beta f \rangle \nabla_\Gamma + \gamma \right) \langle u \rangle \qquad \text{on } \Gamma_0 \tag{3.26}$$

where $\nabla_\Gamma \cdot$ and ∇_Γ are the surface divergence and the surface gradient on Γ, respectively.

3.2.3 Numerical validation of the approximate model

We end this section with a numerical study of the convergence of the approximate crack problem to the original problem as $\delta \to 0$ in the two-dimensional case.

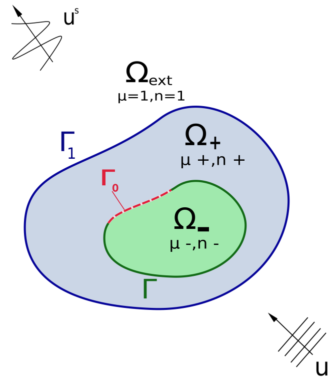


Figure 3.3: The configuration of the crack problem.

Again ignoring the effect of the end points of Γ_0 on the asymptotic expansions, heuristically it is expected that the order of convergence is δ^2. To validate the ATCs, we carried numerical experiments implemented in the finite element library FreeFem++ [51].

In our experiments, we compare the solution of the scattering problem by a finite element method based on directly meshing the opening Ω_δ (i.e. solving (3.1)-(3.4), (3.5)-(3.8) and (3.9) by a finite element method) to the solution of the crack problem (i.e. (3.20)-(3.22), (3.23)-(3.25) and (3.9) by a finite element based on the variational problem (3.36)). Both problems are solved using a FEM code in FreeFem++ [51], where the unbounded domain is truncated and the exact boundary condition in terms of Dirichlet-to-Neumann operator (which is explained in more detail in the following section) is imposed on a circular artificial boundary.

For our numerical example we consider a circular inhomogeneity of radius one with an opening Ω_δ given by (see Figure 3.4)

$$f^-(s) = 0,\ f^+(s) := -l^{-2}(s+l)(s-l); \text{for } s \in (-l,l),\ \text{with } l = 0.2\pi,$$

on the interface $r = 1$. The material properties are chosen to be $n_- = 1, \mu_- = 1$ in Ω_-, $n_+ = 1, \mu^+ = 1$ in Ω_+, $n_\delta = 0.2, \mu_\delta = 0.9$ in Ω_δ, and the wave number $k = 3$, so in this case the wavelength is $\lambda = 2\pi/k \simeq 2$. For fixed $\delta = 0.04\lambda \simeq 0.083$, and different

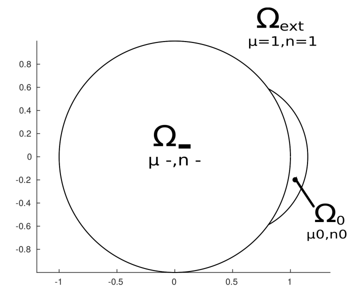


Figure 3.4: The configuration of the delaminated structure used in the numerical experiments

incident directions $\widehat{\mathbf{d}} = (\cos(\theta), \sin(\theta))$, in Figure 3.5 panel (a) we plot the H^1 relative error

$$e(\delta, \widehat{\mathbf{d}}) := \frac{\|u_\delta^{ext}(\cdot, \widehat{\mathbf{d}}) - u^{ext}(\cdot, \widehat{\mathbf{d}})\|_{H^1(B_R \setminus \overline{\Omega})}}{\|u^{ext}(\cdot, \widehat{\mathbf{d}})\|_{H^1(B_R \setminus \overline{\Omega})}}$$

where u_δ^{ext} and u^{ext} correspond to the exact scattering problem (3.1)-(3.4), (3.5)-(3.8), (3.9)) and to the approximate scattering problem (3.20)-(3.22), (3.23)-(3.25), (3.9)), respectively, and B_R is a large ball of radius $R > 0$ containing $\Omega = \Omega_+ \cup \Omega_-$. We observe that the maximum error is, as expected, attained for the incident direction $\widehat{\mathbf{d}} = (1, 0)$, i.e. for the incident plane wave $u^i(\mathbf{x}, \mathbf{y}) = e^{ik\mathbf{x}\cdot\mathbf{y}}$ which hits the opening Ω_δ in the middle in the perpendicular direction. Figure 3.5 panel (b) shows the H^1 relative error $e(\delta, \widehat{\mathbf{d}})$ as a function of the small parameter δ corresponding to the incident direction $\widehat{\mathbf{d}} = (1, 0)$. The plot shows that the numerical convergence rate is close to $O(\delta^{1.7})$ which approximately corresponds to the expected theoretical rate of convergence rate $O(\delta^2)$ for the second order ATCs model. Since for the solution of inverse problem we use far field data, which is defined in Section 3.4, in Figure 3.6 we show numerical results where we compare the far fields of the exact model and the approximate model for the same shape as above. In Figure 3.6 panel (a) is shown the absolute value of the far fields $u_\delta^\infty(\cdot, \widehat{\mathbf{d}})$ and $u^\infty(\cdot, \widehat{\mathbf{d}})$ corresponding to the scattered waves for the ATCs

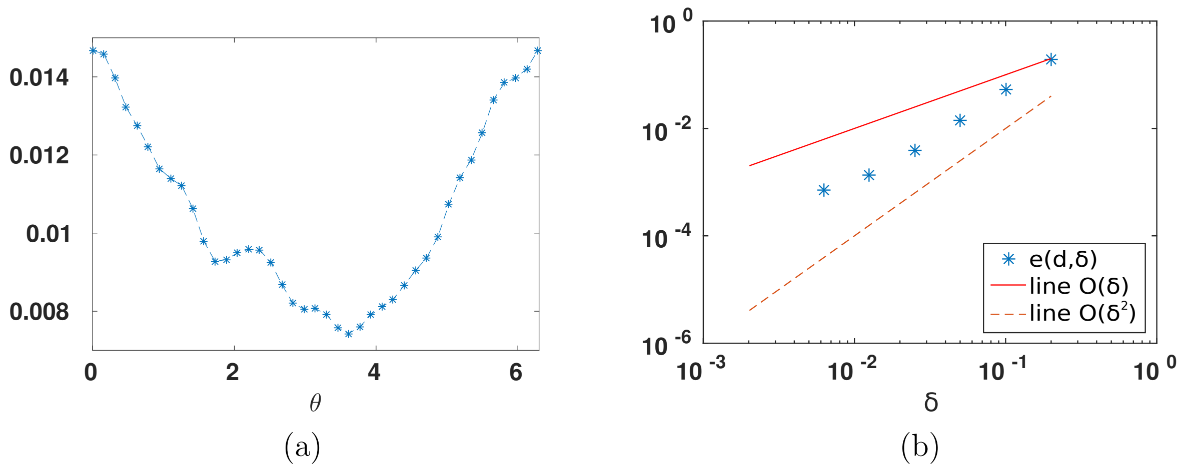


Figure 3.5: Panel (a) shows the H^1 relative error of total fields resulting from different incident direction, whereas panel (b) the H^1 relative error for different values of δ. The approximated rate of convergence is $O(\delta^{1.7})$.

model and the exact model, respectively, again for $\widehat{\mathbf{d}} = (1,0)$. In Figure 3.6 panel (b) we show the relative L^2 error of these far fields

$$e^{\infty}(\delta, \widehat{\mathbf{d}}) := \frac{\|u_\delta^{\infty}(\cdot, \widehat{\mathbf{d}}) - u^{\infty}(\cdot, \widehat{\mathbf{d}})\|_{L^2(\mathbb{S}^1)}}{\|u^{\infty}(\cdot, \widehat{\mathbf{d}})\|_{L^2(\mathbb{S}^1)}}$$

for different values of δ and $\widehat{\mathbf{d}} = (1,0)$. The plot shows that the numerical convergence rate of the far fields in approximately $O(\delta^1)$. And we suggest that this drop of convergence (for $O(\delta^2)$) is due to singularities at the edges of the crack (see Remark 3.2.1).

3.3 The well-posedness of the approximate model

Now we turn our attention to the study of the well-posedness of the approximate crack problem (3.20)-(3.22), (3.23)-(3.25) and (3.9). Although our formal asymptotic calculations are performed only in the two-dimensional case, for the analysis we shall assume that this approximate model is also valid in the three-dimensional case. To study the problem we employ a variational method which provides also the analytical framework for a finite element method to numerically compute the solution. The first

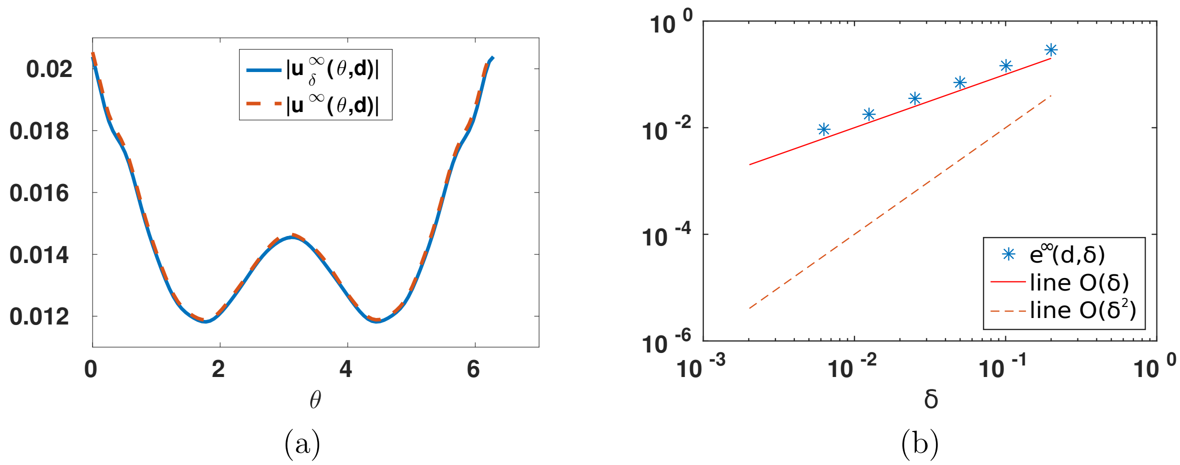


Figure 3.6: Panel (a) shows the plot of the modulus of the far field for both models for $\delta = 0.05$. Panel (b) shows the far field L^2 relative error $e^\infty(\delta, \widehat{\mathbf{d}})$, for different values of δ. The approximated rate of convergence is $O(\delta^1)$.

step is to formulate the problem in a bounded domain and to this end we introduce a large ball B_R of radius $R > 0$ containing $\overline{\Omega}$ and let S_R denote the boundary of B_R. The exterior *Dirichlet-to-Neumann* operator $T_k : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ is defined by

$$T_k : \alpha \mapsto \frac{\partial v}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad S_R$$

where $v \in H^1_{loc}(\mathbb{R}^m \setminus \overline{B}_R)$ solves

$$\begin{aligned} \Delta v + k^2 v = 0 \qquad & \text{in} \qquad \mathbb{R}^m \setminus \overline{B}_R, \\ v = \alpha \qquad & \text{on} \qquad S_R, \\ \lim_{r \to \infty} r^{\frac{m-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0. & \end{aligned}$$

It is well-known that the exterior Dirichlet-to-Neumann operator $T_k : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ satisfies (see e.g [22])

$$\Im \left(\int_{S_R} (T_k u)\, \overline{u} ds \right) \geq 0 \qquad \text{and} \qquad -\Re \left(\int_{S_R} (T_k u)\, \overline{u} ds \right) \geq 0. \tag{3.27}$$

It is standard to show (see e.g. [21] and [22]) that (3.20)-(3.22), (3.23)-(3.25) and (3.9) is equivalent to the problem of finding u^{ext}, u^+, u^- satisfying

$$\Delta u^{ext} + k^2 u^{ext} = 0 \qquad \text{in} \qquad B_R \setminus \overline{\Omega}, \tag{3.28}$$

$$\nabla \cdot \left(\frac{1}{\mu_+} \nabla u^+ \right) + k^2 n_+ u^+ = 0 \qquad \text{in} \qquad \Omega_+, \tag{3.29}$$

$$\nabla \cdot \left(\frac{1}{\mu_-} \nabla u^- \right) + k^2 n_- u^- = 0 \qquad \text{in} \qquad \Omega_-, \tag{3.30}$$

$$\frac{\partial (u^{ext} - u_i)}{\partial \boldsymbol{\nu}} = T_k(u^{ext} - u^i) \qquad \text{on} \qquad S_R, \tag{3.31}$$

$$u^{ext} = u^+ \quad \text{and} \quad \frac{\partial u^{ext}}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_+} \frac{\partial u^+}{\partial \boldsymbol{\nu}} \qquad \text{on} \qquad \Gamma_1, \tag{3.32}$$

$$[u] = 0 \quad \text{and} \quad \left[\frac{1}{\mu} \frac{\partial u}{\partial \boldsymbol{\nu}} \right] = 0 \qquad \text{on} \qquad \Gamma \setminus \overline{\Gamma}_0, \tag{3.33}$$

$$[u] = \alpha \left\langle \frac{1}{\mu} \frac{\partial u}{\partial \boldsymbol{\nu}} \right\rangle \quad \text{and} \quad \left[\frac{1}{\mu} \frac{\partial u}{\partial \boldsymbol{\nu}} \right] = \left(-\nabla_\Gamma \cdot \langle \beta f \rangle \nabla_\Gamma + \gamma \right) \langle u \rangle \qquad \text{on} \qquad \Gamma_0. \tag{3.34}$$

In $\mathbb{R}^2$ the boundary differential operator simplifies to

$$\nabla_\Gamma \cdot \langle \beta f \rangle \nabla_\Gamma w = \frac{\partial}{\partial s} \langle \beta f \rangle \frac{\partial}{\partial s} w.$$

We recall that $\Omega = \Omega_+ \cup \Omega_-$ and the coefficients α, $\langle \beta f \rangle$ and γ, which are bounded functions defined on Γ_0, are given by (3.19). In order to study the well-posedeness of the above problem, we notice that while the energy space H^1 suffices to rigorously define the solution of the differential equations in $\Omega_\pm$ and $B_R \setminus \overline{\Omega}$, it is not enough to define the boundary differential operator on Γ_0 that appears in (3.34). To handle the boundary differential operator on Γ_0 we define the space

$$\mathcal{H} := \left\{ u \in H^1(B_R \setminus \overline{\Gamma}_0) \text{ such that } \sqrt{f^\pm}\, \nabla_\Gamma \langle u \rangle \in L^2(\Gamma_0) \right\}, \tag{3.35}$$

endowed with the norm

$$\|u\|^2_{\mathcal{H}} = \|u\|^2_{H^1(B_R \setminus \overline{\Gamma}_0)} + \left\| \sqrt{f^+}\, \nabla_\Gamma \langle u \rangle \right\|^2_{L^2(\Gamma_0)} + \left\| \sqrt{f^-}\, \nabla_\Gamma \langle u \rangle \right\|^2_{L^2(\Gamma_0)}.$$

Obviously $\mathcal{H}$ is a Hilbert space since the weights $f^\pm \in L^\infty(\Gamma_0)$ are non-negative (note that $f^\pm = 0$ at the boundary of Γ_0 on Γ). Now, multiplying all three equations with $v \in \mathcal{H}$, integrating by parts, using the continuity of transmission conditions across

$\Gamma \backslash \overline{\Gamma}_0$, the boundary condition on S_R, and the approximate transmission condition on Γ_0, we arrive at the following equivalent variational formulation of (3.28)-(3.34): find $u \in \mathcal{H}$ such that

$$A(u,v) = \mathcal{L}(v) \qquad \text{for all } v \in \mathcal{H} \tag{3.36}$$

where

$$\begin{aligned} A(u,v) \;&:=\; \int_{B_R} \frac{1}{\mu} \nabla u \cdot \overline{\nabla v} - k^2 n u \overline{v} \; \mathrm{d}x + \int_{\Gamma_0} \langle \beta f \rangle \nabla_\Gamma \langle u \rangle \nabla_\Gamma \overline{\langle v \rangle} ds \\ &+ \int_{\Gamma_0} \gamma \langle u \rangle \overline{\langle v \rangle} \; \mathrm{d}s + \int_{\Gamma_0} \frac{1}{\alpha} [u] \overline{[v]} \; \mathrm{d}s - \int_{S_R} T_k u \overline{v} \; \mathrm{d}s \end{aligned} \tag{3.37}$$

and

$$\mathcal{L}(v) = - \int_{S_R} \left(T_k u^i \overline{v} - \frac{\partial u^i}{\partial \boldsymbol{\nu}} \overline{v} \right) ds. \tag{3.38}$$

Here $u|_{\Omega_+} = u^+$, $u|_{\Omega_-} = u^-$ and $u|_{B_R \backslash \overline{\Omega}} = u^{ext}$, and

$$\mu := 1, n := 1 \text{ in } B_R \setminus \overline{\Omega}, \;\; \mu := \mu_+, n := n_+ \text{ in } \Omega_+, \;\; \mu := \mu_-, n := n_- \text{ in } \Omega_-. \tag{3.39}$$

We decompose the bounded sesquilinear form $A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by (3.43) as

$$A(u,v) = A_0(u,v) + B(u,v), \tag{3.40}$$

where

$$A_0(u,v) \;:=\; \int_{B_R} \frac{1}{\mu} \nabla u \cdot \overline{\nabla v} + u \overline{v} \; \mathrm{d}x + \int_{\Gamma_0} \langle \beta f \rangle \nabla_\Gamma \langle u \rangle \nabla_\Gamma \overline{\langle v \rangle} \; \mathrm{d}s - \int_{S_R} T_k u \overline{v} ds$$

and

$$B(u,v) \;:=\; - \int_{B_R} (k^2 n + 1) u \overline{v} \; \mathrm{d}x + \int_{\Gamma_0} \gamma \langle u \rangle \overline{\langle v \rangle} \; \mathrm{d}s + \int_{\Gamma_0} \frac{1}{\alpha} [u] \overline{[v]} \; \mathrm{d}s.$$

Let $\mathbb{A}_0 : \mathcal{H} \to \mathcal{H}$ and $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ be the linear operators defined from the sesquilinear forms $A_0(\cdot,\cdot)$ and $B(\cdot,\cdot)$ by means of the Riesz representation theorem

$$(\mathbb{A}_0 u, v)_{\mathcal{H}} = A_0(u,v) \qquad \text{and} \qquad (\mathbb{B} u, v)_{\mathcal{H}} = B(u,v), \qquad \text{for all } u, v \in \mathcal{H}.$$

At this point let us assume that there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\Re\left(\frac{1}{\mu}\right) \geq \epsilon_1$, and $\Re\left(\frac{1}{\mu_\delta} - \frac{1}{\mu^\pm}\right) \geq \epsilon_2$ (which implies that $\Re(\beta^\pm) \geq 2\delta\epsilon_2$). Then we have that

$$\begin{aligned}
\Re\left(A_0(u,u)\right) &= \int_{B_R} \left(\Re\left(\frac{1}{\mu}\right)|\nabla u|^2 + |u|^2\right) \,\mathrm{d}x + \int_{\Gamma_0} \langle \Re(\beta) f\rangle \left|\nabla_\Gamma \langle u\rangle\right|^2 ds \\
&- \Re\left(\int_{S_R} (T_k u)\overline{u}\,\mathrm{d}s\right) \qquad (3.41)\\
&\geq \min(\epsilon_1, 1)\|u\|^2_{H^1(\Omega)} + \delta\epsilon_2 \left\|\sqrt{f^+}\,\nabla_\Gamma \langle u\rangle\right\|^2_{L^2(\Gamma_0)} \\
&+ \delta\epsilon_2 \left\|\sqrt{f^-}\,\nabla_\Gamma \langle u\rangle\right\|^2_{L^2(\Gamma_0)} \geq C\|u\|^2_{\mathcal{H}}
\end{aligned}$$

for some positive constant $C > 0$, which proves that $A_0(\cdot,\cdot)$ is coercive. The boundedness of $A_0(\cdot,\cdot)$ is obvious given the assumptions on the coefficients and the fact that T_k is bounded. Thus, $\mathbb{A}_0 : \mathcal{H} \to \mathcal{H}$ is invertible operator with bounded inverse.

Due to the fact that $\alpha := 2\delta \langle f(\mu_\delta - \mu)\rangle$ is zero at the boundary of Γ_0 in Γ, the operator $\mathbb{B}$ is not bounded in general. We need to impose some restriction on the rate that $f^\pm$ approaches zero at boundary of Γ_0. Indeed we can prove the following result:

Lemma 3.3.1. *Assume $1/\alpha \in L^t(\Gamma_0)$ for $t = 1+\epsilon$ in $\mathbb{R}^2$ and $t = 7/4+\epsilon$ in $\mathbb{R}^3$ for arbitrary small $\epsilon > 0$. Then $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ is a compact bounded linear operator.*

Proof. We check all three terms of the operator $\mathbb{B}$, i.e.

$$(\mathbb{B}_1 u, v)_{\mathcal{H}} = -\int_{B_R} (k^2 n + 1) u\overline{v}\,\mathrm{d}x, \qquad (\mathbb{B}_2 u, v)_{\mathcal{H}} = \int_{\Gamma_0} \gamma \langle u\rangle \overline{\langle v\rangle}\,\mathrm{d}s$$

$$\text{and} \qquad (\mathbb{B}_3 u, v)_{\mathcal{H}} = \int_{\Gamma_0} \frac{1}{\alpha}\,[u]\,\overline{[v]}\,\mathrm{d}s.$$

Noting that $n \in L^\infty(B_R)$, the compactness of $\mathbb{B}_1$ follows from the fact that $H^1(B_R)$ (and consequently $\mathcal{H}$) is compactly embedded in $L^2(B_R)$ and that

$$\|\mathbb{B}_1 u\|_{\mathcal{H}} = \sup_{\|v\|_{\mathcal{H}}=1} \left| -\int_{B_R} (k^2 n + 1) u\overline{v} dx\right| \leq C\|u\|_{L^2(B_R)}.$$

Next, since $\gamma \in L^\infty(\Gamma_0)$, we have that

$$\begin{aligned} \|\mathbb{B}_2 u\|_{\mathcal{H}} &= \sup_{\|v\|_{\mathcal{H}}=1} \left| \int_{\Gamma_0} \gamma \langle u \rangle \overline{\langle v \rangle} \, \mathrm{d}s \right| \le C \sup_{\|v\|_{\mathcal{H}}=1} \| \langle u \rangle \|_{L^2(\Gamma_0)} \| \langle v \rangle \|_{L^2(\Gamma_0)} \\ &\le C \sup_{\|v\|_{\mathcal{H}}=1} \| \langle u \rangle \|_{L^2(\Gamma)} \| \langle v \rangle \|_{H^{1/2}(\Gamma)} \\ &\le C \sup_{\|v\|_{\mathcal{H}}=1} \| \langle u \rangle \|_{L^2(\Gamma)} \|v\|_{H^1(B_R)} \le C \| \langle u \rangle \|_{L^2(\Gamma)}. \end{aligned}$$

for some positive constant $C > 0$, where we have used the continuity of the trace operator from $H^1(B_R)$ to $H^{1/2}(\Gamma)$. Now the compactness of $\mathbb{B}_2$ follows from the the boundedness of the trace operator and compactly embedding of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$.

Due to the fact that $\alpha := 2\delta \langle f(\mu_\delta - \mu) \rangle$ is zero at the boundary of Γ_0 in Γ, the analysis of $\mathbb{B}_3$ is more delicate and we need to appeal to Rellich-Kondrachov embedding theorems for $W^{m,p}$ spaces (see e.g. [1]). To this end we first recall that from Theorem 5.3 of [1], we have that for a bounded domain $\mathcal{O}$ with $\mathcal{C}^1$-boundary $\partial\mathcal{O}$, the trace operator $\gamma : H^1(\mathcal{O}) \to L^q(\partial\mathcal{O})$ is a continuous embedding if $2 \le q < \infty$ for $\mathcal{O} \subset \mathbb{R}^2$, and $2 \le q < 4$ for $\mathcal{O} \subset \mathbb{R}^3$. Hence assuming that Γ_0 is smooth and using this embedding result, for t as in the assumptions of the lemma we have that

$$\begin{aligned} \|\mathbb{B}_3 u\|_{\mathcal{H}} &= \sup_{\|v\|_{\mathcal{H}}=1} \left| \int_{\Gamma_0} \frac{1}{\alpha} [u][\overline{v}] ds \right| \le \sup_{\|v\|_{\mathcal{H}}=1} \left\| \frac{1}{\alpha} \right\|_{L^t(\Gamma_0)} \|[v]\|_{L^p(\Gamma_0)} \|[u]\|_{L^q(\Gamma_0)} \\ &\le C \sup_{\|v\|_{\mathcal{H}}=1} \left\| \frac{1}{\alpha} \right\|_{L^t(\Gamma_0)} \|v\|_{\mathcal{H}} \|[u]\|_{L^q(\Gamma_0)} \le C \left\| \frac{1}{\alpha} \right\|_{L^t(\Gamma_0)} \|[u]\|_{L^q(\Gamma_0)}. \end{aligned} \tag{3.42}$$

where we have used that there is a constant $C > 0$, such that $\|[v]\|_{L^p(\Gamma_0)} \le C\|v\|_{\mathcal{H}}$. Note that for arbitrary small ϵ, p and q are chosen arbitrarily large in $\mathbb{R}^2$ and arbitrarily close to 4 in $\mathbb{R}^3$, in both cases such that $1/t + 1/p + 1/q = 1$. We also remark that for $u \in \mathcal{H}$ we have that $[u] = 0$ in $\Gamma \setminus \Gamma_0$. Now, we use the Rellich-Kondrachov compact embedding theorem (see Theorem 6.3, Part I in [1]). Applying this theorem for $\Omega := \Gamma_0$ which is a 2-d smooth manifold in the case of $\mathbb{R}^3$ or 1-d smooth manifold in the case of $\mathbb{R}^2$ (in our case $m = 1/2$, $p = 2$, $j = 0$, $k = n = 2$ in $\mathbb{R}^3$ or $k = n = 1$ in $\mathbb{R}^2$), implies that the embedding

$$H^{1/2}(\Gamma_0) \hookrightarrow L^q(\Gamma_0)$$

is compact if $1 \le q < 4$ in $\mathbb{R}^3$ or if $1 \le q < \infty$ in $\mathbb{R}^2$. Combining this with the fact that embedding $\mathcal{H} \hookrightarrow H^{1/2}(\Gamma_0)$ is bounded, from (3.42) we deduce that $\mathbb{B}_3$ is compact, and this concludes the proof of the lemma. We remark that here Theorem 6.3, [1] is adapted to the compact manifold Γ_0 covered by a finite number of charts, each with Riemannian metric bounded below and above by the Euclidean metric, by applying standard arguments based on the partition of unity. □

Lemma 3.3.2. *Assume that $0 \le \Im(n^{\pm}) \le \Im(n_\delta)$ and $0 \le \Im(\mu^{\pm}) \le \Im(\mu_\delta)$. Then problem (3.28)-(3.34) has a unique solution.*

Proof. Take $u^i = 0$ in (3.28)-(3.34), and let u be a solution to the homogenous problem. Taking the imaginary part of (3.43) for $v = u$ we have

$$\begin{aligned} 0 \;&=\; \int_{B_R} \Im\left(\frac{1}{\mu}\right)|\nabla u| - k^2\Im(n)|u|^2 \,\mathrm{d}x + \int_{\Gamma_0} \Im\left\langle \beta f\right\rangle \left|\nabla_\Gamma \left\langle u\right\rangle\right|^2 ds \\ &+\; \int_{\Gamma_0} \Im(\gamma)\left|\langle u\rangle\right|^2\;\mathrm{d}s + \int_{\Gamma_0} \Im\left(\frac{1}{\alpha}\right)\left|[u]\right|^2\;\mathrm{d}s - \Im\left(\int_{S_R} T_k u\overline{u}\;\mathrm{d}s\right) \qquad (3.43) \end{aligned}$$

Now, since from the assumptions on the material properties we have that $\Im\left(\frac{1}{\mu^{\pm}}\right) \le 0$, $\Im(n^{\pm}) \ge 0$, $\Im(\langle\beta f\rangle) \le 0$, $\Im(\alpha) \ge 0$ and $\Im(\gamma) \le 0$, the above equation implies

$$\Im\left(\int_{S_R} T_k u\overline{u}\;\mathrm{d}s\right) \le 0.$$

But (3.27) now implies that indeed

$$\Im\left(\int_{S_R} T_k u\overline{u}\;\mathrm{d}s\right) = 0.$$

The definition of the Dirichet-to-Neumann operator and Rellich's lemma (see [21] and [32]) now imply that $u = 0$ and $\partial u/\partial \boldsymbol{\nu} = 0$ on S_R. Finally, from Holmgren's theorem together with the unique continuation principle (which under our geometrical and physical assumptions holds true, see e.g. Theorem 17.2.6 in [52]), we can conclude that $u = 0$ which proves the uniqueness of (3.28)-(3.34). □

In summary, combining Lemma 3.3.1 and Lemma 3.3.2 with the coercivity result (3.41) we obtain the main result of this section.

Theorem 3.3.1 (Well-posedness)**.** *In addition to the geometrical and physical assumptions stated in the Introduction, assume that:*

1. $\Re\left(\frac{1}{\mu}\right) \geq \epsilon_1$, *and* $\Re\left(\frac{1}{\mu_\delta} - \frac{1}{\mu^\pm}\right) \geq \epsilon_2$ *for some constants* $\epsilon_1 > 0$ *and* $\epsilon_2 > 0$,
2. $0 \leq \Im(n^\pm) \leq \Im(n_\delta)$ *and* $0 \leq \Im(\mu^\pm) \leq \Im(\mu_\delta)$ *and*
3. *the profile* $f^\pm$ *go to zero at the boundary of* Γ_0 *in* Γ *such that* $1/\alpha \in L^t(\Gamma_0)$ *for* $t = 1+\epsilon$ *in* $\mathbb{R}^2$ *and* $t = 7/4 + \epsilon$ *in* $\mathbb{R}^3$ *for arbitrary small* $\epsilon > 0$, *where* $\alpha = \langle f(\mu_\delta - \mu)\rangle$.

Then problem (3.28)-(3.34) has a unique solution $u \in \mathcal{H}$ *which depends continuously on the incident wave* u^i *with respect to the* $\mathcal{H}$*-norm.*

Remark 3.3.1. *Since any solution of (3.28)-(3.34) can be extended to a solution of the scattering problem (3.20)-(3.22), (3.23)-(3.25) and (3.9) and vise-versa, Theorem (3.3.1) provides a well-posedness result for the approximate crack problem.*

For later use we need to consider the above scattering problem in the following form: Find $w \in \mathcal{H} \cap H^1_{loc}(\mathbb{R}^m \setminus \Gamma_0)$ such that

$$\nabla \cdot \left(\tfrac{1}{\mu}\nabla w\right) + k^2 n w = 0 \qquad \text{in} \qquad \mathbb{R}^m \setminus \overline{\Gamma}_0, \tag{3.44}$$

$$[w] = \alpha \left\langle \tfrac{1}{\mu}\tfrac{\partial w}{\partial \boldsymbol{\nu}}\right\rangle + \alpha h_1 \qquad \text{on} \qquad \Gamma_0, \tag{3.45}$$

$$\left[\tfrac{1}{\mu}\tfrac{\partial w}{\partial \boldsymbol{\nu}}\right] = \left(-\nabla_\Gamma \cdot \langle \beta f\rangle \nabla_\Gamma + \gamma\right)\langle w\rangle + h_2 \qquad \text{on} \qquad \Gamma_0, \tag{3.46}$$

$$\lim_{r\to\infty} r^{\frac{m-1}{2}}\left(\tfrac{\partial w}{\partial r} - ikw\right) = 0, \tag{3.47}$$

where h_1 and h_2 are

$$\begin{cases} h_1 := \left\langle \dfrac{1}{\mu}\dfrac{\partial v}{\partial \boldsymbol{\nu}}\right\rangle - \dfrac{1}{\alpha}[v], \\ \\ h_2 := \left(-\nabla_\Gamma \cdot \langle \beta f\rangle \nabla_\Gamma + \gamma\right)\langle v\rangle - \left[\dfrac{1}{\mu}\dfrac{\partial v}{\partial \boldsymbol{\nu}}\right], \end{cases} \tag{3.48}$$

for some $v \in \mathcal{H}$ with $\nabla \cdot ((1/\mu)\nabla v) \in L^2(B_R \setminus \Gamma_0)$. For the later use we define the following trace space on Γ_0 of function $u \in \mathcal{H}$,

$$\mathcal{H}(\Gamma_0) := \left\{ u \in H^{1/2}(\Gamma_0) \ \text{ such that } \ \sqrt{f^\pm}\nabla_\Gamma u \in L^2(\Gamma_0)\right\} \tag{3.49}$$

and its dual $\mathcal{H}^{-1}(\Gamma_0)$ with respect to the following duality pairing

$$(u,v)_{\mathcal{H}(\Gamma_0),\mathcal{H}^{-1}(\Gamma_0)} := (u,v)_{H^{1/2}(\Gamma_0),\tilde{H}^{-1/2}(\Gamma_0)} + \left(f^{\pm}\nabla_\Gamma u, \nabla_\Gamma v\right)_{L^2(\Gamma_0),L^2(\Gamma_0)}. \qquad (3.50)$$

Here $\widetilde{H}^{1/2}(\Gamma_0)$ and $\widetilde{H}^{-1/2}(\Gamma_0)$ consist of functions in $H^{1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$ that can be extended by zero in the entire Γ as $H^{1/2}$ and $H^{-1/2}$ functions, respectively. They are duals of $H^{-1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$, respectively. Hence $h_1 \in H^{-1/2}(\Gamma_0)$ and $h_2 \in \mathcal{H}^{-1}(\Gamma_0)$.

3.4 The inverse problem of reconstructing the delaminated part Γ_0

In this section we turn our attention to the main goal of this study, which is the reconstruction of the delaminated portion Γ_0 of the interface Γ between two materials from measured scattering data. Our reconstruction method is a modified linear sampling method, adapted to our problem where we already know the interface Γ and only look for the delaminated part Γ_0. The linear sampling method and factorization method have been used to reconstruct cracks or screens with various types of boundary conditions [13], [18], [20], [57] and [87] (see also the monographs [21] and [23]). Although numerically both the linear sampling method and factorization method provide similar reconstruction results, the factorization method is mathematically more satisfactory. Here we develop the linear sampling method since our complicated jump conditions modeling the delaminated part Γ_0 fail to satisfy the standard assumptions under which the factorization method works (see [28]). For other inversion methods applied to similar types of inverse problems in acoustic and elasticity we refer the reader to [5, 6, 9].

We assume that the interrogating incident fields are plane waves given by $u^i(\mathbf{x},\widehat{\mathbf{d}}) = e^{ik\widehat{\mathbf{d}}\cdot\mathbf{x}}$ where the unit vector $\widehat{\mathbf{d}}$ is the incident direction. The corresponding scattered field $u^s(\mathbf{x},\widehat{\mathbf{d}})$, i.e. the solution of (3.20)-(3.22), (3.23)-(3.25) and (3.9) with $u^i := e^{ik\widehat{\mathbf{d}}\cdot\mathbf{x}}$ satisfies (see [32] and [21])

$$u^s(\mathbf{x},\widehat{\mathbf{d}}) = \gamma_m \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|^{(m-1)/2}} u^\infty(\widehat{\mathbf{x}},\widehat{\mathbf{d}}) + O\left(\frac{1}{|\mathbf{x}|}\right), \qquad \widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|, \quad |\mathbf{x}| \to \infty,$$

where

$$\gamma_m = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \quad \text{if } m = 2 \quad \text{and} \quad \gamma_m = \frac{1}{4\pi} \quad \text{if } m = 3. \tag{3.51}$$

The function $u^\infty(\mathbf{x}, \widehat{\mathbf{d}})$ which is an analytic function of $\widehat{\mathbf{x}}$ on the unit sphere $\mathbb{S}^{m-1} := \{\mathbf{x} \in \mathbb{R}^m, \ |\mathbf{x}| = 1\}$, is referred to as the far field pattern of the scattered field $u^s(\mathbf{x}, \widehat{\mathbf{d}})$.

The *inverse problem* we consider here is to determine the delaminated portion Γ_0 of the boundary Γ from a knowledge of $u^\infty(\mathbf{x}, \widehat{\mathbf{d}})$ for $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{d}}$ on the unit sphere $\mathbb{S}^{m-1}$. Although in applications to nondestructive testing it is possible to have measurements all around, we remark that the inversion algorithm that we shall develop next can also be justified and implemented for limited aperture data (see Section 4.5 in [21]) as well as for near field data. However, the quality of the reconstruction is likely to be poor for small apertures which is usually the case for qualitative methods [46]. We also remark that for many problems in nondestructive testing, it is reasonable to assume that the background medium is known as we do here, since the background corresponds to the healthy object to be tested. In the cases when the background is not know and for simple defects, qualitative methods could be used to determine interfaces between homogeneous regions of the background media along with the defect (see [85] and some references therein).

3.4.1 A mixed reciprocity principle

We start by proving a mixed reciprocity result in order to deal with the non-homogeneous background. This generalizes similar results first used in [75], and later developped also in [46], [17] and [27] (see also [7] for a similar type of calculations).

To this end we let $u_b(\cdot, \widehat{\mathbf{d}})$ be the total field due to the background, i.e. in absence of the delamination Γ_0, corresponding to the incident plain wave $u^i(\cdot, d)$. More precisely, $u_b(\cdot, \widehat{\mathbf{d}})$ is the unique solution in $H^1_{loc}(\mathbb{R}^m)$ of

$$\begin{aligned} \nabla \cdot \left(\tfrac{1}{\mu}\nabla u_b\right) + k^2 n u_b = 0 \quad \text{in } \mathbb{R}^m, \\ u_b = u_b^s + u^i, \\ \lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\tfrac{\partial u_b^s}{\partial r} - iku_b^s\right) = 0, \end{aligned} \tag{3.52}$$

where μ and n, both in $L^\infty(\Omega)$, are defined by (3.39). Note that the continuity of the field and co-normal derivatives across Γ_1 and Γ are implicit in this formulation. Next let $\mathbb{G}_b(\cdot,\cdot)$ be the Green's function associated with the background media, i.e $\mathbb{G}_b(\cdot,\mathbf{z}) \in H^1_{loc}(\mathbb{R}^m \setminus \{z\})$ satisfying

$$\begin{aligned} \nabla \cdot \left(\tfrac{1}{\mu}\nabla \mathbb{G}_b(\cdot,\mathbf{z})\right) + k^2 n \mathbb{G}_b(\cdot,\mathbf{z}) = -\delta(\cdot - \mathbf{z}), \qquad \text{in } \mathbb{R}^m \setminus \{z\}, \\ \lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\tfrac{\partial \mathbb{G}_b(\cdot,\mathbf{z})}{\partial r} - ik\mathbb{G}_b(\cdot,\mathbf{z})\right) = 0, \end{aligned} \tag{3.53}$$

where again the continuity of the field and co-normal derivatives across Γ_1 and Γ is understood. We denote by $\mathbb{G}_b^\infty(\cdot,\mathbf{z}) \in L^2(\mathbb{S}^{m-1})$ the far-field pattern of the radiating field $\mathbb{G}_b(\cdot,\mathbf{z})$.

Theorem 3.4.1 (Mixed Reciprocity principle)**.** *The following relation holds*

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}},\mathbf{z}) = \gamma_m u_b(\mathbf{z}, -\widehat{\mathbf{x}}) \quad \textit{for all } z \in \mathbb{R}^m \quad \textit{and} \quad \widehat{\mathbf{x}} \in \mathbb{S}^{m-1},$$

where γ_m is defined by (3.51).

Proof. Let us first consider $z \in \Omega_{\text{ext}} := \mathbb{R}^m \setminus \overline{\Omega}$. Let $\Phi(\cdot,\mathbf{z})$ denote the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$ given by

$$\Phi(\mathbf{x},\mathbf{z}) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{z}|) & \text{in } \mathbb{R}^2, \\[2em] \dfrac{1}{4\pi}\dfrac{e^{ik|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|} & \text{in } \mathbb{R}^3. \end{cases}$$

Since $\mathbb{G}_b(\cdot,\mathbf{z}) - \Phi(\cdot,\mathbf{z})$ is a non-singular radiating solution to $\Delta u + k^2 u = 0$ in Ω_{ext}, an application of Green's second identity together with the Sommerfeld radiation condition implies that for all $\mathbf{x} \in \Omega_{\text{ext}}$

$$\begin{aligned} (\mathbb{G}_b - \Phi)(\mathbf{x},\mathbf{z}) &= \int_{\Gamma_1} \left\{ (\mathbb{G}_b - \Phi)(\mathbf{y},\mathbf{z}) \frac{\partial \Phi}{\partial \boldsymbol{\nu}_y}(\mathbf{x},\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}) \frac{\partial (\mathbb{G}_b - \Phi)}{\partial \boldsymbol{\nu}_y}(\mathbf{y},\mathbf{z}) \right\} \, \mathrm{d}s(\mathbf{y}) \\ &= \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{y},\mathbf{z}) \frac{\partial \Phi}{\partial \boldsymbol{\nu}_y}(\mathbf{x},\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}) \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}(\mathbf{y},\mathbf{z}) \right\} \, \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{3.54}$$

where we have used the fact that, since $z \in \Omega_{\text{ext}}$,

$$\int_{\Gamma_1} \left\{ \Phi(\mathbf{y},\mathbf{z}) \frac{\partial \Phi}{\partial \boldsymbol{\nu}_y}(\mathbf{x},\mathbf{y}) - \Phi(\mathbf{y},\mathbf{z}) \frac{\partial \Phi}{\partial \boldsymbol{\nu}_y}(\mathbf{x},\mathbf{y}) \right\} \ \mathrm{d}s(\mathbf{y}) = 0.$$

Then, from (3.54), and using the fact that $\Phi^\infty(\widehat{\mathbf{x}},\mathbf{z}) = \gamma_m u^i(\mathbf{z},-\widehat{\mathbf{x}}) := \gamma_m e^{-i\widehat{\mathbf{x}}\cdot\mathbf{z}}$ we obtain for all $\mathbf{x} \in \Omega_{\text{ext}}$

$$\begin{aligned} &\mathbb{G}_b^\infty(\widehat{\mathbf{x}},\mathbf{z}) - \gamma_m u^i(\mathbf{z},-\widehat{\mathbf{x}}) = \\ &\qquad \gamma_m \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{y},\mathbf{z}) \frac{\partial u^i}{\partial \boldsymbol{\nu}_y}(\mathbf{y},-\widehat{\mathbf{x}}) - u^i(\mathbf{y},-\widehat{\mathbf{x}}) \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}(\mathbf{y},\mathbf{z}) \right\} \ \mathrm{d}s(\mathbf{y}). \end{aligned} \tag{3.55}$$

On the other hand, the scattered field due to the background $u_b^s(\cdot,-\widehat{\mathbf{x}})$ is also a radiating solution of $\Delta u + k^2 u = 0$ in Ω_{ext} hence we have that

$$\int_{\Gamma_1} \left\{ (\Phi - \mathbb{G}_b)(\mathbf{y},\mathbf{z}) \frac{\partial u_b^s}{\partial \boldsymbol{\nu}_y}(\mathbf{y},-\widehat{\mathbf{x}}) - u_b^s(\mathbf{y},-\widehat{\mathbf{x}}) \frac{\partial (\Phi - \mathbb{G}_b)}{\partial \boldsymbol{\nu}_y}(\mathbf{y},\mathbf{z}) \right\} \ \mathrm{d}s(\mathbf{y}) = 0.$$

Now the integral representation formula for $u_b^s(\cdot,-\widehat{\mathbf{x}})$ in Ω_{ext} (see [21]) yields

$$\begin{aligned} u_b^s(\mathbf{z},-\widehat{\mathbf{x}}) &= \int_{\Gamma_1} \left\{ u_b^s(\mathbf{y},-\widehat{\mathbf{x}}) \frac{\partial \Phi}{\partial \boldsymbol{\nu}_y}(\mathbf{y},\mathbf{z}) - \Phi(\mathbf{y},\mathbf{z}) \frac{\partial u_b^s(\mathbf{y},-\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} \right\} \ \mathrm{d}s(\mathbf{y}) \qquad (3.56) \\ &= \int_{\Gamma_1} \left\{ u_b^s(\mathbf{y},-\widehat{\mathbf{x}}) \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}(\mathbf{y},\mathbf{z}) - \mathbb{G}_b(\mathbf{y},\mathbf{z}) \frac{\partial u_b^s(\mathbf{y},-\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} \right\} \ \mathrm{d}s(\mathbf{y}). \end{aligned}$$

In addition, using the transmission conditions across the interfaces Γ_1 and the equations for u_b and $\mathbb{G}_b(\cdot,\cdot)$ we obtain

$$\begin{aligned} &\textstyle\int_{\Gamma_1} \left\{ u_b(\mathbf{y},-\widehat{\mathbf{x}}) \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}(\mathbf{z},\mathbf{y}) - \mathbb{G}_b(\mathbf{z},\mathbf{y}) \frac{\partial u_b(\mathbf{y},-\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} \right\} \ \mathrm{d}s(\mathbf{y}) \\ &= \textstyle\int_{\Gamma_1} \left\{ u_b^+(\mathbf{y},-\widehat{\mathbf{x}}) \frac{1}{\mu^+} \frac{\partial \mathbb{G}_b^+}{\partial \boldsymbol{\nu}_y}(\mathbf{z},\mathbf{y}) - \mathbb{G}_b^+(\mathbf{z},\mathbf{y}) \frac{1}{\mu^+} \frac{\partial u_b^+(\mathbf{y},-\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} \right\} \ \mathrm{d}s(\mathbf{y}) \qquad (3.57) \\ &= \textstyle\int_{\Omega} \left\{ u_b(\mathbf{y},-\widehat{\mathbf{x}}) \nabla \cdot \left(\frac{1}{\mu} \nabla \mathbb{G}_b \right)(\mathbf{z},\mathbf{y}) - \mathbb{G}_b(\mathbf{z},\mathbf{y}) \nabla \cdot \left(\frac{1}{\mu} \nabla u_b \right)(\mathbf{y},-\widehat{\mathbf{x}}) \right\} \ \mathrm{d}s(\mathbf{y}) = 0 \end{aligned}$$

Thus from (3.56) and (3.57), since $u_b = u_b^s + u^i$ we have that

$$u_b^s(\mathbf{z},-\widehat{\mathbf{x}}) = \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{z},\mathbf{y}) \frac{\partial u^i(\mathbf{y},-\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} - u^i(\mathbf{y},-\widehat{\mathbf{x}}) \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}(\mathbf{z},\mathbf{y}) \right\} \ \mathrm{d}s(\mathbf{y}). \tag{3.58}$$

Finally (3.55) provides

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}},\mathbf{z}) \ = \ \gamma_m u_b(\mathbf{z},-\widehat{\mathbf{x}}).$$

Next let $\mathbf{z} \in \Omega_+ \cup \Omega_-$. Then $\mathbb{G}_b(\cdot, \mathbf{z})$ is a smooth radiating solution of $\Delta u + k^2 u = 0$ in Ω_{ext}, and hence Green's representation formula implies

$$\mathbb{G}_b(\mathbf{x}, \mathbf{z}) = \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \frac{\partial \Phi}{\partial \boldsymbol{\nu}_y}(\mathbf{x}, \mathbf{y}) - \Phi_k(\mathbf{x}, \mathbf{y}) \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}(\mathbf{y}, \mathbf{z}) \right\} \, \mathrm{d}s(\mathbf{y}). \tag{3.59}$$

Evaluating the far field pattern yields

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) = \gamma_m \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \frac{\partial e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \boldsymbol{\nu}_y} - e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial \mathbb{G}_b(\mathbf{y}, \mathbf{z})}{\partial \boldsymbol{\nu}_y} \right\} \, \mathrm{d}s(\mathbf{y}). \tag{3.60}$$

Moreover, since $u_b^s(\cdot, -\widehat{\mathbf{x}})$ is also a radiating solution to the Helmholtz equation in Ω_{ext}, we have that

$$\gamma_m \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \frac{\partial u_b^s(\mathbf{y}, -\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} - u_b^s(\mathbf{y}, -\widehat{\mathbf{x}}) \frac{\partial \mathbb{G}_b(\mathbf{y}, \mathbf{z})}{\partial \boldsymbol{\nu}_y} \right\} \, \mathrm{d}s(\mathbf{y}) = 0, \tag{3.61}$$

Hence adding (3.60) and (3.61), recalling that $u_b(\mathbf{y}, -\widehat{\mathbf{x}}) = u_b^s(\mathbf{y}, -\widehat{\mathbf{x}}) + e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}$ and applying Green's second identity and the transmission conditions across Γ_1 and Γ proves that

$$\begin{aligned}
\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) &= \gamma_m \int_{\Gamma_1} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \frac{\partial e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \boldsymbol{\nu}_y} - e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial \mathbb{G}_b(\mathbf{y}, \mathbf{z})}{\partial \boldsymbol{\nu}_y} \right\} \, \mathrm{d}s(\mathbf{y}) \\
&= \gamma_m \int_{\Omega_+ \cup \Omega_-} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \nabla \cdot \left(\frac{1}{\mu} \nabla u_b \right) (\mathbf{y}, -\widehat{\mathbf{x}}) \right. \\
&- \left. u_b(\mathbf{y}, -\widehat{\mathbf{x}}) \nabla \cdot \left(\frac{1}{\mu} \nabla \mathbb{G}_b \right) (\mathbf{y}, \mathbf{z}) \right\} dy \\
&+ \gamma_m \int_{\Gamma} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \left[\frac{1}{\mu} \frac{\partial u_b}{\partial \boldsymbol{\nu}(\mathbf{y})} \right] (\mathbf{y}, -\widehat{\mathbf{x}}) - u_b(\mathbf{y}, -\widehat{\mathbf{x}}) \left[\frac{1}{\mu} \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y} \right] (\mathbf{y}, \mathbf{z}) \right\} \, \mathrm{d}s(\mathbf{y}).
\end{aligned}$$

Now we use the continuity of $\frac{1}{\mu} \frac{\partial u_b}{\partial \boldsymbol{\nu}_y}$ and $\frac{1}{\mu} \frac{\partial \mathbb{G}_b}{\partial \boldsymbol{\nu}_y}$ across Γ and the fact that u_b and $\mathbb{G}_b$ satisfy the same equation in $(\Omega_+ \cup \Omega_-) \setminus \overline{B_\epsilon(\mathbf{z})}$, where $B_\epsilon(\mathbf{z})$ is a small ball of radius ϵ centered at $\mathbf{z}$ and included either in Ω_+ or Ω_-, to obtain

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) = \gamma_m \int_{B_\epsilon(\mathbf{z})} \left\{ \mathbb{G}_b(\mathbf{y}, \mathbf{z}) \nabla \cdot \left(\frac{1}{\mu} \nabla u_b \right) (\mathbf{y}, -\widehat{\mathbf{x}}) - u_b(\mathbf{y}, -\widehat{\mathbf{x}}) \nabla \cdot \left(\frac{1}{\mu} \nabla \mathbb{G}_b \right) (\mathbf{y}, \mathbf{z}) \right\} dy.$$

Letting ϵ tend to zero and using the equation for u_b and the first equation in (3.53) for $\mathbf{x} \in B_\epsilon(\mathbf{z})$ finally implies

$$\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \mathbf{z}) = \gamma_m \, u_b(\mathbf{z}, -\widehat{\mathbf{x}})$$

where we have used (3.56). Finally, by the continuity of $\mathbb{G}_b$ across Γ_1 and Γ, we can now conclude that $\mathbb{G}_b^\infty(\widehat{\mathbf{x}}, \cdot) = \gamma_m \, u_b(\cdot, -\widehat{\mathbf{x}})$ holds everywhere in $\mathbb{R}^m$. $\square$

3.4.2 The linear sampling method

We now propose and analyze a version of the Linear Sampling Method (LSM) to detect the delaminated part Γ_0 on the known interface Γ. As mentioned earlier, the data needed for our inversion scheme is the multistatic far field pattern $u^\infty(\hat{\mathbf{x}}, \widehat{\mathbf{d}})$, $\hat{\mathbf{x}}, \widehat{\mathbf{d}} \in \mathbb{S}^{m-1}$. This far field data allows us to define the standard far field operator $\mathcal{F} : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ given by

$$(\mathcal{F}g)\,(\widehat{\mathbf{x}}) = \int_{\mathbb{S}^{m-1}} u^\infty(\mathbf{x}, \widehat{\mathbf{d}}) g(\widehat{\mathbf{d}}) \;\; \mathrm{d}s(\widehat{\mathbf{d}}). \tag{3.62}$$

By linearity $\mathcal{F}g$ is the far field pattern of the scattered field u^s satisfying the scattering problem (3.20)-(3.22), (3.23)-(3.25) and (3.9) with $u^i := v_g$, where v_g is the so-called Herglotz wave function defined by

$$v_g(\mathbf{x}) = \int_{\mathbb{S}^{m-1}} g(\widehat{\mathbf{d}}) e^{ik\mathbf{x}\cdot\widehat{\mathbf{d}}} \;\; \mathrm{d}s(\widehat{\mathbf{d}}). \tag{3.63}$$

On the other hand the far field pattern $u_b^\infty(\widehat{\mathbf{x}}, \widehat{\mathbf{d}})$ of the scattered field due to the background, i.e. the solution $u_b^s(\cdot, d)$ of (3.52), defines the background far field operator $\mathcal{F}_b : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$

$$(\mathcal{F}_b g)\,(\widehat{\mathbf{x}}) = \int_{\mathbb{S}^{m-1}} u_b^\infty(\mathbf{x}, \widehat{\mathbf{d}}) g(\widehat{\mathbf{d}}) \;\; \mathrm{d}s(\widehat{\mathbf{d}}). \tag{3.64}$$

Note that $\mathcal{F}_b g$ can be computed since it is assume that the undamaged configuration of the scatterer is known *a priori*. Similarly, by linearity $\mathcal{F}_b g$ is the far field pattern of the solution u_b^s with $u^i := v_g$. Also by linearity, the total field $u_{b,g}$ corresponding to the scattering by the background media due to v_g as incident field, i.e solution of (3.52) with $u^i := v_g$, can be written as

$$u_{b,g}(\mathbf{x}) := \int_{\mathbb{S}^{m-1}} u_b(\mathbf{x}, \widehat{\mathbf{d}}) g(\widehat{\mathbf{d}}) \;\; \mathrm{d}s(\widehat{\mathbf{d}}). \tag{3.65}$$

Finally, we define the far field operator solely due to the delamination $\mathcal{F}_D : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ which is given by

$$\mathcal{F}_D g = \mathcal{F}g - \mathcal{F}_b g. \tag{3.66}$$

Obviously $\mathcal{F}_D g$ can be seen as the far field pattern of the scattered field due to the defect Γ_0 when the incident field is $u_{b,g}$ given by (3.65). From this point we assume that we know $\mathcal{F}_D$, and we will use it to develop the linear sampling method to reconstruct Γ_0. To this end, we define the bounded linear operator $\mathscr{H} : L^2(\mathbb{S}^{m-1}) \to H^{-1/2}(\Gamma_0) \times \mathcal{H}^{-1}(\Gamma_0)$ by

$$\mathscr{H} g \;=\; \left(\alpha \, \frac{1}{\mu} \frac{\partial u_{b,g}}{\partial \boldsymbol{\nu}} \bigg|_{\Gamma_0}, K u_{b,g} \right), \tag{3.67}$$

where $K : \mathcal{H}(\Gamma_0) \to \mathcal{H}^{-1}(\Gamma_0)$ corresponds to one part of the boundary data on Γ_0 and is given by (see (3.48) and (3.50))

$$(K\phi, \psi)_{\mathcal{H}(\Gamma_0), \mathcal{H}^{-1}(\Gamma_0)} = \int_{\Gamma_0} \left\{ \langle \beta f \rangle \nabla_\Gamma \phi \cdot \nabla_\Gamma \overline{\psi} + \gamma \phi \overline{\psi} \right\} ds.$$

The conjugate transpose operator $K^* : \mathcal{H}(\Gamma_0) \to \mathcal{H}^{-1}(\Gamma_0)$ is defined by

$$(K^*\phi, \psi) = \int_{\Gamma_0} \left\{ \langle \overline{\beta} f \rangle \nabla_\Gamma \phi \cdot \nabla_\Gamma \overline{\psi} + \overline{\gamma} \phi \overline{\psi} \right\} ds := \left(\overline{K} \phi, \psi \right).$$

Note that $\mathscr{H} g$ maps $u_{b,g}$ to the corresponding transmission conditions given by (3.48), since both the field $u_{b,g}$ and its co-normal derivative are continuous on Γ_0 (so the terms in (3.48) with jumps disappear) and we simply write the average by the common value on either side of the curve, i.e $\left\langle \frac{1}{\mu} \frac{\partial u_{b,g}}{\partial \boldsymbol{\nu}} \right\rangle = \frac{1}{\mu^\pm} \frac{\partial u^\pm_{b,g}}{\partial \boldsymbol{\nu}}$, and $\langle u_{b,g} \rangle = u^\pm_{b,g}$. We remark that for smooth Γ_0 and smooth coefficients $\mu^\pm$ and $n^\pm$, we can assume by the regularity of the solution of the transmission problem that $u_{b,g} \in \mathcal{H}$ and hence its trace on Γ_0 is in $\mathcal{H}(\Gamma_0)$.

Lemma 3.4.1. *The operator $\mathscr{H} : L^2(\mathbb{S}^{m-1}) \to H^{-1/2}(\Gamma_0) \times \mathcal{H}^{-1}(\Gamma_0)$ has dense range. Assume in addition to the assumptions of Theorem 3.3.1 that $\Re(n - n_\delta) > 0$ (or more generally that there is no non-trivial $u_{b,g}$ such that $K u_{b,g} = 0$), then $\mathscr{H}$ is injective.*

Proof. We first check the injectivity. Let $g \in L^2(\mathbb{S}^{m-1})$ such that $\mathscr{H} g = 0$. Then both $\frac{1}{\mu} \frac{\partial u_{b,g}}{\partial \boldsymbol{\nu}} |_{\Gamma_0}$ and $u_{b,g}|_{\Gamma_0} = 0$. The latter follows by taking the real part of $K u_{b,g} = 0$ and the fact that $\Re(\langle \beta f \rangle) > 0$ and $\Re(\gamma) > 0$. Then, by Holmgren's theorem we conclude that $u_{b,g} = 0$ in a region extending on both sides of Γ_0, and by analytic continuation we obtain that $u_{b,g} \equiv 0$ vanishes identically. Since $u_{b,g}$ is sum of radiating scattering wave and the Herglotz wave function v_g which is an entire solution to Helmholtz equation, the latter implies $v_g \equiv 0$ yielding $g = 0$. Next, to show that $\mathscr{H}$ has dense range it suffices to prove that $\mathscr{H}^*$ is injective, where $\mathscr{H}^* : \widetilde{H}^{1/2}(\Gamma_0) \times \mathcal{H}(\Gamma_0) \to L^2(\mathbb{S}^{m-1})$ is the transpose-conjugate operator associated with $\mathscr{H}$. To this end, suppose that (ζ, η) in $\widetilde{H}^{1/2}(\Gamma_0) \times \mathcal{H}(\Gamma_0)$. Then

$$
\begin{aligned}
(\mathscr{H} g, (\zeta, \eta)) &= \left(\frac{\alpha}{\mu}\frac{\partial u_{b,g}}{\partial \boldsymbol{\nu}}, \zeta\right) + (K u_{b,g}, \eta) = \left(\frac{\alpha}{\mu}\frac{\partial u_{b,g}}{\partial \boldsymbol{\nu}}, \zeta\right) + \left(u_{b,g}, \overline{K}\eta\right) \\
&= \int_{\Gamma_0} \left\{\frac{\alpha}{\mu}\frac{\partial u_{b,g}}{\partial \boldsymbol{\nu}_y}\overline{\zeta} + u_{b,g} K\overline{\eta}\ \mathrm{d}s_y\right\} \\
&= \int_{\mathbb{S}^{m-1}} g(\widehat{\mathbf{x}}) \int_{\Gamma_0} \left\{\overline{\zeta}\frac{\alpha}{\mu}\frac{\partial u_b(\mathbf{y}, \widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y} + K\overline{\eta} u_b(\mathbf{y}, \widehat{\mathbf{x}})\right\} \mathrm{d}s_y\ \mathrm{d}s_{\widehat{\mathbf{x}}} \\
&= (g, \mathscr{H}^*(\zeta, \eta)).
\end{aligned} \tag{3.68}
$$

Thus

$$
\mathscr{H}^*(\zeta, \eta) = \int_{\Gamma_0} \left\{\frac{\overline{\alpha}\mu}{\overline{\mu}}\frac{1}{\mu}\frac{\partial u_b(\mathbf{y}, -\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}_y}\zeta + u_b(\mathbf{y}, -\widehat{\mathbf{x}})\overline{K}\eta\right\} \mathrm{d}s_y. \tag{3.69}
$$

From the mixed reciprocity relation Theorem 3.4.1, we have that that $\mathscr{H}^*(\zeta, \eta)$ is the far field pattern associated with the scattered wave

$$
w^s(\mathbf{x}) = \gamma_m^{-1} \int_{\Gamma_0} \left\{\zeta\frac{\overline{\alpha}\mu}{\overline{\mu}}\frac{1}{\mu}\frac{\partial \mathbb{G}_b(\mathbf{x}, \mathbf{y})}{\partial \boldsymbol{\nu}_y} + \overline{K}\eta\mathbb{G}_b(\mathbf{x}, \mathbf{y})\right\} \mathrm{d}s_y
$$

where γ_m is defined in (3.51). Moreover, since the singularity of the free space Green's function $\mathbb{G}_b(\cdot,\cdot)$ is of the same order as the fundamental solution $\Phi(\cdot,\cdot)$, w^s is given by the following representation formula (see e.g. [63])

$$
w^s(\mathbf{x}) = \int_{\Gamma_0} \left\{ [w^s]\frac{1}{\mu}\frac{\partial \mathbb{G}_b(\mathbf{x}, \mathbf{y})}{\partial \boldsymbol{\nu}_y} - \left[\frac{1}{\mu}\frac{\partial w^s}{\partial \boldsymbol{\nu}}\right]\mathbb{G}_b(\mathbf{x}, \mathbf{y})\right\} \mathrm{d}s_y,
$$

and thus

$$
[w^s] = \gamma_m^{-1}\frac{\overline{\alpha}\mu}{\overline{\mu}}\zeta \qquad \text{and} \qquad \left[\frac{1}{\mu}\frac{\partial w^s}{\partial \boldsymbol{\nu}}\right] = -\gamma_m^{-1}\overline{K}\eta. \tag{3.70}
$$

Therefore, if $\mathscr{H}^*(\zeta, \eta) = 0$, then by Rellich's lemma together with the unique continuation principle and Holmgren's theorem, $w^s = 0$ in $\mathbb{R}^m \backslash \overline{\Gamma}_0$, so $[w^s] = 0$ and $\left[\frac{1}{\mu}\frac{\partial w^s}{\partial \boldsymbol{\nu}}\right] = 0$, implying that $\zeta = \eta = 0$. □

Next, define the bounded linear operator $\mathscr{G} : H^{-1/2}(\Gamma_0) \times \mathcal{H}^{-1}(\Gamma_0) \to L^2(\mathbb{S}^{m-1})$ by

$$\mathscr{G} : (h_1, h_2) \mapsto w^\infty$$

where w^∞ is the far field pattern of the corresponding radiating solution w to (3.44)-(3.47). Notice here that the the well-posedness of the problem guarantees that the operator $\mathscr{G}$ is well defined and bounded, since in the variational formulation the source terms h_1, h_2 always define a bounded linear functional in the space $\mathcal{H}$. It is clear from the definition of $\mathscr{H}$ and $\mathscr{G}$ that we have the factorization $\mathcal{F}_D = \mathscr{G}\mathscr{H}$.

Since for our inverse problem we know the interface Γ and are looking for the delaminated part Γ_0, we define the test function as follows: for any $L \subset \Gamma$, given $(\alpha_L, \beta_L) \in L^2(L) \times \widetilde{H}^1(L)$ we define

$$\phi_L^\infty(\widehat{\mathbf{x}}) := \gamma_m \int_L \left\{ \alpha_L(\mathbf{y}) u_b(\mathbf{y}, -\widehat{\mathbf{x}}) + \beta_L(\mathbf{y}) \frac{1}{\mu} \frac{\partial u_b(\mathbf{y}, -\widehat{\mathbf{x}})}{\partial \boldsymbol{\nu}(\mathbf{y})} \right\} \, \mathrm{d}s(\mathbf{y}) \tag{3.71}$$

where $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. Then, we can prove the following

Lemma 3.4.2. *Let $L \subset \Gamma$ and $(\alpha_L, \beta_L) \in L^2(L) \times \widetilde{H}^1(L)$, not simultaneously zero. Then $L \subset \Gamma_0$ if and only if $\phi_L^\infty \in Range(\mathscr{G})$.*

Proof. Let's first assume that $L \subset \Gamma_0$. Then the corresponding extensions by zero in Γ_0, $(\widetilde{\alpha}_L, \widetilde{\beta}_L)$, are in $L^2(\Gamma_0) \times \widetilde{H}^1(\Gamma_0)$, and the potential

$$\phi_0(\mathbf{x}) := \int_{\Gamma_0} \left\{ \widetilde{\alpha}_L(\mathbf{y}) \mathbb{G}_b(\mathbf{x}, \mathbf{y}) + \widetilde{\beta_L}(\mathbf{y}) \frac{1}{\mu} \frac{\partial \mathbb{G}_b(\mathbf{x}, \mathbf{y})}{\partial \boldsymbol{\nu}(\mathbf{y})} \right\} \, \mathrm{d}s(\mathbf{y})$$

belongs to $H^1_{loc}(\mathbb{R}^m \backslash \overline{\Gamma}_0)$ and satisfies

$$[\phi_0] = \widetilde{\beta}_L, \quad \left[\frac{1}{\mu} \frac{\partial \phi_0}{\partial \boldsymbol{\nu}} \right] = -\widetilde{\alpha}_L \quad \text{on} \quad \Gamma_0. \tag{3.72}$$

Let's now denote by $\mathbf{S}_{\Gamma_0}$ and $\mathbf{K}_{\Gamma_0}$ the restriction to Γ_0 of the generalized single and double layer potentials, defined by

$$(\mathbf{S}_{\Gamma_0}\psi)(\mathbf{x}) := \int_{\Gamma_0} \psi(\mathbf{y}) \mathbb{G}_b(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma_0$$

and

$$(\mathbf{K}_{\Gamma_0}\psi)(\mathbf{x}) := \int_{\Gamma_0} \psi(\mathbf{y}) \frac{\partial}{\partial \boldsymbol{\nu}(\mathbf{y})} \mathbb{G}_b(\mathbf{x},\mathbf{y}) \ \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma_0.$$

In [21], it is shown that $\mathbf{S}_{\Gamma_0} : \tilde{H}^{-\frac{1}{2}+s}(\Gamma_0) \to H^{\frac{1}{2}+s}(\Gamma_0)$ and $\mathbf{K}_{\Gamma_0} : \tilde{H}^{\frac{1}{2}+s}(\Gamma_0) \to H^{\frac{1}{2}+s}(\Gamma_0)$ are continuous for every $-1 \leq s \leq 1$ (here $\widetilde{H}^r(\Gamma_0)$ denotes the space of functions that can be extended by zero to the whole Γ as functions in $H^r(\Gamma)$). Since, by the transmission conditions (3.72), we know that $\left[\frac{1}{\mu}\frac{\partial \phi_0}{\partial \boldsymbol{\nu}}\right] \in L^2(\Gamma_0)$ and $[\phi_0] \in \widetilde{H}^1(\Gamma_0)$, together with the fact that $\langle \phi_0 \rangle = -\mathbf{S}_{\Gamma_0}\left[\frac{1}{\mu}\frac{\partial \phi_0}{\partial \boldsymbol{\nu}}\right] + \mathbf{K}_{\Gamma_0}[\phi_0]$, we have that $\langle \phi_0 \rangle \in H^1(\Gamma_0)$ and hence the potential ϕ_0 belongs to $\mathcal{H}$. Therefore, ϕ_0 satisfies (3.44)-(3.47) with h_1 and h_2 defined by (3.48) for $v = -\phi_0 \in \mathcal{H}$, implying that $\mathscr{G}(h_1, h_2) = \phi_L^\infty$. To prove the converse, let's suppose that $L \not\subset \Gamma_0$ but that there exists a pair $(\alpha_L, \beta_L) \in L^2(L) \times \widetilde{H}^1(L)$, not simultaneously zero, such that $\phi_L^\infty \in Range(\mathscr{G})$. By definition of $\mathscr{G}$, there exists (h_1, h_2) in $H^{-1/2}(\Gamma_0) \times \mathcal{H}(\Gamma_0)$ such that $\phi_L^\infty = w^\infty$, where w satisfies (3.44)-(3.47). Therefore, ϕ_L^∞ is the far field pattern of the two potentials:

$$\phi_L(\mathbf{x}) = \gamma_m^{-1} \int_L \left\{ \alpha_L(\mathbf{y}) \mathbb{G}_b(\mathbf{x},\mathbf{y}) + \beta_L(\mathbf{y}) \frac{1}{\mu} \frac{\partial \mathbb{G}_b(\mathbf{x},\mathbf{y})}{\partial \boldsymbol{\nu}(\mathbf{y})} \right\} \ \mathrm{d}s(\mathbf{y})$$

and

$$w(\mathbf{x}) = \int_{\Gamma_0} \left\{ \left[\frac{1}{\mu}\frac{\partial w}{\partial \boldsymbol{\nu}(\mathbf{y})}\right](\mathbf{y}) \mathbb{G}_b(\mathbf{x},\mathbf{y}) + [w](\mathbf{y}) \frac{1}{\mu} \frac{\partial \mathbb{G}_b(\mathbf{x},\mathbf{y})}{\partial \boldsymbol{\nu}(\mathbf{y})} \right\} \ \mathrm{d}s(\mathbf{y}).$$

By Rellich's lemma, unique continuation, and Holmgren's theorem, $w = \phi_L$ identically in $\mathbb{R}^m \backslash \overline{\Gamma_0 \cup L}$. However this is a contradiction, because given any point $x_0 \in L \backslash \overline{\Gamma}_0$, both w and the co-normal derivative $\frac{1}{\mu}\frac{\partial w}{\partial \boldsymbol{\nu}_L}$ are continuous at x_0, whereas either ϕ_L or the co-normal derivative $\frac{1}{\mu}\frac{\partial \phi_L}{\partial \boldsymbol{\nu}_L}$ have a jump across L at x_0 (since either α_L or β_L doesn't vanish at that point). $\square$

Lemma 3.4.3. *Assume in addition to the assumptions of Theorem 3.3.1 that* $\Re(n - n_\delta) > 0$ *(or more generally there is no non-trivial* $u_{b,g}$ *such that* $Ku_{b,g} = 0$*). Then* $\mathcal{F}_D : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ *is injective and has dense range.*

Proof. Since $\mathcal{F}_D = \mathscr{G}\mathscr{H}$, the injectivity follows from Lemma 3.4.1 and the fact that the operator $\mathscr{G}$ is injective due to the well-posedness of (3.44)-(3.47). Next, since the range of $\mathscr{H}$ is dense in $H^{-1/2}(\Gamma_0) \times \mathcal{H}^{-1}(\Gamma_0)$ it suffices to show that the range of $\mathscr{G}$ is dense. From Lemma 3.4.2, in particular we have that functions $P\psi$ of the form

$$(P\psi)(\widehat{\mathbf{x}}) := \int_{\Gamma_0} \psi(\mathbf{y}) u_b(\mathbf{y}, -\widehat{\mathbf{x}})\, dy = \gamma_m^{-1} \int_{\Gamma_0} \psi(\mathbf{y}) \mathbb{G}_b^{\infty}(\widehat{\mathbf{x}}, \mathbf{y})\, dy$$

are in the range of $\mathscr{G}$ for all $\psi \in L^2(\Gamma_0)$. The set $\{P\psi \text{ for all } \psi \in L^2(\Gamma_0)\}$ is dense in $L^2(\mathbb{S}^{m-1})$. Indeed, let us consider $P : L^2(\Gamma_0) \to L^2(\mathbb{S}^{m-1})$. Its adjoint $P^* : L^2(\mathbb{S}^{m-1}) \to L^2(\Gamma_0)$ is given by

$$(P^* g)(\mathbf{y}) = \int_{\mathbb{S}^{m-1}} g(\widehat{\mathbf{x}}) \overline{u_b(\mathbf{y}, -\widehat{\mathbf{x}})} d\widehat{\mathbf{x}} = \overline{u_{b,h}}(\mathbf{y})$$

where $h(\widehat{\mathbf{x}}) := \overline{g}(-\widehat{\mathbf{x}})$ and $u_{b,h}$ is given by (3.65). Now the total field due to the background medium $u_{b,h}$ corresponding to the Herglotz wave function v_h as incident wave can not be zero unless $h = 0$, since the background problem is well posed under the assumptions of Theorem 3.3.1. This implies that P^* is injective which finishes the proof. $\square$

Now we are ready to characterize Γ_0 in terms of the behavior of the approximate solution to the *far-field equation*

$$\mathcal{F}_D g = \phi_L^{\infty}.$$

The following main theorem is a summary of the all the above results.

Theorem 3.4.2 (Linear Sampling Method)**.** *Let $\mathcal{F}_D : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ be the far field operator corresponding given by (3.66). Then:*

1. *For an arbitrary arc $L \subset \Gamma_0$ and $\epsilon > 0$, there exists a function $g_L^{\epsilon} \in L^2(\mathbb{S}^{m-1})$ such that*
$$\|\mathcal{F}_D g_L^{\epsilon} - \phi_{\infty}^{L}\|_{L^2(\mathbb{S}^{m-1})} < \epsilon,$$
and, as $\epsilon \to 0$, the corresponding solution $u_{b,g_L^{\epsilon}}$ to the background problem (3.52) converges in $\mathcal{H}$ to the unique solution u_L of (3.44)-(3.47) with $h_1 = \alpha \left\langle \frac{1}{\mu} \frac{\partial \phi_L^{\infty}}{\partial \boldsymbol{\nu}} \right\rangle$ and $h_2 = K \langle \phi_L^{\infty} \rangle$ on Γ_0.

2. *For $L \not\subset \Gamma_0$ and $\epsilon > 0$, every function $g_L^{\epsilon} \in L^2(\mathbb{S}^{m-1})$ such that*
$$\|\mathcal{F}_D g_L^{\epsilon} - \phi_{\infty}^{L}\|_{L^2(\mathbb{S}^{m-1})} < \epsilon,$$
is such that the corresponding solution $u_{b,g_L^{\epsilon}}$ to the background problem (3.52) satisfies
$$\lim_{\epsilon \to 0} \|u_{b,g_L^{\epsilon}}\|_{\mathcal{H}} = \infty \quad \text{and} \quad \lim_{\epsilon \to 0} \|g_L^{\epsilon}\|_{L^2(\mathbb{S}^{m-1})} = \infty.$$

This theorem constitutes the foundation of the linear sampling method which we will implement in the next section.

3.5 Numerical examples for the inverse problem

In this section we show how the linear sampling method that we have just developed can be applied numerically, and show its viability by some numerical examples. From the statement of Theorem 3.4.2, we know that the approximate solution of the far-field equation $\mathcal{F}_D\widetilde{g}_L = \phi_L^\infty$ can be used to detect the delaminated part Γ_0. Unfortunately, the far field equation is ill-posed since the far-field operator $\mathcal{F}_D$ is compact, and of course the discrete counterpart, $Ag_L = f_L$, will inherit the ill-posedness as ill-conditioning. Therefore, it has to be solved by means of a regularization method.

Let us first discuss the construction of the discrete far-field operator A and the right hand side f_L. In all the numerical examples that we present in this section, the discrete counterpart of the far-field operator is the matrix $A \in \mathbb{C}^{40\times 40}$, such that $A_{ij} = u^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{d}}_j) - u_b^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{d}}_j)$, where $u^\infty(\cdot, \widehat{\mathbf{d}}_j)$ and $u_b^\infty(\cdot, \widehat{\mathbf{d}}_j)$ are the the far-field pattern of the scattering problem with and without delamination, respectively, when the incident one is $u^{inc}(\mathbf{x}, \widehat{\mathbf{d}}_j) = e^{ik\mathbf{x}\cdot\widehat{\mathbf{d}}_j}$. Here we take $\widehat{\mathbf{d}}_j = (\cos(2\pi j/40), \sin(2\pi j/40))$, and $\widehat{\mathbf{x}}_i = (\cos(2\pi i/40), \sin(2\pi i/40))$, for $i, j = 0, 1, ..., 39$.

The far-field patterns for the approximate model given by (3.20)-(3.22), (3.23)-(3.25) and (3.9), are computed using a finite element based on the variational problem (3.36) implemented in FreeFem++ [51]. In all our simulations P_1 elements were used. The mesh refinement in FreeFem++ is imposed by specifying the number of nodes on the boundaries involved. In our examples, the number of nodes on the exterior boundary S_R was set as $p_{ext} = 40\pi R/\lambda$, in the boundary of the homogeneity Γ_1 as $N_{\Gamma_1} = 12\pi r_1/d_x$, and on the interface Γ as $N_\Gamma = 5\pi r_1/d_x$, where $\lambda = \frac{2\pi}{k}$ is the wavelength, $d_x = \lambda/20$ and r_1 is the charachteristic length of the inner layer Ω_-.

The Dirichlet-to-Neumann (DtN) map on the exterior boundary S_R was implemented by Nicolas Chaulet [29]. Both for the DtN and the far-field calculation, a truncated expression of their expansion in terms the first $2N+1$ Fourier basis elements

$\{e^{in\theta}\}_{n=-N}^{N}$ were considered, where $N = p_{ext}/10$.

In order to investigate the stability of the reconstruction method with respect to noise, we added some random noise to the computed far field for the approximate crack problem, so we actually consider $\widetilde{A}_{ij} = A_{ij}(1+\epsilon\zeta_{ij})$, where $\{\zeta_{ij}\}$ is a collection of independent random variables with uniform distribution over the interval $[-0.5, 0.5]$, and $\epsilon > 0$ is a constant chosen so that the relative noise $\rho := ||A - \widetilde{A}||_2/||A||_2$ attains the desired value. In each example ρ is computed and specified.

Since f_L is the discrete version of the right hand side of equation (3.73) and we have some freedom to choose the densities α_L and β_L, we decided to consider α_L as an approximation of $\delta_{\mathbf{z}}$ (where $\delta_{\mathbf{z}}$ is the Dirac delta located on $\mathbf{z} \in \Gamma$) and $\beta_L = 0$. Then, for a given finite set of *sample* points $\{\mathbf{z}_j\} \subset \Gamma$, our discrete right hand side simplifies to

$$(f_{\mathbf{z}_j})_k = u_b(\mathbf{z}_j, -\widehat{\mathbf{d}}_k).$$

Since Γ is already known there are many other possibilities for choosing the sampling arc L and test functions α_L, β_L but we have not tried them here. Nevertheless, as the numerical examples show, our choice give reasonable reconstructions. In all the numerical examples that we present, we chose a collection of equally distributed points along the interface Γ, $\{\mathbf{z}_k\}_{k=1}^{64}$. In order to "solve" each of the 64 ill-conditioned linear equations

$$\widetilde{A}_\rho g_k = f_{\mathbf{z}_k},$$

we use the well-known Tikhonov regularization method, that consists in solving the following minimization problems instead

$$g_k^{\lambda^*} = \text{argmin}_{g\in\mathbb{C}^{40}}\{||\widetilde{A}_\rho g - f_{\mathbf{z}_k}||^2 + \lambda^*||g||^2\},$$

where the regularization parameter was arbitrarily chosen as $\lambda^* = 10^{-10}$. The solution of these problems was made using the free Matlab package *regtools* (see [50]).

As stated in Theorem (3.4.2), the value of $||g_k^{\lambda^*}||^{-1}$ is large if $\mathbf{z}_k$ is in the crack support Γ_0, and small otherwise. Therefore, it can be used to identify the location of Γ_0. In the reconstructions that we present, we show results for four different noise levels ρ, in three different settings (a circle with one single crack, a kite with one single crack, and a kite with two cracks). For visualization purposes, in our reconstructions the separation of the dotted lines $\widetilde{\Gamma}_{\pm}$ is chosen to be proportional to $\Theta(\mathbf{z}_k) = ||g_k^{\lambda^*}||^{-1}$, with the parametrization:

$$\mathbf{x}_{\widetilde{\Gamma}_{\pm}}(t) = \mathbf{x}_{\Gamma}(t) \pm \eta_* \Theta(\mathbf{x}_{\Gamma}(t)) \boldsymbol{\nu}(t),$$

where $\mathbf{x}_{\Gamma}$ is the parametrization of Γ, and we arbitrarily set $\eta_* = 0.04$ as a constant that modulates the size of Θ for pure visualization purposes. The openings of the dotted lines $\widetilde{\Gamma}_{\pm}$ correspond, therefore, to the predicted location of the cracks by the linear sampling method just developed in section 3.4. All the numerical experiments presented here were made for layered obstacles with parameters $n_- = 4$, $n_+ = 2$, $\mu_- = \mu_+ = 1$, $\mu_\delta = 0.9, n_\delta = 0.2$, and wave number $k = 3$. Numerical examples

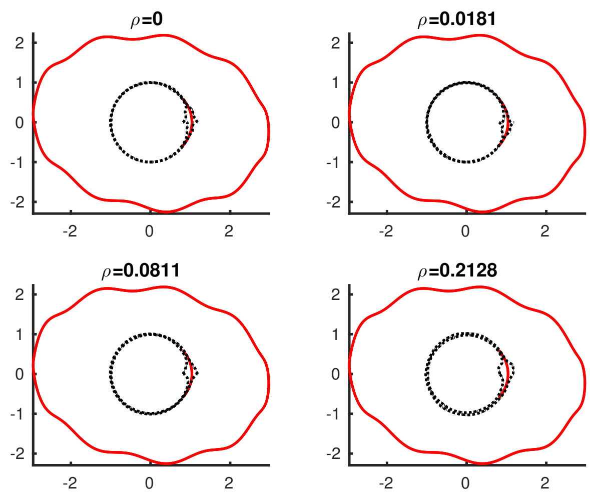


Figure 3.7: Reconstruction of a single crack Γ_0 in a circular interface, for four levels of noise ρ. The solid line at the circular interface is the exact location of the crack, and the opening between the dotted lines $\mathbf{x}_{\widetilde{\Gamma}_{\pm}}$ is the predicted location of Γ_0. The outer lighter coloured curve is Γ_1.

are presented in Figure 3.7, Figure 3.8 and Figure 3.9 indicate that our reconstruction method provides reasonable reconstructions of Γ_0 even in the presence of noise.

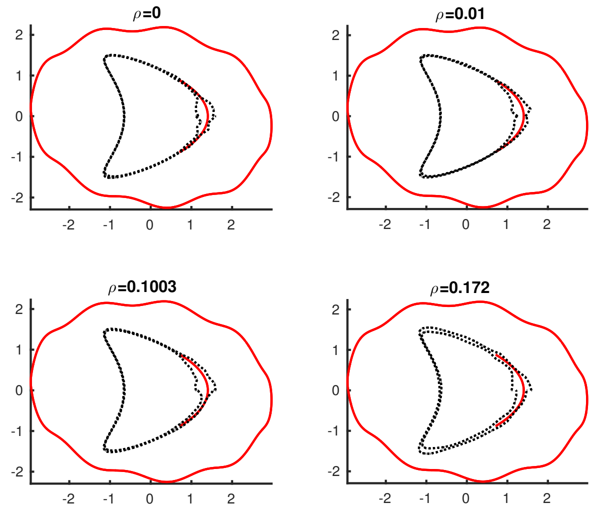


Figure 3.8: Reconstruction of a single crack Γ_0 in a kite-shaped interface of a two-layered media, for four levels of noise ρ. The solid line at the kite-shaped interface is the exact location of the crack, and the opening between the dotted lines $\mathbf{x}_{\widetilde{\Gamma}_\pm}$ is the predicted location of Γ_0. The outer lighter colored curve is Γ_1.

Conclusion

We have derived a asymptotic model for the delamination of a two materials that successfully approximates scattering from thin delaminated regions. This model was shown to be well-posed and was then used to derive a new inverse scattering technique based on a modified linear sampling method that we showed can detect delamination in model problems. The extension of these ideas to the 3D electromagnetic problem are considered in the next chapter.

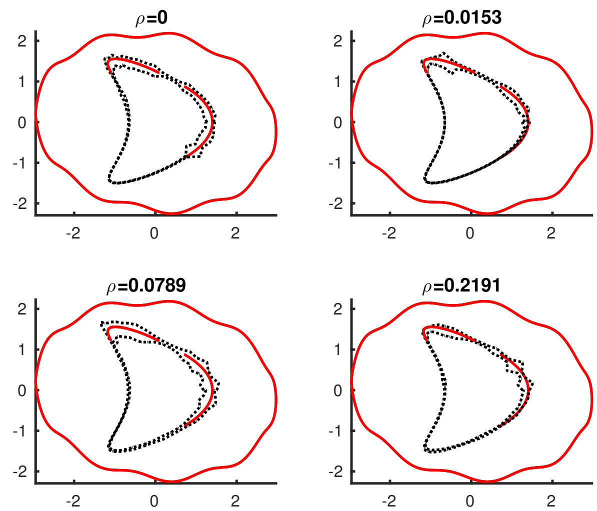


Figure 3.9: Reconstruction of two cracks $\Gamma_0^1 \cup \Gamma_0^2$ in a kite-shaped interface of a two-layered media for four levels of noise ρ. The solid line at the kite-shaped interface is the exact location of the crack, and the opening between the dotted lines $\mathbf{x}_{\widetilde{\Gamma}_\pm}$ is the predicted location of Γ_0. The outer lighter colored curve is Γ_1.

Chapter 4

NONDESTRUCTIVE TESTING OF THE DELAMINATED INTERFACE BETWEEN TWO MATERIALS: THE ELECTROMAGNETIC CASE

4.1 The problem

In this fourth chapter, we will turn our attention to the problem of detection of delamination, but in the context of electromagnetic inverse scattering (e.g. microwaves). The applications of this method would include, for example, the detection of debonding in integrated electric circuits [39, 53], and could potentially be used for the identification of thin biological tissues connected to early stages of cancer developement [43, 84].

The inherent technical difficulties associated with the analysis of Maxwell's equations have forced us to restrict ourselves to the specific case of the detection of planar delaminations of constant thickness.

We will study the scattering of an electromagnetic wave by a layered isotropic penetrable obstacle, $\Omega \subset \mathbb{R}^3$, that is schematically depicted in Figure 4.1. Being consistent with the notation of previous chapters, we denote by $\Gamma_1 = \partial\Omega$ the boundary of Ω, and by $\Omega_{ext} := \mathbb{R}^3 \setminus \overline{\Omega}$ the exterior domain. In what follows we assume that Γ is a smooth surface.

In the undamaged or *background* state, we consider Ω to be composed by two layers of different materials, Ω^b_- and Ω^b_+, where Ω^b_- is simply connected and Ω^b_+ is just connected. The boundary of Ω^b_-, denoted by Γ, is the common interface of the two layers Ω^b_- and Ω^b_+, and it is an orientable $\mathcal{C}^2$ regular surface (see panel (a) in Fig. 4.1).

In the damaged or *defective* state, the two layers have separated, and the thin delamination Ω_δ has appeared. The section $\Gamma_0 := \Gamma \cap \Omega_\delta$ is precisely where the original layers have separated. In this defective configuration $\Omega = int(\overline{\Omega_+ \cup \Omega_- \cup \Omega_\delta})$.

We will assume throughout this chapter that Γ_0 is an open surface with Lipschitz continuous relative boundary $\partial\Gamma_0$. It will also be assumed that Γ_0 is part of a planar section of Γ, and that Ω_δ is of constant thickness (see panel (b) in Fig. 4.1 and Fig. 4.2). In practice, this is unlikely to happen, because actually delaminations usually occur at bending interfaces, but it will constitute a first approach to the problem that we expect to generalize in future work.

Under these geometrical assumptions, Ω_δ has a cylindrical shape and its boundary, $\partial\Omega_\delta$, can be split into three components: the top and bottom surfaces Γ_+ and Γ_- parallel to Γ_0, and the side $\mathscr{S}$ (see Figs. 4.2 and 4.3). The four different domains,

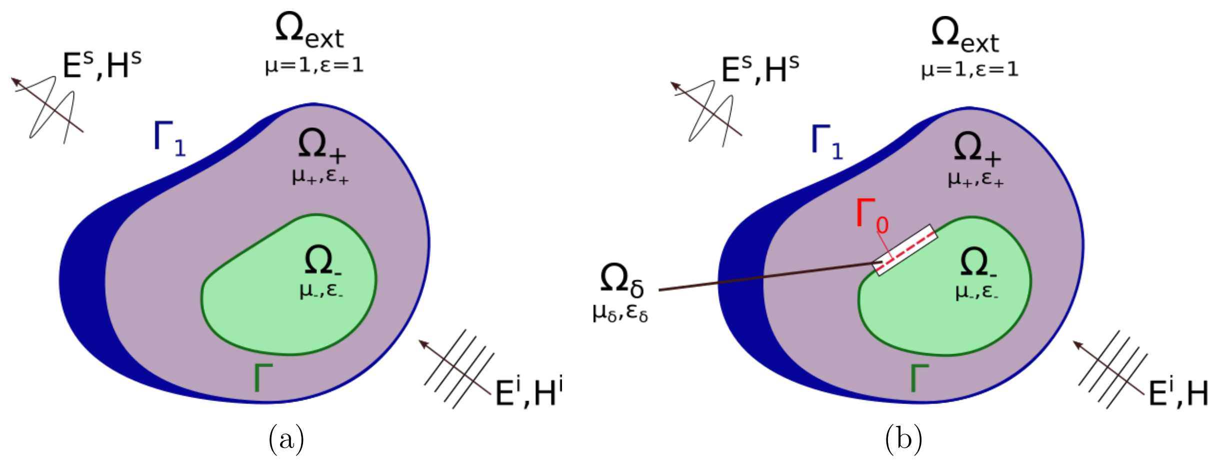


Figure 4.1: Panel (a) Cross section of the undamaged state. Panel (b) Cross section of the damaged or defective obstacle. The thin layer Ω_δ represents the delamination.

Ω_{ext}, Ω_+, Ω_- and Ω_δ, have different physical properties characterized by their electric permittivity and magnetic permeability. After normalizing with respect to the material properties of the homogeneous medium Ω_{ext}, these material properties are expressed in terms of the relative magnetic permeability μ and the relative electric permittivity ϵ, so that $\epsilon = \mu = 1$ in Ω_{ext}, and are assumed to be piece-wise continuous scalar fields in the other sub-domains, Ω_+, Ω_- and Ω_δ, which will respectively be denoted by:

$$\mu = \begin{cases} \mu_+ & \text{in} \quad \Omega_+ \\ \mu_- & \text{in} \quad \Omega_- \\ \mu_\delta & \text{in} \quad \Omega_\delta \end{cases} \quad \text{and} \quad \epsilon = \begin{cases} \epsilon_+ & \text{in} \quad \Omega_+ \\ \epsilon_- & \text{in} \quad \Omega_- \\ \epsilon_\delta & \text{in} \quad \Omega_\delta \end{cases}$$

Assumption 4.1.1. *Throughout this chapter we will asume the following properties:*

- *The functions $\mu : \mathbb{R}^3 \to \mathbb{R}$ and $\epsilon : \mathbb{R}^3 \to \mathbb{C}$ are piece-wise smooth functions in $\mathbb{R}^3$. Moreover, $\Re(\epsilon) > 0$, $\Im(\epsilon) \geq 0$, and $0 < \mu^{-1} < C$ for some constant $C > 0$.*
- *The material properties in the thin layer, μ_δ and ϵ_δ, are constant.*
- *There is an open neighborhood $\mathcal{N}$ of Ω_δ where the functions $\mu_\pm$ and $\epsilon_\pm$, are constant.*

Being consistent with the notation of the previous chapters, we denote by $\boldsymbol{\nu}$ the unit normal vector on Γ_1 pointing towards Ω_{ext}, and on $\Gamma \setminus \overline{\Gamma_0}$ towards Ω_+.

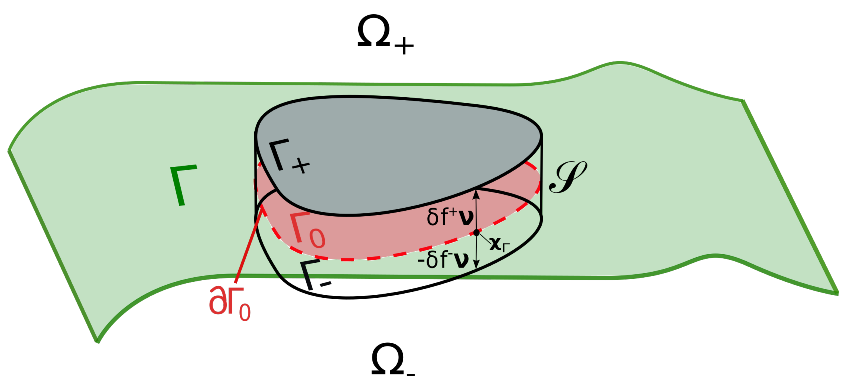


Figure 4.2: Zoom on the planar delamination. Panel (b) Normal vectors on the boundary of the delamination.

The equations that model the scattering of the total electromagnetic fields $(\mathbf{E}, \mathbf{H})$ in the frequency domain are given by

$$\nabla \times \mathbf{H} + ik\epsilon\mathbf{E} = \mathbf{0} \quad \text{in} \quad \Omega_\delta \cup \Omega_+ \cup \Omega_- \cup \Omega_{ext}, \tag{4.1}$$

$$\nabla \times \mathbf{E} - ik\mu\mathbf{H} = \mathbf{0} \quad \text{in} \quad \Omega_+ \cup \Omega_\delta \cup \Omega_- \cup \Omega_{ext}, \tag{4.2}$$

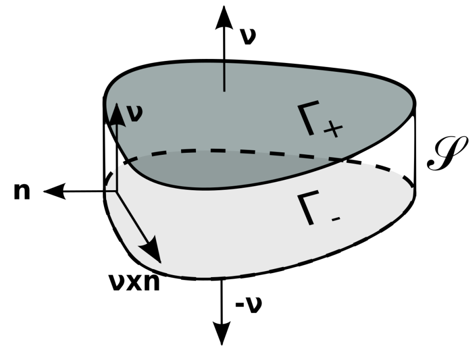


Figure 4.3: Normal vectors on the boundary of the delamination.

where both $\boldsymbol{\nu}\times\mathbf{H}$ and $\boldsymbol{\nu}\times\mathbf{E}$ are continuous across the interfaces Γ_1, $\Gamma_\pm$ $\mathscr{S}$, and $(\Gamma\backslash\overline{\Gamma}_0)$. In the unbounded domain Ω_{ext} the total fields can be decomposed as $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ and $\mathbf{H} = \mathbf{H}^s + \mathbf{H}^i$, where $(\mathbf{E}^i, \mathbf{H}^i)$ denotes the incident fields, and $(\mathbf{E}^s, \mathbf{H}^s)$ are the radiating fields that satisfy the Silver-Müller radiation condition:

$$\lim_{r\to\infty} r\left(\mathbf{H}^s \times \widehat{\mathbf{x}} - \mathbf{E}^s\right) = \mathbf{0}, \tag{4.3}$$

where $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$, $r = |\mathbf{x}|$, and the convergence is uniform in all directions $\widehat{\mathbf{x}} \in \mathbb{S}^2$. Existence of a unique solution for the full model is well known (cf. [65]).

4.2 The asymptotic model

Following the same ideas as in Chapter 3, we will substitute the full model (4.1)-(4.3), by an ATCs model, to avoid solving a differential equation in Ω_δ.
However, as opposed to Chapter 3, the second order ATCs crack-type model for electromagnetic scattering (Model I in Appendix 3) is unstable, as already discussed by Chun et al. in [30] for the time domain. This is reflected in the fact that in the frequency domain, the signs of the coefficients appearing in the corresponding ATCs model, are not compatible with the main operators.
Therefore, the ATCs model that we use in this analysis differs from the one presented in Chapter 3, in the fact that the jumps and average values of the fields are taken with

respect to traces of the fields on the two different surfaces Γ_- and Γ_+. This model corresponds to the set of ATCs in Corollary C.3.1. This set of ATCs are called sometimes Chun's-type ATCs [30, 40] and were derived in a much more general setting in Proposition C.3.1, and are similar to the models analyzed in [30] and in [38].
Although the fully detailed derivation of the ATCs model is in Appendix C, we sketch here the main ideas for the reader's convenience.

4.2.1 Elements of Differential Geometry

In order to establish the asymptotic model, we introduce some notation and concepts from basic differential geometry. The notation is consistent with the more general configuration explained in detail in Appendix C, and is based on [47].
As mentioned before, $\boldsymbol{\nu}$ denotes the constant unit normal vector defined on the smooth surface Γ, pointing into Ω_+^b. For each $\mathbf{x}_\Gamma \in \Gamma$ there exists a local parametrization $\boldsymbol{\xi} = (\xi_1, \xi_2) \mapsto \mathbf{x}_\Gamma$ which without loss of generality we assume induces a positive orientation of Γ, consistent with $\boldsymbol{\nu}$.
Let $0 < \eta_*$ be a real number such that in the open neighborhood of Γ given by

$$\mathcal{N} := \{\mathbf{x} \in \mathbb{R}^3 \mid \min_{\mathbf{y}\in\Gamma} |\mathbf{x} - \mathbf{y}| < \eta_*\}, \tag{4.4}$$

the mapping

$$(\mathbf{x}_\Gamma, \eta) \mapsto \mathbf{x} = \mathbf{x}_\Gamma + \eta\boldsymbol{\nu}(\mathbf{x}_\Gamma), \tag{4.5}$$

is an isomorphism. Moreover, define the vector field $\breve{\boldsymbol{\nu}}$ in $\mathcal{N}$ by

$$\breve{\boldsymbol{\nu}}(\mathbf{x}_\Gamma + s\boldsymbol{\nu}) := \boldsymbol{\nu}(\mathbf{x}_\Gamma), \text{ for all } \mathbf{x}_\Gamma \in \Gamma \text{ and } |\eta| < \eta_*,$$

then the *curvature tensor* defined by $\mathcal{C}_{\mathbf{x}_\Gamma} := \nabla_\Gamma \breve{\boldsymbol{\nu}}(\mathbf{x}_\Gamma)$ is identically zero for all $\mathbf{x}_\Gamma$ on Γ_0. The tangential vectors $\{\boldsymbol{\tau}_\alpha := \partial_{\xi_\alpha}\mathbf{x}_\Gamma\}_{\alpha=1,2}$ are called the *covariant* basis of the tangent plane $T_{\mathbf{x}_\Gamma}$ to Γ at $\mathbf{x}_\Gamma$.

Let $\mathbf{v}$ is a $\mathcal{C}^\infty(\Gamma_0)^3$ vector field, then we define the tangential and normal projection of $\mathbf{v}$, respectively, by:

$$\Pi_{||}\mathbf{v} = \boldsymbol{\nu} \times (\mathbf{v} \times \boldsymbol{\nu}) \quad \text{and} \quad \Pi_N\mathbf{v} = \boldsymbol{\nu} \cdot \mathbf{v}. \tag{4.6}$$

Using compact notation, the tangential and normal projections on Γ_0 will be denoted by $\mathbf{v}_T := (\boldsymbol{\nu}\times\mathbf{v})\times\boldsymbol{\nu}$ and $\mathbf{v}_N := \boldsymbol{\nu}\cdot\mathbf{v}$, respectively. Analogously, for the parallel surfaces $\Gamma_\pm$ to Γ_0, if $\mathbf{v}^\pm$ are in $\mathcal{C}^\infty(\Gamma_\pm)^3$, then we write for short their respective tangential and normal projections on $\Gamma_\pm$ as: $\mathbf{v}_T^\pm := (\boldsymbol{\nu}\times\mathbf{v}^\pm)\times\boldsymbol{\nu}$ and $\mathbf{v}_N^\pm := \boldsymbol{\nu}\cdot\mathbf{v}^\pm$.

Remark 4.2.1. *It is well known that the projections $\Pi_{||}$ and Π_N defined by (4.6) have continuous extensions $\gamma_T^\pm : \mathbf{H}(curl, \mathcal{N}\cap\Omega_\pm) \to \mathbf{H}^{-1/2}(curl_\Gamma, \Gamma_0)$ and $\gamma_N^\pm : \mathbf{H}(div, \mathcal{N}\cap\Omega_\pm) \to H^{-1/2}(\Gamma_0)$, respectively (see [65]).*

Surface differential operators.

1. Given a scalar field u defined on Γ_0, one can compute its surface gradient defined by

$$\nabla_\Gamma u(\mathbf{x}_\Gamma) := \nabla\breve{u}(\mathbf{x}_\Gamma),$$

where the scalar field $\breve{u} : \mathcal{N} \to \mathbb{C}$ is defined by $\breve{u}(\mathbf{x}_\Gamma + \eta\boldsymbol{\nu}(\mathbf{x}_\Gamma)) := u(\mathbf{x}_\Gamma)$. In terms of the covariant basis $\{\boldsymbol{\tau}_\alpha\}$, it can be written as $\nabla_\Gamma = (\partial_{\xi_1}\cdot)\boldsymbol{\tau}_1 + (\partial_{\xi_2}\cdot)\boldsymbol{\tau}_2$.

2. By definition, the adjoint operator of ∇_Γ is $-div_\Gamma$, which for all smooth vector fields $\mathbf{v}$ defined on Γ_0 satisfies $div_\Gamma\mathbf{v} = \partial_{\xi_1}(\mathbf{v}\cdot\boldsymbol{\tau}_1) + \partial_{\xi_2}(\mathbf{v}\cdot\boldsymbol{\tau}_2)$.

3. Important surface differential operators for the upcoming analysis, will be the scalar and vectorial surface curl operators, respectively denoted by $curl_\Gamma$ and $\overrightarrow{curl}_\Gamma$, defined as follows: given a smooth tangential vector $\boldsymbol{\eta} = \eta_1\boldsymbol{\tau}_1 + \eta_2\boldsymbol{\tau}_2 \in (\mathcal{C}^\infty(\Gamma_0))^3$ and a smooth scalar field $\rho \in \mathcal{C}^\infty(\Gamma_0)$,

$$curl_\Gamma\boldsymbol{\eta} := \partial_1\eta_2 - \partial_2\eta_1 \quad \text{and} \quad \overrightarrow{curl}_\Gamma\rho := \partial_2\rho\boldsymbol{\tau}_1 - \partial_1\rho\boldsymbol{\tau}_2. \tag{4.7}$$

4. In [47] it is shown that if $\mathbf{v}$ is a smooth enough vector field in $\mathbb{R}^3$ the differential operator curl can be expressed in terms of the curvilinear coordinates $(\mathbf{x}_\Gamma, \eta)$ on the planar surface as

$$\nabla\times\mathbf{v} = (curl_\Gamma\mathbf{v}_T)\,\boldsymbol{\nu} + \overrightarrow{curl}_\Gamma\mathbf{v}_N + \boldsymbol{\nu}\times\partial_\eta\mathbf{v}. \tag{4.8}$$

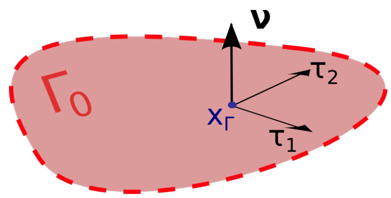


Figure 4.4: The reference coordinates $\mathbf{x}_\Gamma$ defined on Γ.

4.2.1.1 The formal asymptotic analysis

In order to establish the setting for the asymptotic analysis, we will assume that the delamination is thin enough so that $\overline{\Omega}_\delta \subset \mathcal{N}$ (where $\mathcal{N}$ is defined by (4.4)), then, as shown in Fig. 4.2, the two boundaries $\Gamma_\pm$ of Ω_δ can be written in our new curvilinear coordinates as follows:

$$\Gamma_\pm := \left\{ \mathbf{x}_{\Gamma_\pm} = \mathbf{x}_\Gamma \pm \delta f^\pm \boldsymbol{\nu} \, : \, \mathbf{x}_\Gamma \in \Gamma_0 \right\}, \tag{4.9}$$

where $0 < \delta \ll 1$ is the thickness of the delamination and $f^+, f^- \geq 0$ are constants such that $f^+ + f^- = 1$.

4.2.1.2 The ansatz for the outer and inner fields

If the parameter δ is small enough, then we formally assume that the following asymptotic expansions of the fields are valid in a neighborhood $\mathcal{N}_0$ of Ω_δ such that $\overline{\mathcal{N}}_0 \subset \mathcal{N}$:

$$(\mathbf{E}^\pm(\mathbf{x}_\Gamma, \eta), \mathbf{H}^\pm(\mathbf{x}_\Gamma, \eta)) = \sum_{l=0}^{\infty} \delta^l (\mathbf{E}_l^\pm(\mathbf{x}_\Gamma, \eta), \mathbf{H}_l^\pm(\mathbf{x}_\Gamma, \eta)) \quad \text{in } \Omega^\pm, \tag{4.10}$$

where each term $(\mathbf{E}_l^\pm(\mathbf{x}_\Gamma, \eta), \mathbf{H}_l^\pm(\mathbf{x}_\Gamma, \eta))$ in the asymptotic expansion is assumed to be analytic and independent of δ, for all $l \geq 0$.

The ansatz for the asymptotic expansion inside the delamination Ω_δ is slightly different because here Ω_δ plays the role of a boundary layer and we expect rapid changes on the fields. Thus we will regularize the singular asymptotic problem by considering the usual stretching of the normal variable $\zeta = \frac{\eta}{\delta}$ (see for example [10],[47],[71]), and this leads to

$$(\mathbf{E}^\delta, \mathbf{H}^\delta)(\mathbf{x}_\Gamma, \zeta) = \sum_{l=0}^{\infty} \delta^l (\mathbf{E}_l(\mathbf{x}_\Gamma, \zeta), \mathbf{H}_l(\mathbf{x}_\Gamma, \zeta)) \quad \text{in } \Omega_\delta, \tag{4.11}$$

where, again, none of the terms $(\mathbf{E}_l, \mathbf{H}_l)$, $l \geq 0$, depend on δ.

4.2.1.3 The approximate transmission conditions (ATCs)

In order to state the new model we need to introduce some more notation. Given a function u (either scalar or vectorial), such that $u = u^+$ in Ω_+ and $u = u_-$ in Ω_-, whose traces are well-defined on Γ_+ and Γ_-, then we denote the jump and average value of u by:

$$[\![u]\!] = u^+|_{\Gamma_+} - u^-|_{\Gamma_-} \text{ and } \langle\!\langle u \rangle\!\rangle = \frac{1}{2}(u^+|_{\Gamma_+} + u^-|_{\Gamma_-}). \tag{4.12}$$

Using this notation, from the asymptotic expansion (4.10) the tangential traces of our fields satisfy (at least formally) the following *exact* transmission conditions:

$$[\![\boldsymbol{\nu} \times \mathbf{E}]\!](\mathbf{x}_\Gamma) = \sum_{j=0}^{\infty} \delta^j [\![\boldsymbol{\nu} \times \mathbf{E}_j]\!](\mathbf{x}_\Gamma), \tag{4.13}$$

$$[\![\boldsymbol{\nu} \times \mathbf{H}]\!](\mathbf{x}_\Gamma) = \sum_{j=0}^{\infty} \delta^j [\![\boldsymbol{\nu} \times \mathbf{H}_j]\!](\mathbf{x}_\Gamma), \tag{4.14}$$

for all $\mathbf{x}_\Gamma \in \Gamma_0$.

For any $n \geq 0$, the n-th order ATCs associated with this problem are defined by the transmission conditions that we obtain after dropping the $O(\delta^{n+1})$ terms (see [71],[49]):

$$[\![\boldsymbol{\nu} \times \mathbf{E}]\!](\mathbf{x}_\Gamma) = \sum_{j=0}^{n} \delta^j [\![\boldsymbol{\nu} \times \mathbf{E}_j]\!](\mathbf{x}_\Gamma), \tag{4.15}$$

$$[\![\boldsymbol{\nu} \times \mathbf{H}]\!](\mathbf{x}_\Gamma) = \sum_{j=0}^{n} \delta^j [\![\boldsymbol{\nu} \times \mathbf{H}_j]\!](\mathbf{x}_\Gamma), \tag{4.16}$$

for all $\mathbf{x}_\Gamma \in \Gamma_0$. In the problem setting that we are currently analyzing, it can be shown after some calculations (see Proposition C.3.1), that under the hypothesis of a planar delamination of constant thickness, the second order ATCs are:

$$[\![\boldsymbol{\nu} \times \mathbf{E}]\!] = \widetilde{\mathcal{A}}_1 \langle\!\langle \mathbf{H}_T \rangle\!\rangle \quad \text{and} \quad [\![\boldsymbol{\nu} \times \mathbf{H}]\!] = \widetilde{\mathcal{A}}_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle, \tag{4.17}$$

where

$$\widetilde{\mathcal{A}}_1 \langle\!\langle \mathbf{H}_T \rangle\!\rangle = \delta\widetilde{\alpha}_1 \langle\!\langle \mathbf{H}_T \rangle\!\rangle + \delta\widetilde{\beta}_1 \overrightarrow{curl}_\Gamma \left(curl_\Gamma \langle\!\langle \mathbf{H}_T \rangle\!\rangle \right) \quad \text{on} \quad \Gamma_0, \tag{4.18}$$

$$\widetilde{\mathcal{A}}_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle = \delta\widetilde{\alpha}_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle + \delta\widetilde{\beta}_2 \overrightarrow{curl}_\Gamma \left(curl_\Gamma \langle\!\langle \mathbf{E}_T \rangle\!\rangle \right) \quad \text{on} \quad \Gamma_0, \tag{4.19}$$

and $\widetilde{\alpha}_1 = 2ik\mu_\delta$, $\widetilde{\alpha}_2 = -2ik\epsilon_\delta$, $\widetilde{\beta}_1 = \frac{2}{ik\epsilon_\delta}$, and $\widetilde{\beta}_2 = -\frac{2}{ik\mu_\delta}$.

Therefore the second order ATCs model that we will study in this chapter consists of equations (4.1) and (4.2) in the domains $\Omega_- \cup \Omega_+ \cup \Omega_{ext}$, and the transmission conditions defined by (4.17).

In terms of only the electric field $\mathbf{E}$, by equation (4.1), the ATCs model gives rise to the problem: Seek the field $\mathbf{E} \in \mathbf{H}(curl, \mathbb{R}^3 \setminus \overline{\Omega}_\delta)$ that satisfies

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{E} \right) - k^2 \epsilon \mathbf{E} = \mathbf{0} \quad \text{in} \quad \Omega_+ \cup \Omega_- \cup \Omega_{ext}, \tag{4.20}$$

$$[\![\boldsymbol{\nu} \times \mathbf{E}]\!] = \delta\alpha_1 \langle\!\langle \left(\mu^{-1} \nabla \times \mathbf{E} \right)_T \rangle\!\rangle - \delta\beta_1 \overrightarrow{curl}_\Gamma curl_\Gamma \langle\!\langle \left(\mu^{-1} \nabla \times \mathbf{E} \right)_T \rangle\!\rangle \text{ on } \Gamma_0, \tag{4.21}$$

$$[\![\boldsymbol{\nu} \times \left(\mu^{-1} \nabla \times \mathbf{E} \right)]\!] = \delta\alpha_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle - \delta\beta_2 \overrightarrow{curl}_\Gamma \; curl_\Gamma \langle\!\langle \mathbf{E}_T \rangle\!\rangle \text{ on } \Gamma_0, \tag{4.22}$$

$$\mathbf{n} \times \left(\mu^{-1} \nabla \times \mathbf{E} \right) = \mathbf{0} \text{ on } \mathscr{S}, \tag{4.23}$$

where the coefficients appearing in the transmission conditions have the expressions $\alpha_1 = 2\mu_\delta$, $\alpha_2 = 2k^2\epsilon_\delta$, $\beta_1 = \frac{2}{k^2\epsilon_\delta}$, and $\beta_2 = \frac{2}{\mu_\delta}$, and where, in Ω_{ext}, $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ is the total field, $\mathbf{E}^i$ is an incident field, and $\mathbf{E}^s$ satisfies the Silver-Müller radiation condition:

$$\lim_{r \to \infty} r \left((\nabla \times \mathbf{E}^s) \times \widehat{\mathbf{x}} - ik\mathbf{E}^s \right) = \mathbf{0}, \tag{4.24}$$

where $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$ and $r = |\mathbf{x}|$.

4.3 The variational formulation

4.3.1 The boundary operators

As usual, in order to place our problem within a Fredholm operator framework, we will work in a bounded domain. Let B_R be an arbitrary ball of radius $R > 0$ that contains the obstacle $\overline{\Omega}$, and denote by S_R its boundary. Then, muliplying equation (4.20) by a test function $\mathbf{v} \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ and integrating by parts in B_R, we get the following expression:

$$\begin{aligned}
&\int_{B_R} \mu^{-1}\nabla \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{v}} - k^2 \epsilon \mathbf{E} \cdot \overline{\mathbf{v}} \ \mathrm{d}\,\mathbf{y} \\
+ \ &\int_{\Gamma_0} \langle\!\langle \left(\mu^{-1}\nabla \times \mathbf{E}\right)_T \rangle\!\rangle \cdot \overline{[\![\boldsymbol{\nu} \times \mathbf{v}]\!]} \ \mathrm{d}s(\mathbf{y}) - \int_{\Gamma_0} [\![\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{E}\right)]\!] \cdot \overline{\langle\!\langle \mathbf{v} \rangle\!\rangle_T} \ \mathrm{d}s(\mathbf{y}) \\
+ \ & ik\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}), \mathbf{v}_T\rangle_{S_R} = -\int_{S_R} (\widehat{\mathbf{x}} \times \mathbf{E}^i) \cdot \overline{\mathbf{v}} \ \mathrm{d}s(\mathbf{y}) \\
+ \ & ik\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}^i), \mathbf{v}_T\rangle_{S_R}, \qquad (4.25)
\end{aligned}$$

where $\langle \cdot, \cdot \rangle_{S_R}$ is the duality pairing between $\mathbf{H}^{-1/2}(div_{S_R}, S_R)$ and $\mathbf{H}^{-1/2}(curl_{S_R}, S_R)$ (and that, by pivoting with $\mathbf{L}^2_t(S_R)$, can be substituted by the usual $\mathbf{L}^2_t(S_R)$-inner product), and $G_e : \mathbf{H}^{-1/2}(div_{S_R}, S_R) \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$ is the well-known exterior electric-to-magnetic Calderón operator (see [65],[32]), defined by $G_e(\boldsymbol{\lambda}) = \widehat{\mathbf{x}} \times \mathbf{H}^s$, where $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy

$$\begin{aligned}
ik\mathbf{E}^s + \nabla \times \mathbf{H}^s &= \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \\
ik\mathbf{H}^s - \nabla \times \mathbf{E}^s &= \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \\
\widehat{\mathbf{x}} \times \mathbf{E}^s &= \boldsymbol{\lambda} \quad \text{on } S_R, \\
\lim_{r\to\infty} r(\mathbf{H}^s \times \widehat{\mathbf{x}} - \mathbf{E}^s) &= \mathbf{0}, \qquad (4.26)
\end{aligned}$$

where again $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$ and $r = |\mathbf{x}|$.

4.3.2 The ATCs operators

In order to use the ATCs (4.21)-(4.22) in (4.25), we have to, first of all, invert the transmission condition (4.21) on Γ_0. To this end, it will be useful to prove Lemma

4.3.1 and to recall Lemma 4.3.2, whose proof can be found in [3].

Denote by $\widehat{\mathbf{n}}$ the unit normal vector defined on $\partial\Omega_\delta$, and pointing to the exterior of Ω_δ. Therefore (see Figure 4.3):

$$\widehat{\mathbf{n}} = \begin{cases} \boldsymbol{\nu} & \text{on } \Gamma_+, \\ \mathbf{n} & \text{on } \mathscr{S}, \\ -\boldsymbol{\nu} & \text{on } \Gamma_-. \end{cases}$$

Define in addition

$$\begin{aligned} \widetilde{H}^{-1/2}(\Gamma_\pm) \;:=\; & \Big\{ \mathbf{v} \in H^{-1/2}(\Gamma_\pm) \,|\, \widetilde{\mathbf{v}} \in H^{-1/2}(\partial\Omega_0), \\ & \text{where } \widetilde{\mathbf{v}} = \mathbf{v} \text{ in } \Gamma_\pm, \\ & \text{and } \widetilde{\mathbf{v}} = \mathbf{0} \text{ in } \partial\Omega_0 \setminus \overline{\Gamma_\pm} \Big\}, \end{aligned} \tag{4.27}$$

endowed with the $H^{-1/2}(\Gamma_\pm)$ norm.

Analogously, we define $\widetilde{H}^{-1/2}(\Gamma_0)$ (see also Appendix D), as follows:

$$\begin{aligned} \widetilde{H}^{-1/2}(\Gamma_0) \;:=\; & \Big\{ \mathbf{v} \in H^{-1/2}(\Gamma_0) \,|\, \widetilde{\mathbf{v}} \in H^{-1/2}(\Gamma), \\ & \text{where } \widetilde{\mathbf{v}} = \mathbf{v} \text{ in } \Gamma_0, \\ & \text{and } \widetilde{\mathbf{v}} = \mathbf{0} \text{ in } \Gamma \setminus \overline{\Gamma_0} \Big\}, \end{aligned} \tag{4.28}$$

endowed with the $H^{-1/2}(\Gamma_0)$ norm.

Using this notation we have the following result.

Lemma 4.3.1. *Let $\mathbf{u} \in \mathbf{H}(curl, \Omega \setminus \overline{\Omega}_\delta)$. Then,*

1. $\widehat{\mathbf{n}} \cdot \nabla \times \mathbf{u}|_{\partial\Omega_\delta} = curl_{\partial\Omega_\delta}(\widehat{\mathbf{n}} \times (\mathbf{u} \times \widehat{\mathbf{n}}))$ *in* $H^{-1/2}(\partial\Omega_\delta)$.
2. *If* $\mathbf{u}$ *is such that* $\mathbf{n} \times (\mu^{-1}\nabla \times \mathbf{u})|_{\mathscr{S}} = \mathbf{0}$*, then*

$$\boldsymbol{\nu} \cdot \nabla \times \mathbf{u}|_{\Gamma_\pm} = curl_\Gamma \mathbf{u}|_{\Gamma_\pm} \in \widetilde{H}^{-1/2}(\Gamma_\pm),$$

and therefore $curl_\Gamma \langle\!\langle \mathbf{u} \rangle\!\rangle_T \in \widetilde{H}^{-1/2}(\Gamma_0)$.

Proof. To see 1., notice that $\nabla \times \mathbf{u} \in \mathbf{H}(div, \Omega \setminus \overline{\Omega}_\delta)$ and thus $\widehat{\mathbf{n}} \cdot \nabla \times \mathbf{u} \in \mathbf{H}^{-1/2}(\partial\Omega_\delta)$. Thus given $\xi \in H^1(\Omega_\delta)$ whose support does not intersect Γ_1, using Green's identities,

$$\langle \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{u},\, \xi \rangle_{H^{-1}(\Omega_\delta), H^1(\Omega_\delta)} \;=\; \langle \nabla \times \mathbf{u},\, \nabla(\chi\boldsymbol{\psi}) \rangle_{H^{-1/2}(\partial\Omega_\delta), H^{1/2}(\partial\Omega_\delta)} \tag{4.29}$$

where $\boldsymbol{\psi} \in H^{3/2}_{loc}(\mathbb{R}^3 \setminus \overline{\Omega}_\delta)$ is a lifting of ξ, and $\chi \in \mathcal{C}^\infty$ is a cut-off function such that $\chi = 1$ is a neighborhood of Ω_δ and its support is compact.
Thus, pivoting with $L^2(\Omega \setminus \overline{\Omega}_\delta)^3$ and using Stokes formulas,

$$\begin{aligned} \langle \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{u},\, \xi \rangle_{H^{-1}(\Omega\setminus\overline{\Omega}_\delta), H^1(\Omega\setminus\overline{\Omega}_\delta)} &= \langle \widehat{\mathbf{n}} \times \mathbf{u},\, \nabla_{\partial\Omega_\delta}\xi \rangle_{L^2(\partial\Omega_\delta)^3} && (4.30)\\ &= \langle \mathbf{u},\, \nabla_{\partial\Omega_\delta}\xi \times \widehat{\mathbf{n}} \rangle_{L^2(\partial\Omega_\delta)^3}, && (4.31) \end{aligned}$$

and since $\nabla_{\partial\Omega_\delta}\xi \times \widehat{\mathbf{n}} = \overrightarrow{curl}_{\partial\Omega_\delta}\xi$ on , we get that $\widehat{\mathbf{n}} \cdot \nabla \times \mathbf{u} = curl_{\partial\Omega_\delta}(\widehat{\mathbf{n}} \times (\mathbf{u} \times \widehat{\mathbf{n}}))$ in $\mathbf{H}^{-1/2}(\partial\Omega_\delta)$.
To see 2., suppose that in addition $\mathbf{n} \times (\mu^{-1}\nabla \times \mathbf{u})|_{\mathscr{S}} = \mathbf{0}$. This in particular implies that $\boldsymbol{\nu}$ (which is normal to the planar faces $\Gamma_\pm$) satisfies $\boldsymbol{\nu} \cdot \nabla \times \mathbf{u}|_{\mathscr{S}} = \mathbf{0}$.
Therefore $\boldsymbol{\nu} \times (\nabla \times \mathbf{u}) \in \mathbf{H}^{-1/2}(\partial\Omega_\delta))$ is well defined and its restrictions to the planar sides satisfy $curl_{\partial\Omega_\delta}\mathbf{u}|_{\Gamma_\pm} = \pm curl_{\Gamma_\pm}\mathbf{u} \in \widetilde{H}^{-1/2}(\Gamma_\pm)$. Moreover, $curl_\Gamma \langle\!\langle \mathbf{u} \rangle\!\rangle_T \in \widetilde{H}^{-1/2}(\Gamma_0)$. □

Given the surface differential operator $D_{\Gamma_\pm} = curl_{\Gamma_\pm}$ or $D_{\Gamma_\pm} = div_{\Gamma_\pm}$, we define the spaces

$$\widetilde{H}^{-1/2}(D_{\Gamma_\pm}, \Gamma_\pm) := \Big\{ \mathbf{u} \in \widetilde{H}^{-1/2}(\Gamma_\pm) \,|\, D_{\Gamma_\pm}\mathbf{u} \in \widetilde{H}^{-1/2}(\Gamma_\pm) \Big\}, \tag{4.32}$$

endowed with the graph norm

$$\|\mathbf{u}\|^2_{\widetilde{H}^{-1/2}(D_{\Gamma_\pm},\Gamma_\pm)} = \|\mathbf{u}\|^2_{\widetilde{H}^{-1/2}(\Gamma_\pm)} + \left\| D_{\Gamma_\pm}\mathbf{u} \right\|^2_{\widetilde{H}^{-1/2}(\Gamma_\pm)}. \tag{4.33}$$

The following result is an immediate consequence of Lemma 2.2 in [3]:

Lemma 4.3.2. *Let* $\mathbf{E}$ *be a solution to (4.20)-(4.23). Then*

$$\begin{aligned} (\mu_\pm^{-1}\nabla \times \mathbf{E})_T|_{\Gamma_\pm} &\in \widetilde{\mathbf{H}}^{-1/2}(curl_{\Gamma_\pm}, \Gamma_\pm) \quad \text{and}\\ \boldsymbol{\nu} \times (\mu_\pm^{-1}\nabla \times \mathbf{E})|_{\Gamma_\pm} &\in \widetilde{\mathbf{H}}^{-1/2}(div_{\Gamma_\pm}, \Gamma_\pm). \end{aligned} \tag{4.34}$$

For our analysis, define the space

$$\mathcal{H}_0(\Gamma_0) := \widetilde{\mathbf{H}}^{-1/2}(curl_\Gamma, \Gamma_0) \cap \mathbf{H}(curl_\Gamma, \Gamma_0), \tag{4.35}$$

with the norm

$$\|\mathbf{u}\|^2_{\mathbf{H}(curl_\Gamma,\Gamma_0)} = \|\mathbf{u}\|^2_{L^2(\Gamma_0)} + \|curl_\Gamma \mathbf{u}\|^2_{L^2(\Gamma_0)}\,. \tag{4.36}$$

Remark 4.3.1. It is immediate that $(\mathcal{H}_0(\Gamma_0), \|\cdot\|_{\mathcal{H}_0(\Gamma_0)})$ is a Hilbert space, with $\|\cdot\|_{\mathcal{H}_0(\Gamma_0)}$ corresponds to the graph norm:

$$\|\mathbf{u}\|_{\mathcal{H}_0(\Gamma_0)} := \Big(\|\mathbf{u}\|^2_{H^{-1/2}(\Gamma_0)} + \|curl_\Gamma \mathbf{u}\|^2_{H^{-1/2}(\Gamma_0)} + \|\mathbf{u}\|^2_{L^2(\Gamma_0)} + \|curl_\Gamma \mathbf{u}\|^2_{L^2(\Gamma_0)} \Big)^{1/2}.$$

But since for every $\mathbf{u} \in \mathcal{H}_0(\Gamma_0)$,

$$\begin{aligned}\|\mathbf{u}\|^2_{H^{-1/2}(\Gamma_0)} \;\; &+ \;\; \|curl_\Gamma \mathbf{u}\|^2_{H^{-1/2}(\Gamma_0)} + \|\mathbf{u}\|^2_{L^2(\Gamma_0)} + \|curl_\Gamma \mathbf{u}\|^2_{L^2(\Gamma_0)} \\ &\leq (C+1)(\|\mathbf{u}\|^2_{L^2(\Gamma_0)} + \|curl_\Gamma \mathbf{u}\|^2_{L^2(\Gamma_0)}),\end{aligned}$$

where $C > 0$ is the norm of the embedding $L^2(\Gamma_0) \subset H^{-1/2}(\Gamma_0)$, we know also that $(\mathcal{H}_0(\Gamma_0), \|\cdot\|_{\mathbf{H}(curl_\Gamma,\Gamma_0)})$ is a Hilbert space.
Moreover, denote by $\mathcal{H}_0(\Gamma_0)^*$ the dual space of $\mathcal{H}_0(\Gamma_0)$ with respect to the pivot space $L^2(\Gamma_0)^3$. Then since the embedding $\mathcal{H}_0(\Gamma_0) \subset \widetilde{\mathbf{H}}^{-1/2}(curl_\Gamma, \Gamma_0)$ is bounded, $\mathbf{H}^{-1/2}(div_\Gamma, \Gamma_0) \subset \mathcal{H}_0(\Gamma_0)^*$ is bounded as well.

Define $\mathcal{A}_i : \mathcal{H}_0(\Gamma_0) \to \mathcal{H}_0(\Gamma_0)^*$ by

$$\mathcal{A}_i \mathbf{u} = \alpha_i \mathbf{u} - \beta_i \overrightarrow{curl}_\Gamma curl_\Gamma \mathbf{u}. \tag{4.37}$$

Observe that pivoting with $\mathbf{L}^2_t(\Gamma_0)$,

$$\Big\langle \mathcal{A}_i \mathbf{u}, \mathbf{v} \Big\rangle_{\mathcal{H}_0(\Gamma_0)^*, \mathcal{H}_0(\Gamma_0)} \;\; = \;\; \int_{\Gamma_0} \alpha_i \mathbf{u} \cdot \overline{\mathbf{v}} \;\, ds(\mathbf{y}) - \int_{\Gamma_0} \beta_i curl_\Gamma \mathbf{u} \; \overline{curl_\Gamma \mathbf{v}} \;\, ds(\mathbf{y}), \tag{4.38}$$

for all $\mathbf{u}$, $\mathbf{v} \in \mathcal{H}_0(\Gamma_0)$. The analysis of the invertibility of (4.21) will be based on variational techniques and a Helmholtz decomposition of $\mathcal{H}_0(\Gamma_0)$ will be used. Inspired by the sesquilinear form (4.38) first define the Hilbert space:

$$S_\Gamma := \Big\{ p \in H^1(\Gamma_0) \Big| \int_{\Gamma_0} p \;\, ds(\mathbf{y}) = 0 \Big\}, \tag{4.39}$$

equipped with the norm

$$||p||_{S_\Gamma} := ||\nabla_\Gamma p||_{\mathbf{L}^2_t(\Gamma_0)}. \tag{4.40}$$

Lemma 4.3.3. *Let $\mathbf{u} \in \mathcal{H}_0(\Gamma_0)$, then the problem of finding $p \in S_\Gamma$ such that*

$$\int_{\Gamma_0} \alpha_1 \nabla_\Gamma p \cdot \overline{\nabla_\Gamma q} \ \ ds(\mathbf{y}) = \int_{\Gamma_0} \mathbf{u} \cdot \overline{\nabla_\Gamma q} \ \ ds(\mathbf{y}) \quad \text{for all } q \in S_\Gamma, \tag{4.41}$$

is well posed.

Proof. The sesquilinear form $\beta : \nabla_\Gamma S_\Gamma \times \nabla_\Gamma S_\Gamma \to \mathbb{C}$ defined by

$$\beta(\nabla_\Gamma p, \nabla_\Gamma q) := \int_{\Gamma_0} \alpha_1 \nabla_\Gamma p \cdot \overline{\nabla_\Gamma q} \ \ \mathrm{d}s(\mathbf{y})$$

is coercive:

$$\beta(\nabla_\Gamma p, \nabla_\Gamma p) := \int_{\Gamma_0} \alpha_1 |\nabla_\Gamma p|^2 \ \ \mathrm{d}s(\mathbf{y}) \geq \mu_\delta ||\nabla_\Gamma p||^2_{L^2},$$

so the proof follows by the Lax-Milgram theorem. $\square$

Now define the orthogonal space to $\nabla_\Gamma S_\Gamma$ for the Helmholtz decomposition:

$$V_\Gamma := \left\{ \mathbf{u}_0 \in \mathcal{H}_0(\Gamma_0) \,\middle|\, \int_{\Gamma_0} \alpha_1 \mathbf{u}_0 \cdot \overline{\nabla_\Gamma q} \ \ \mathrm{d}s(\mathbf{y}) = 0 \text{ for all } q \in S_\Gamma \right\}, \tag{4.42}$$

endowed with the $|| \cdot ||_{\mathcal{H}_0(\Gamma_0)}$ norm.

Theorem 4.3.1. *(Helmholtz decomposition of $\mathcal{H}_0(\Gamma_0)$)*

$$\mathcal{H}_0(\Gamma_0) = \nabla_\Gamma S_\Gamma \oplus V_\Gamma. \tag{4.43}$$

Proof. (i) Clearly, both $\nabla_\Gamma S_\Gamma$ and V_Γ are closed subspaces of $\mathcal{H}_0(\Gamma_0)$.

(ii) Moreover, given $\mathbf{u} \in \mathcal{H}_0(\Gamma_0)$, from Lemma 4.3.3 we know that there is a unique $p \in S_\Gamma$ that solves (4.3.3). Then $\mathbf{u}_0 := \mathbf{u} - \nabla_\Gamma p$ belongs to V_Γ, so that $\mathcal{H}_0(\Gamma_0) \subset \nabla_\Gamma S_\Gamma + V_\Gamma$.

(iii) Finally, if $\mathbf{u}_0 \in \nabla_\Gamma S_\Gamma \cap V_\Gamma$, then $\mathbf{u}_0 = \nabla_\Gamma p$ and then solves problem (4.41) for zero right hand side, implying that $\mathbf{u}_0 = \mathbf{0}$. Therefore $\mathcal{H}_0(\Gamma_0) \subset \nabla_\Gamma S_\Gamma \oplus V_\Gamma$.

$\square$

Remark 4.3.2. *Since $div_\Gamma \mathbf{u}_0 = 0$ in Γ_0, from the standard Helmholtz decomposition of $\mathbf{L}^2_t(\Gamma)$ (see [67]) we know that for all $\mathbf{u}_0 \in V_\Gamma$, there is $q \in H^1(\Gamma)$ such that $\mathbf{u}_0 = \overrightarrow{curl}_\Gamma q|_{\Gamma_0}$, thus the embedding $V_\Gamma \subset \mathbf{H}^{1/2}(\Gamma_0)$ holds and it is bounded, which implies that V_Γ is compactly embedded in $\mathbf{L}^2_t(\Gamma_0)$.*

Necessary for the upcoming analysis, we will assume throughout this paper that the following holds:

Assumption 4.3.1. *The quantity* $\omega = k^2\epsilon_\delta\mu_\delta$ *is not a Dirichlet eigenvalue of* $\overrightarrow{curl}_\Gamma curl_\Gamma$ *in* Γ_0.

Proposition 4.3.1. *The operator* $\mathcal{A}_1 : \mathcal{H}_0(\Gamma_0) \to \mathcal{H}_0(\Gamma_0)^*$ *is invertible. Moreover, if* $q \in H^1(\Gamma_0)$*, then* $\mathcal{A}_1^{-1}\overrightarrow{curl}_\Gamma q$ *is in* V_Γ.

Proof. Given any $\mathbf{w}_0 \in V_\Gamma$, consider the problem of finding $\mathbf{u}_0 \in \mathcal{H}_0(\Gamma_0)$ such that

$$\Big\langle \mathbf{u}_0, \mathbf{v}_0 \Big\rangle_{\mathcal{H}_0(\Gamma_0)^*, \mathcal{H}_0(\Gamma_0)} = \int_{\Gamma_0} \mathbf{w} \cdot \overline{\mathbf{v}_0} \ \mathrm{d}s(\mathbf{y}), \quad \text{for all } \mathbf{v}_0 \in V_\Gamma. \tag{4.44}$$

It is equivalent to find $\mathbf{u}_0 \in V_\Gamma$ such that

$$\begin{aligned} \int_{\Gamma_0} \Big\{ \mathbf{u}_0 \cdot \overline{\mathbf{v}_0} \ + \ \beta_1 curl_\Gamma \mathbf{u}_0 \, \overline{curl_\Gamma \mathbf{v}_0} \Big\} \ \mathrm{d}s(\mathbf{y}) & \\ - \int_{\Gamma_0} \Big\{ (\alpha_1 + 1)\mathbf{u}_0 \cdot \overline{\mathbf{v}_0} \Big\} \ \mathrm{d}s(\mathbf{y}) &= - \int_{\Gamma_0} \mathbf{w} \cdot \overline{\mathbf{v}_0} \ \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.45}$$

for all $\mathbf{v}_0 \in V_\Gamma$. By Remark 4.3.2, the second integral term is a compact sesquilinear dorm in $V_\Gamma \times V_\Gamma$, whereas the first integral is obviously a coercive term. Therefore, the well-posedness of this problem is equivalent to the uniqueness of its solution, and this is a direct consequence of Assumption 4.3.1.

Therefore, combining this with Lemma 4.3.3, we conclude that for any $\mathbf{w} = \mathbf{w}_0 + \xi$ in $\mathcal{H}_0(\Gamma_0)$ (where the decomposition is in $\nabla_\Gamma S_\Gamma \oplus V_\Gamma$), the unique solution $\mathbf{u} \in \mathcal{H}_0(\Gamma_0)$ to

$$\Big\langle \mathbf{u}, \mathbf{v} \Big\rangle_{\mathcal{H}_0(\Gamma_0)^*, \mathcal{H}_0(\Gamma_0)} = \int_{\Gamma_0} \mathbf{w} \cdot \overline{\mathbf{v}} \ \mathrm{d}s(\mathbf{y}), \quad \text{for all } \mathbf{v} \in \mathcal{H}_0(\Gamma_0) \tag{4.46}$$

is $\mathbf{u} = \mathbf{u}_0 + \nabla_\Gamma p$, where $\mathbf{u}_0 \in V_\Gamma$ solves (4.44), and $p \in S_\Gamma$ is the unique solution of the variational problem (4.41).

Finally, if $q \in H^1(\Gamma_0)$, then $\mathbf{w}_0 = \overrightarrow{curl}_\Gamma q$ is in V_Γ, and thus it is clear that $\mathcal{A}_1^{-1}\overrightarrow{curl}_\Gamma q$ is in V_Γ. □

4.3.3 The variational formulation of the asymptotic model

Having established conditions for the invertibility of $\mathcal{A}_1$ in Proposition 4.3.1, we can formally write:

$$\mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \mathbf{E}]\!] = \langle\!\langle \mu^{-1}\nabla \times \mathbf{E} \rangle\!\rangle_T. \tag{4.47}$$

Let B_R be the open ball of radius $R > 0$ be large enough so that $\overline{\Omega} \subset B_R$. Thus, after multiplying equation (4.20) by a test function $\mathbf{v}$ and integrating by parts in B_R, we deduce that a variational formulation of problem (4.20)-(4.24) is: Seek $\mathbf{E} \in \mathcal{H}_0$ such that

$$a(\mathbf{E}, \mathbf{v}) \;=\; \mathcal{L}(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathcal{H}_0, \tag{4.48}$$

where,

$$a(\mathbf{E}, \mathbf{v}) = a^+(\mathbf{E}, \mathbf{v}) + b(\mathbf{E}, \mathbf{v}) + ik\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}), \mathbf{v}_T\rangle_{S_R}, \tag{4.49}$$

and

$$\begin{aligned} a^+(\mathbf{E}, \mathbf{v}) :=& \int_{B_R^\delta} \left(\mu^{-1}\nabla \times \mathbf{E} \cdot \overline{\nabla \times \mathbf{v}}\right) \; \mathrm{d}\,\mathbf{y} \\ &+ \int_{\Gamma_0} \delta\beta_2 curl_\Gamma \langle\!\langle \mathbf{E}_T \rangle\!\rangle \, \overline{curl_\Gamma \langle\!\langle \mathbf{v}_T \rangle\!\rangle} \; \mathrm{d}s(\mathbf{y}) \end{aligned} \tag{4.50}$$

$$\begin{aligned} b(\mathbf{E}, \mathbf{v}) :=& -\int_{B_R^\delta} k^2\epsilon \mathbf{E} \cdot \overline{\mathbf{v}} \; \mathrm{d}\,\mathbf{y} - \int_{\Gamma_0} \delta\alpha_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle \cdot \langle\!\langle \overline{\mathbf{v}} \rangle\!\rangle_T \; \mathrm{d}s(\mathbf{y}) \\ &+ \frac{1}{\delta}\int_{\Gamma_0} \boldsymbol{\lambda}_E \cdot \overline{[\![\boldsymbol{\nu} \times \mathbf{v}]\!]} \; \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.51}$$

$$\mathcal{L}(\mathbf{v}) = \int_{S_R} (\widehat{\mathbf{x}} \times (\nabla \times \mathbf{E}^i)) \cdot \overline{\mathbf{v}} - ik\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}_i), \mathbf{v}_T\rangle_{S_R}, \tag{4.52}$$

where $B_R^\delta := B_R \setminus \overline{\Omega_\delta}$, $\boldsymbol{\lambda}_E = \mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \mathbf{E}]\!]$, i.e.,

$$\int_{\Gamma_0} [\![\boldsymbol{\nu} \times \mathbf{E}]\!] \cdot \overline{\boldsymbol{\eta}} \; \mathrm{d}s(\mathbf{y}) = \int_{\Gamma_0} \alpha_1 \boldsymbol{\lambda}_E \cdot \overline{\boldsymbol{\eta}} \; \mathrm{d}s(\mathbf{y}) - \int_{\Gamma_0} \beta_1 curl_\Gamma \boldsymbol{\lambda}_E \, \overline{curl_\Gamma \boldsymbol{\eta}} \; \mathrm{d}s(\mathbf{y}) \tag{4.53}$$

for all $\boldsymbol{\eta} \in \mathcal{H}_0(\Gamma_0)$, and the solutions space is

$$\mathcal{H}_0 := \left\{ \mathbf{u} \in \mathbf{H}(curl, B_R^\delta) \,\middle|\, \langle\!\langle \mathbf{u}_T \rangle\!\rangle \in \mathbf{H}(curl_\Gamma, \Gamma_0) \text{ and } \mathbf{n} \times \left(\mu^{-1}\nabla \times \mathbf{u}\right)\Big|_{\mathscr{S}} = \mathbf{0} \right\} \tag{4.54}$$

endowed with the norm

$$||\mathbf{u}||^2_{\mathcal{H}_0} := ||\mathbf{u}||^2_{\mathbf{H}(curl, B_R^\delta)} + ||\langle\!\langle \mathbf{u}_T \rangle\!\rangle||^2_{\mathbf{H}(curl_\Gamma, \Gamma_0)}. \tag{4.55}$$

4.3.4 A Helmholtz decomposition of $\mathcal{H}_0$

Recalling the definition (4.39) of S_Γ, define now:

$$\begin{aligned} S &:= \{p \in H^1(B_R^\delta) \,|\, \langle\!\langle p \rangle\!\rangle \in S_\Gamma\} \\ &= \Big\{ p \in H^1(B_R^\delta) \,\Big|\, \nabla_\Gamma \langle\!\langle p \rangle\!\rangle \in \mathbf{L}_t^2(\Gamma_0), \text{ and} \\ &\qquad \int_{\Gamma_0} \langle\!\langle p \rangle\!\rangle \; \mathrm{d}s(\mathbf{y}) = \mathbf{0} \Big\}, \end{aligned} \tag{4.56}$$

endowed with the norm

$$||p||_S^2 := ||\nabla p||_{\mathcal{H}_0}^2 = ||\nabla p||_{\mathbf{L}^2(B_R^\delta)}^2 + ||\nabla_\Gamma \langle\!\langle p \rangle\!\rangle||_{\mathbf{L}_t^2(\Gamma_0)}^2. \tag{4.57}$$

We now prove that the variational problem is well-posed in V_0.

Proposition 4.3.2. *Assume in addition to Assumptions 4.1.1 and 4.3.1that there is a constant $\epsilon_{min} > 0$ such that $\Re(\epsilon_\pm) \geq \epsilon_{min} > 0$ and that the constant material properties in Ω_δ satisfy $\Re(\epsilon_\delta) > 0$ and $\mu_\delta > 0$. Then the problem of finding $p \in S$ such that*

$$a(\nabla p, \nabla q) = \ell(\nabla q) \text{ for all } q \in S, \tag{4.58}$$

is well posed for all $\ell \in (\nabla S)^*$.

Proof. Let p, $q \in S$. Then $[\![p]\!], [\![q]\!] \in S_\Gamma$ and thus by Proposition 4.3.1, $\mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \nabla_\Gamma p]\!] = \mathcal{A}_1^{-1}\overrightarrow{curl}_\Gamma[\![p]\!] = \boldsymbol{\lambda}_{\nabla p} \in V_\Gamma$ and $\mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \nabla_\Gamma q]\!] = \mathcal{A}_1^{-1}\overrightarrow{curl}_\Gamma[\![q]\!] = \boldsymbol{\lambda}_{\nabla q} \in V_\Gamma$.
Moreover, $a(\nabla p, \nabla q)$ can be decomposed into

$$a(\nabla p, \nabla q) = -\widetilde{\alpha}(\nabla p, \nabla q) - \widetilde{\beta}(\nabla p, \nabla q) + ik\langle G_e(\widehat{\mathbf{x}} \times \nabla p), \nabla q\rangle_{S_R}$$

where

$$\begin{aligned} \widetilde{\alpha}(\nabla p, \nabla q) &:= \int_{B_R^\delta} k^2 \epsilon \nabla \mathbf{p} \cdot \overline{\nabla q} \; \mathrm{d}\,\mathbf{y} + \int_{\Gamma_0} \delta\alpha_2 \nabla_\Gamma \langle\!\langle p \rangle\!\rangle \cdot \overline{\nabla_\Gamma \langle\!\langle q \rangle\!\rangle} \; \mathrm{d}s(\mathbf{y}) \\ &+ \frac{1}{\delta} \int_{\Gamma_0} \overline{\beta_1}\, \overline{curl_\Gamma \boldsymbol{\lambda}_{\nabla p}} \cdot curl_\Gamma \boldsymbol{\lambda}_{\nabla q} \; \mathrm{d}s(\mathbf{y}) \end{aligned} \tag{4.59}$$

$$\widetilde{\beta}(\nabla p, \nabla q) := -\frac{1}{\delta} \int_{\Gamma_0} \overline{\alpha_1}\, \overline{\boldsymbol{\lambda}_{\nabla p}} \cdot \boldsymbol{\lambda}_{\nabla q} \; \mathrm{d}s(\mathbf{y}). \tag{4.60}$$

From the expressions of α_1, α_2 and β_1,

$$\begin{aligned}\Re(\widetilde{\alpha}(\nabla p, \nabla p)) &= \int_{B_R^\delta} k^2 \Re(\epsilon)|\nabla p|^2 \ \mathrm{d}\,\mathbf{y} + \int_{\Gamma_0} 2\delta\mu_\delta \nabla_\Gamma |\langle\!\langle p \rangle\!\rangle|^2 \ \mathrm{d}s(\mathbf{y}) \\ &+ \frac{1}{\delta}\int_{\Gamma_0} \frac{2\Re(\epsilon_\delta)}{k^2|\epsilon_\delta|^2} |curl_\Gamma \boldsymbol{\lambda}_{\nabla p}|^2 \ \mathrm{d}s(\mathbf{y}) \\ &\geq \widetilde{C}(||\nabla p||^2_{\mathbf{L}^2(B_R^\delta)} + ||\nabla_\Gamma \langle\!\langle p \rangle\!\rangle||^2_{\mathbf{L}^2_t(\Gamma_0)}),\end{aligned} \tag{4.61}$$

where $\widetilde{C} = \min\left\{k^2\epsilon_{min},\, 2\delta\mu_\delta\right\} > 0$ is a constant independent of p. Thus $\widetilde{\alpha}$ is a coercive sesquilinear form in $\nabla S \times \nabla S$.

On the other hand, it is clear that $\widetilde{\beta}(\cdot,\cdot)$ is bounded and, from the boundedness of $\mathcal{A}_1^{-1}$ (Proposition 4.3.1), and the boundedness of the embedding $\widetilde{\mathbf{H}}^{-1/2}(div_\Gamma, \Gamma_0) \subset \mathcal{H}_0(\Gamma_0)^*$, we get that

$$\begin{aligned}||\boldsymbol{\lambda}_{\nabla p}||_{\mathcal{H}_0(\Gamma_0)} &\leq C||\overrightarrow{curl}_\Gamma [\![p]\!]||_{\mathcal{H}_0(\Gamma_0)^*} \leq C_1 ||\overrightarrow{curl}_\Gamma [\![p]\!]||_{\mathbf{H}^{-1/2}(div_\Gamma,\Gamma_0)} \\ &\leq C_2||\nabla_\Gamma [\![p]\!]||_{H^{-1/2}(\Gamma_0)} \leq C_3 ||\nabla p||_{\mathbf{L}^2(B_R^\delta)}.\end{aligned} \tag{4.62}$$

So

$$|\widetilde{\beta}(\nabla p, \nabla q)| \ \leq \ c||\boldsymbol{\lambda}_{\nabla p}||_{\mathbf{L}^2_t(\Gamma_0)} ||q||_S. \tag{4.63}$$

Let (p_n) be a bounded sequence in S, then by (4.62) $(\boldsymbol{\lambda}_{\nabla p_n})$ is also a bounded sequence in V_Γ (see Proposition 4.3.1). Since the space V_Γ is compactly embedded in $\mathbf{L}^2_t(\Gamma_0)$ (Remark 4.3.2), then, from (4.63), the Riesz operator $\widetilde{B} : S \to \mathbb{C}$ defined by

$$\langle\!\langle \widetilde{B}(p), q \rangle\!\rangle_{S^*,S} := \widetilde{\beta}(\nabla p, \nabla q)$$

has a strongly convergent subsequence $(\widetilde{B}(p_{n_k}))$ in S^*. Thus the operator $\widetilde{B} : S \to \mathbb{C}$ is compact.

The remaining part to analyze is related to the Calderón operator G_e. In Lemmas 9.23 and 9.24 of [65], it is shown that there exists an operator $\widetilde{G_e} : \mathbf{H}^{-1/2}(div_{S_R}, S_R) \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$ such that

$$\langle \widetilde{G_e}(\boldsymbol{\xi}), \boldsymbol{\xi} \times \widehat{\mathbf{x}} \rangle_{S_R} < 0, \text{ for all } \xi \in \mathbf{H}^{-1/2}(div_{S_R}, S_R),$$

and that the operator $G_e + ik\widetilde{G_e} : \mathbf{H}^{-1/2}_{div_{S_R}}(div_{S_R}, S_R) \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$ is compact, where

$$\begin{aligned}\mathbf{H}^{-1/2}_{div_{S_R}}(div_{S_R}, S_R) &:= \\ \left\{ \boldsymbol{\lambda} = \sum_{n=1}^{\infty}\sum_{m=-n}^{m=n} b_{n,m}\mathbf{V}_n^m \right. & \left| \quad \sum_{n=1}^{\infty}\sum_{m=-n}^{m=n} \frac{1}{\sqrt{1+n(n+1)}}|b_{n,m}|^2 < \infty \right\} \end{aligned} \tag{4.64}$$

endowed with the norm

$$\|\boldsymbol{\lambda}\|^2_{\mathbf{H}^{-1/2}_{div_{S_R}}(div_{S_R}, S_R)} := \sum_{n=1}^{\infty}\sum_{m=-n}^{m=n} \frac{1}{\sqrt{1+n(n+1)}}|b_{n,m}|^2 \tag{4.65}$$

for $\boldsymbol{\lambda} = \sum_{n=1}^{\infty}\sum_{m=-n}^{m=n} b_{n,m}\mathbf{V}_n^m$.

Moreover, from similar arguments to those given in [65], $\widehat{\mathbf{x}}\times\nabla p \in \mathbf{H}^{-1/2}_{div_{S_R}}(div_{S_R}, S_R)$ for all $p \in S$, and therefore, we can conclude that the splitting into a negative and a compact term:

$$ik\langle G_e(\widehat{\mathbf{x}}\times\nabla p), \nabla q\rangle_{S_R} = k^2\langle \widetilde{G_e}(\widehat{\mathbf{x}}\times\nabla p), \nabla q\rangle_{S_R} + ik\langle (G_e + ik\widetilde{G_e})(\widehat{\mathbf{x}}\times\nabla p), \nabla q\rangle_{S_R} \tag{4.66}$$

makes sense. Then $-\widetilde{\alpha}(\cdot,\cdot) + k^2\langle \widetilde{G_e}(\widehat{\mathbf{x}}\times\cdot), \cdot\rangle_{S_R}$ is a sesquilinear form in $\nabla S \times \nabla S$ associated with a coercive Riesz operator, and the sesquilinear form in $\nabla S \times \nabla S$ defined by $\widetilde{\beta}(\cdot,\cdot) + ik\langle (G_e + ik\widetilde{G_e})(\widehat{\mathbf{x}}\times\cdot), \cdot\rangle_{S_R}$ has a compact associated Reisz operator. In summary, the Reisz operator associated with the complete sesquilinear form $a(\cdot,\cdot)|_{\nabla S\times\nabla S}$ is Fredholm with index zero, meaning that the problem is well-posed if and only if it is uniquely solvable. To check this, observe that

$$\begin{aligned}\Im(a(\nabla p, \nabla p)) &= -\int_{B_R^\delta} k^2\Im(\epsilon)|\nabla p|^2 \, \mathrm{d}\,\mathbf{y} - \int_{\Gamma_0} 2\delta k^2\Im(\epsilon_\delta)\nabla_\Gamma\langle\!\langle p\rangle\!\rangle \cdot \overline{\nabla_\Gamma\langle\!\langle q\rangle\!\rangle} \, \mathrm{d}s(\mathbf{y}) \\ &- \frac{1}{\delta}\int_{\Gamma_0} \frac{\Im(\epsilon_\delta)}{k^2|\epsilon_\delta|^2}|curl_\Gamma\boldsymbol{\lambda}_{\nabla p}|^2 \, \mathrm{d}s(\mathbf{y}) \\ &+ \Im\langle ikG_e(\widehat{\mathbf{x}}\times\nabla p), \nabla p\rangle_{S_R}. \end{aligned} \tag{4.67}$$

By Rellich's lemma (see Lemma 9.28 in [65]), since

$$\Im\langle ikG_e(\widehat{\mathbf{x}}\times\nabla p), \nabla p\rangle_{S_R} \geq 0,$$

then $\nabla_{S_R} p = 0$ on S_R, and then all the terms in the right hand side of equation (4.67) vanish. In particular:

$$\nabla_\Gamma \langle\!\langle p \rangle\!\rangle = \mathbf{0} \quad \text{and} \quad curl_\Gamma \boldsymbol{\lambda}_{\nabla p} = 0. \tag{4.68}$$

So by (4.61), $\nabla p = \mathbf{0}$ in B_R^δ. $\square$

Now consider the second space for the Helmholtz decomposition of $\mathcal{H}_0$:

$$\begin{aligned} V_0 \;&:=\; \left\{ \mathbf{u}_0 \in \mathcal{H}_0 \,\middle|\, b(\mathbf{u}_0, \nabla q) + ik\langle G_e(\widehat{\mathbf{x}} \times \mathbf{u}_0), \nabla q\rangle_{S_R} = 0 \text{ for all } q \in S \right\} &(4.69)\\ &= \left\{ \mathbf{u}_0 \in \mathcal{H}_0 \,\middle|\, div(\epsilon \mathbf{u}_0) = 0 \text{ in } B_R^\delta, \right. \\ &\quad [\![\epsilon \boldsymbol{\nu} \cdot \mathbf{u}_0]\!] = -\frac{\delta \alpha_2}{k^2} div_\Gamma \langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T, \\ &\quad \langle\!\langle \epsilon \boldsymbol{\nu} \cdot \mathbf{u}_0 \rangle\!\rangle = \frac{1}{\delta k^2} curl_\Gamma \left(\boldsymbol{\lambda}_{\mathbf{u}_0}\right), \\ &\quad \mathbf{n} \cdot \mathbf{u}_0 = 0 \text{ on } \mathscr{S} \text{ and } \mathbf{n} \times (\mu^{-1} \nabla \times \mathbf{u}_0)|_{\mathscr{S}} = \mathbf{0}, \\ &\quad \left. \widehat{\mathbf{x}} \cdot \mathbf{u}_0 = -\frac{i}{k}\, div_\Gamma\, G_e(\widehat{\mathbf{x}} \times \mathbf{u}_0) \text{ on } S_R \right\}. &(4.70) \end{aligned}$$

Theorem 4.3.2. *(Helmholtz decomposition of $\mathcal{H}_0$)*

$$\mathcal{H}_0 = V_0 \oplus \nabla S$$

Proof. It is obvious that both ∇S and V_0 are closed subspaces of $\mathcal{H}_0$. Now, given $\mathbf{u} \in \mathcal{H}_0$, we know from Proposition 4.3.2 that there is unique $p \in S$ such that

$$a(\nabla p, \nabla q) = a(\mathbf{u}, \nabla q) \text{ for all } q \in S, \tag{4.71}$$

and then if we define $\mathbf{u}_0 := \mathbf{u} - \nabla p$, it satisfies by definition that $\mathbf{u}_0 \in V_0$. Thus $X \subset V_0 + \nabla S$. Now, if $\mathbf{u}_0 = \nabla p \in V_0 \cap \nabla S$, then by definition of V_0,

$$a(\nabla p, \nabla q) = b(\nabla p, \nabla q) + ik\langle G_e(\widehat{\mathbf{x}} \times \nabla p), \nabla q\rangle_{S_R} = 0 \text{ for all } q \in S \tag{4.72}$$

and from Proposition 4.3.2, this means that $\mathbf{u}_0 = \nabla p = \mathbf{0}$. Therefore $\mathcal{H}_0 = V_0 \oplus \nabla S$. $\square$

4.4 Well-posedness

The Helmholtz decomposition in Theorem 4.3.2 can be used to decompose the variational problem (4.48) into two decoupled problems, one in ∇S and another one in V_0. We use these facts to prove the well-posedness of the variational problem (4.48) on $\mathcal{H}_0$.

We start by deriving a decomposition of the Calderón map G_e in the following lemma, which is essentially the same as Lemma 10.5 in [65].

Lemma 4.4.1. *The electric-to-magnetic Calderón operator G_e can be decomposed as $G_e = G_e^1 + G_e^2$, where $G_e^1 \circ \gamma_T : V_0 \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$ is compact and $G_e^2 : \mathbf{H}^{-1/2}(div_{S_R}, S_R) \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$ satisfies*

$$ik\langle G_e^2(\boldsymbol{\xi}), \boldsymbol{\xi} \times \widehat{\mathbf{x}}\rangle_{S_R} \geq 0 \textit{ for all } \xi \in \mathbf{H}^{-1/2}(div_{S_R}, S_R).$$

Proof. This proof is a slight variation of the proof of Lemma 10.5 in [65], but we present here the details for the reader's convenience. In [65], it is shown that the Calderón map G_e can be split in terms of spherical harmonics as

$$\begin{aligned} G_e(\boldsymbol{\xi}) &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(-ikR\frac{b_{n,m}}{\delta_n}\mathbf{U}_n^m + a_{n,m}\frac{(\delta_n - \widetilde{\delta}_n)}{ikR}\mathbf{V}_n^m\right) \\ &+ \frac{1}{ikR}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{n,m}\widetilde{\delta}_n\mathbf{V}_n^m \\ &= G_e^1(\boldsymbol{\xi}) + G_e^2(\boldsymbol{\xi}) \end{aligned} \tag{4.73}$$

where

$$\boldsymbol{\xi} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{n,m}\mathbf{U}_n^m + b_{n,m}\mathbf{V}_n^m$$

$$\delta_n = \frac{1}{ikR}\left(1 + \frac{kRh_n^{(1)'}(kR)}{h_n^{(1)}(kR)}\right), \quad \text{and} \quad \widetilde{\delta}_n = \frac{1}{ikR}\left(1 + \frac{kRj_n'(kR)}{j_n(kR)}\right), \tag{4.74}$$

and where, of course, we assume that $R > 0$ has been chosen so that $0 < |\widetilde{\delta}_n| < \infty$. In turn, the operator G_e^1 can be splitted as $G_e^1 = G_e^{1,U} + G_e^{1,V}$, where

$$\begin{aligned} G_e^{1,U}(\boldsymbol{\xi}) &= \sum_{n=1}^{\infty} \sum_{m=-n}^{n} -ikR \frac{b_{n,m}}{\delta_n} \mathbf{U}_n^m, \text{ and} \\ G_e^{1,V}(\boldsymbol{\xi}) &= \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{n,m} \frac{(\delta_n - \widetilde{\delta}_n)}{ikR} \mathbf{V}_n^m. \end{aligned} \tag{4.75}$$

Moreover, it is proved that

$$\delta_n - \widetilde{\delta}_n = O\left(\frac{1}{n}\right), \text{ for all } n \geq 1,$$

which ensures that $G_e^{1,V}$ is compact from $\mathbf{H}^{-1/2}(div_{S_R}, S_R)$ into itself, and, from the boundedness of the tangential trace operator $\gamma_T : \mathbf{H}(curl, \Omega) \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$, we have that $G_e^{1,V} \circ \gamma_T : V \to \mathbf{H}^{-1/2}(div_{S_R}, S_R)$ is compact. Now, given $\mathbf{u}_0 \in V$, if we consider a cut-off function χ, such that $\chi = 1$ in a neighborhood $\mathcal{N}_R$ of S_R, and such that $supp(\chi) \cap \overline{\Omega}_- = \emptyset$, then the restriction $(\chi \mathbf{u}_0)|_{\mathcal{N}_R} \in \widetilde{V}$, where the space $\widetilde{V} := \{u \in \mathcal{H}_0 \,|\, \int_{B_R} \epsilon \mathbf{u} \cdot \nabla p \ \mathrm{d}\,\mathbf{y} + ik\langle G_e(\widehat{\mathbf{x}} \times u), \nabla p\rangle_{S_R} = 0 \text{ for all } p \in S\}$ is compactly embedded in $(L^2(B_R))^3$ (see Lemma 10.4 in [65]). Thus, we have the following

$$\begin{aligned} ||(G_e^{1,U} \circ \gamma_T)\mathbf{u}_0||_{\mathbf{H}^{-1/2}(div_{S_R}, S_R)} &\leq ||(G_e^{1,U} \circ \gamma_T)(\chi \mathbf{u}_0)|_{\mathcal{N}_R}||_{\mathbf{H}^{-1/2}(div_{S_R}, S_R)} \\ &\leq ||G_e^{1,U}(\widehat{\mathbf{x}} \times \mathbf{u}_0)||_{\mathbf{H}^{-1/2}(div_{S_R}, S_R)} \\ &\leq C||div_{S_R} G_e(\widehat{\mathbf{x}} \times \mathbf{u}_0)||_{H^{-1/2}(S_R)}. \end{aligned} \tag{4.76}$$

Now, from the variational characterization of $\widetilde{V}$ (similar to (4.69)), and from the trace theorems for $\mathbf{H}(div, B_R)$ (see Theorem 3.24 in [65]),

$$\begin{aligned} ||\nabla_{S_R} \cdot G_e(\widehat{\mathbf{x}} \times \mathbf{u}_0)||_{H^{-1/2}(S_R)} &= ||\widehat{\mathbf{x}} \cdot \mathbf{u}_0||_{H^{-1/2}(S_R)} \\ &\leq \sqrt{||\chi \mathbf{u}_0||^2_{(L^2(B_R))^3} + ||\nabla \cdot (\epsilon \chi \mathbf{u}_0)||^2_{L^2(B_R)}} \\ &= ||\chi \mathbf{u}_0||_{(L^2(B_R))^3}, \end{aligned} \tag{4.77}$$

so the compactness of the embedding of $\widetilde{V} \subset (L^2(B_R))^3$ together with the boundedness of the multiplication by χ from V_0 to $\widetilde{V}$, implies that $(G_e^{1,U} \circ \gamma_T$ is compact from V_0 to $\mathbf{H}^{-1/2}(div_{S_R}, S_R)$.

Finally, the proof of the property for G_e^2 is immediate because

$$\langle G_e(\widehat{\mathbf{x}} \times \boldsymbol{\xi}), \boldsymbol{\xi}\rangle_{S_R} = \frac{1}{R}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} -|b_{n,m}|^2 \widetilde{\delta}_n \geq 0. \tag{4.78}$$

$\square$

Proposition 4.4.1. *Assume in addition to Assumption 4.1.1, that there is a constant $\epsilon_{min} > 0$ such that $\Re(\epsilon_\pm) \geq \epsilon_{min} > 0$ and $\Im(\epsilon_\pm) \geq \epsilon_{min} > 0$, and that the constant material properties in Ω_δ are such that $\Re(\epsilon_\delta) > 0$, $\Im(\epsilon_\delta) > 0$ and $\mu_\delta > 0$ (in particular Assumption 4.3.1 is true). Then the problem of finding $\mathbf{u}_0 \in V_0$ such that*

$$a(\mathbf{u}_0, \mathbf{v}_0) = \ell(\mathbf{v}_0) \text{ for all } \mathbf{v}_0 \in V_0, \tag{4.79}$$

is well posed for all $\ell \in V_0^*$.

Proof. Define the auxiliary sesquilinear forms $a_0, b_0 : V_0 \times V_0 \to \mathbb{C}$ by

$$\begin{aligned} a_0(\mathbf{u}_0, \mathbf{v}_0) &:= \int_{B_R^\delta} \left(\mu^{-1}\nabla \times \mathbf{u}_0 \cdot \overline{\nabla \times \mathbf{v}_0}\right) \, \mathrm{d}\,\mathbf{y} \\ &+ \int_{\Gamma_0} \delta\beta_2 curl_\Gamma \langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T \overline{curl_\Gamma \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_T}\, ds + \frac{1}{\delta}\int_{\Gamma_0} \alpha_1 \boldsymbol{\lambda}_{\mathbf{v}_0} \cdot \overline{\boldsymbol{\lambda}\mathbf{u}_0}\, ds \quad (4.80) \\ b_0(\mathbf{u}_0, \mathbf{v}_0) &:= -\int_{B_R^\delta} k^2 \epsilon \mathbf{u}_0 \cdot \overline{\mathbf{v}_0} \, \mathrm{d}\,\mathbf{y} - \int_{\Gamma_0} \delta\alpha_2 \langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T \cdot \langle\!\langle \overline{\mathbf{v}_0} \rangle\!\rangle_T \, ds \\ &- \frac{1}{\delta}\int_{\Gamma_0} \overline{\beta_1} curl_\Gamma \boldsymbol{\lambda}_{\mathbf{v}_0} \overline{curl_\Gamma \boldsymbol{\lambda}_{\mathbf{u}_0}}\, ds. \quad (4.81) \end{aligned}$$

Then

$$\begin{aligned} \Re(a_0(\mathbf{u}_0, \mathbf{u}_0)) &:= \int_{B_R^\delta} \mu^{-1}|\nabla \times \mathbf{u}_0|^2 \, \mathrm{d}\,\mathbf{y} \\ &+ \int_{\Gamma_0} 2\mu_\delta^{-1}|curl_\Gamma \langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T|^2\, ds + \frac{1}{\delta}\int_{\Gamma_0} 2\mu_\delta |\boldsymbol{\lambda}_{\mathbf{u}_0}|^2\, ds \quad (4.82) \\ \Re(b_0(\mathbf{u}_0, \mathbf{u}_0)) &:= -\int_{B_R^\delta} k^2 \Re(\epsilon)|\mathbf{u}_0|^2 \, \mathrm{d}\,\mathbf{y} - \int_{\Gamma_0} \delta 2k^2\Re(\epsilon_\delta)|\langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T|^2\, ds \\ &- \frac{1}{\delta}\int_{\Gamma_0} 2\frac{\Re(\epsilon_\delta)}{k^2|\epsilon_\delta|^2}|curl_\Gamma \boldsymbol{\lambda}_{\mathbf{u}_0}|^2\, ds. \quad (4.83) \\ \Im(b_0(\mathbf{u}_0, \mathbf{u}_0)) &:= -\int_{B_R^\delta} k^2 \Im(\epsilon)|\mathbf{u}_0|^2 \, \mathrm{d}\,\mathbf{y} - \int_{\Gamma_0} \delta 2k^2\Im(\epsilon_\delta)|\langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T|^2\, ds \\ &- \frac{1}{\delta}\int_{\Gamma_0} 2\frac{\Im(\epsilon_\delta)}{k^2|\epsilon_\delta|^2}|curl_\Gamma \boldsymbol{\lambda}_{\mathbf{u}_0}|^2\, ds, \quad (4.84) \end{aligned}$$

Define $\tilde{a}(\cdot,\cdot) : V_0 \times V_0 \to \mathbb{C}$ as the sequilinear form that contains all the terms involved in $a(\cdot,\cdot)$ except for the term associated with the Calderón operator:

$$\tilde{a}(\mathbf{u}_0, \mathbf{v}_0) = a_0(\mathbf{u}_0, \mathbf{v}_0) + b_0(\mathbf{u}_0, \mathbf{v}_0).$$

Therefore,

$$\begin{aligned}(1+\tilde{\gamma})|\tilde{a}(\mathbf{u}_0,\mathbf{u}_0)| &\geq |\Re(\tilde{a}(\mathbf{u}_0,\mathbf{u}_0))| + \tilde{\gamma}|\Im(\tilde{a}(\mathbf{u}_0,\mathbf{u}_0))| \\ &\geq Re(a_0(\mathbf{u}_0,\mathbf{u}_0)) + \Re(b_0(\mathbf{u}_0,\mathbf{u}_0)) - \tilde{\gamma}\Im(b_0(\mathbf{u}_0,\mathbf{u}_0)) \\ &\geq Re(a_0(\mathbf{u}_0,\mathbf{u}_0)) + c(\|\mathbf{u}_0\|^2_{L^2(B_R^\delta)^3} \\ &+ \|\langle\!\langle \mathbf{u}_0 \rangle\!\rangle\|^2_{L^2(\Gamma_0)^3} + \|curl_\Gamma \boldsymbol{\lambda}_{\mathbf{u}_0}\|^2_{L^2(\Gamma_0)^3}) \\ &\geq \tilde{c}(||\mathbf{u}_0||^2_{\mathbf{H}(curl,B_R^\delta)} + ||\langle\!\langle \mathbf{u}_0 \rangle\!\rangle_T||^2_{\mathbf{H}_{ff}(curl_\Gamma,\Gamma_0)}) = \tilde{c}||\mathbf{u}_0||^2_X,\end{aligned} \tag{4.85}$$

where we have taken $\tilde{\gamma} > 0$ large enough so that there is a constant $c > 0$ such that

$$\begin{aligned}&\Re(b_0(\mathbf{u}_0,\mathbf{u}_0)) - \tilde{\gamma}\Im(b_0(\mathbf{u}_0,\mathbf{u}_0)) \geq \\ &c(\|\mathbf{u}_0\|^2_{L^2(B_R^\delta)^3} + \|\langle\!\langle \mathbf{u}_0 \rangle\!\rangle\|^2_{L^2(\Gamma_0)^3} + \|curl_\Gamma \boldsymbol{\lambda}_{\mathbf{u}_0}\|^2_{L^2(\Gamma_0)^3}),\end{aligned} \tag{4.86}$$

and $\tilde{c} = \min\left\{\mu_{max}^{-1},\, \frac{2}{\mu_\delta},\, \frac{2\mu_\delta}{\delta},\, c\right\}$.

Thus $\tilde{a}$ is coercive in V_0. From Lemma 4.4.1, we know that then $\tilde{a}(\mathbf{u}_0, \mathbf{v}_0) + ik\langle G_e^2(\widehat{\mathbf{x}} \times \mathbf{u}_0), \mathbf{v}_0\rangle_{S_R}$ is coercive and $ik\langle G_e^1(\widehat{\mathbf{x}} \times \mathbf{u}_0), \mathbf{v}_0\rangle_{S_R}$ is compact in V_0. Finally,

$$\begin{aligned}\Im(a(\mathbf{u}_0,\mathbf{u}_0)) &= \Im(b_0(\mathbf{u}_0,\mathbf{u}_0)) + \Im\langle ikG_e(\widehat{\mathbf{x}} \times \mathbf{u}_0), \mathbf{u}_0\rangle_{S_R} \\ &\leq \Im(b_0(\mathbf{u}_0,\mathbf{u}_0)) \leq -\tilde{\tilde{c}}||\mathbf{u}_0||_{\mathbf{L}^2(B_R^\delta)},\end{aligned} \tag{4.87}$$

where $\tilde{\tilde{c}} = \min\left\{k^2\epsilon_{min},\, 2\delta k^2\Im(\epsilon_\delta),\, \frac{2\Im(\epsilon_\delta)}{k^2\delta|\epsilon_\delta|^2}\right\}$ and again in the second line we have used Rellich lemma (Lemma 9.28 in [65]), so $a(\mathbf{u}_0, \mathbf{u}_0) = 0$ if and only if $\mathbf{u}_0 = \mathbf{0}$. □

In summary, using Propositions 4.3.2 and 4.4.1 and Lemma 4.4.1, we have proved the following well-posedness result:

Theorem 4.4.1. *Assume in addition to Assumption 4.1.1 that there is a constant* $\epsilon_{min} > 0$ *such that* $\Re(\epsilon_\pm) \geq \epsilon_{min} > 0$, $\Im(\epsilon_\pm) \geq \epsilon_{min} > 0$ *and, that the constant material properties in* Ω_δ *are such that* $\Re(\epsilon_\delta) > 0$, $\Im(\epsilon_\delta) > 0$ *and* $\mu_\delta > 0$ *(so in particular Assumption 4.3.1 is satisfied). Then* $\mathbf{u} = \mathbf{u}_0 + \nabla p$ *is the unique solution to the problem*

$$a(\mathbf{u}, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathcal{H}_0, \tag{4.88}$$

for all $\mathcal{L} \in \mathcal{H}_0^*$, *where* $p \in S$ *and* $\mathbf{u}_0 \in V$ *are defined by propositions 4.3.2 and 4.4.1, with* $\ell = \mathcal{L}|_{\nabla S}$ *and* $\ell = \mathcal{L}|_{V_0}$, *respectively.*

4.5 Validation of the asymptotic model

In order to validate our approximate model, we have carried out a numerical error analysis based on the comparison between the so called full model and the asymptotic model that we have presented and analyzed throughout this work. For this analysis, we compared the numerical solutions of the full model (4.1)-(4.2) and (4.3) with the numerical solution of the Chun's-type ATCs model (4.20)-(4.24), as the thickness of the delamination δ tends to zero.

In each case, we used a finite element method implemented in the Netgen/Ngsolve package [79].

To approximate the Calderón map G_e, we used a spherical perfectly matched layer (PML) surrounding the obstacle. Then, instead of solving for the total field $\mathbf{E}$ everywhere, we solve for the scattered field $\mathbf{E}^s$ in $B_R \setminus \overline{\Omega}$ and for the total field $\mathbf{E}$ only in Ω. We here describe the variational formulation used for each of the two models (except for the PML which is standard).

We start with **the full model**, which can be written as the simple transmission problem where we seek the fields $\mathbf{E}^s \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathbf{E} \in \mathbf{H}(curl, \Omega)$ that satisfy,

$$\nabla \times \nabla \times \mathbf{E}^s - k^2 \mathbf{E}^s = \mathbf{0} \quad \text{in} \quad \Omega_{ext}, \tag{4.89}$$

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{E}\right) - k^2 \epsilon \mathbf{E} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{4.90}$$

$$\boldsymbol{\nu} \times \mathbf{E}^i + \boldsymbol{\nu} \times \mathbf{E}^s = \boldsymbol{\nu} \times \mathbf{E} \quad \text{on } \Gamma_1 \tag{4.91}$$

$$\boldsymbol{\nu} \times (\nabla \times \mathbf{E}^i) + \boldsymbol{\nu} \times (\nabla \times \mathbf{E}^s) = \boldsymbol{\nu} \times (\nabla \times \mathbf{E}) \quad \text{on } \Gamma_1, \tag{4.92}$$

and, in addition, $\mathbf{E}^s$ satisfies the Silver-Müller radiation condition (4.24).

As usual, to derive the variational formulation we multiply the differential equations by a test function $\mathbf{v}$ and integrating by parts in the ball B_R we get,

$$\begin{aligned}
& \int_{B_R \setminus \overline{\Omega}} \nabla \times \mathbf{E}^s \cdot \nabla \times \overline{\mathbf{v}} - k^2 \mathbf{E}^s \cdot \overline{\mathbf{v}} \;\, \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Omega} \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{v}} - k^2 \epsilon \mathbf{E} \cdot \overline{\mathbf{v}} \;\, \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Gamma_1} \left\langle \mu^{-1} \nabla \times \mathbf{E} \right\rangle \cdot \overline{[\boldsymbol{\nu} \times \mathbf{v}]} \;\, \mathrm{d}s(\mathbf{y}) - \int_{\Gamma_1} \left[\mu^{-1} \nabla \times \mathbf{E}\right] \cdot \langle \boldsymbol{\nu} \times \overline{\mathbf{v}} \rangle \;\, \mathrm{d}s(\mathbf{y}) \\
+ & \; ik \left\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}^s), \mathbf{v} \right\rangle_{S_R} = 0,
\end{aligned} \tag{4.93}$$

where we have substituted the radiation condition (4.24) by the Calderón operator G_e, and the brackets $[\mathbf{u}]$, $\langle \mathbf{u} \rangle$ respectively denote the jump and average values of a function $\mathbf{u}$ at the interface Γ:

$$[\mathbf{u}] = \mathbf{u}_{ext}|_{\Gamma_1} - \mathbf{u}_+|_{\Gamma_1} \quad \text{and} \quad \langle \mathbf{u} \rangle = \frac{\mathbf{u}_{ext}|_{\Gamma_1} + \mathbf{u}_+|_{\Gamma_1}}{2}, \tag{4.94}$$

where $\mathbf{u}|_{\Omega_{ext}} = \mathbf{u}_{ext}$ and $\mathbf{u}|_{\Omega} = \mathbf{u}_+$.

Thus using the transmission condition (4.92),

$$\begin{aligned}
& \int_{B_R \setminus \overline{\Omega}} \nabla \times \mathbf{E}^s \cdot \nabla \times \overline{\mathbf{v}} - k^2 \mathbf{E}^s \cdot \overline{\mathbf{v}} \;\, \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Omega} \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{v}} - k^2 \epsilon \mathbf{E} \cdot \overline{\mathbf{v}} \;\, \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Gamma_1} \mu^{-1} \nabla \times \mathbf{E} \cdot \overline{[\boldsymbol{\nu} \times \mathbf{v}]} \;\, \mathrm{d}s(\mathbf{y}) + ik \left\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}^s), \mathbf{v} \right\rangle_{S_R} \\
= & \; \frac{1}{2} \int_{\Gamma_1} \nabla \times \mathbf{E}^i \cdot \overline{[\boldsymbol{\nu} \times \mathbf{v}]} \;\, \mathrm{d}s(\mathbf{y}) + \int_{\Gamma_1} \nabla \times \mathbf{E}^i \cdot \langle \boldsymbol{\nu} \times \overline{\mathbf{v}} \rangle \;\, \mathrm{d}s(\mathbf{y}).
\end{aligned}$$

On one hand, using a Nitsche's method [69, 60] to enforce the continuity transmission condition (4.91) and on the other hand to get a symmetric form, we add and substract the following two terms to (4.95):

a) The Nitsche's term $\frac{\gamma}{h_{max}} \int_{\Gamma_0} [\mathbf{w}]_T \cdot \overline{[\mathbf{v}]_T} \; \mathrm{d}s(\mathbf{y})$ for a constant $\gamma \in \mathbb{C}$, and mesh resolution h_{max}.

b) A symmetrizing term $\int_{\Gamma_1} [\boldsymbol{\nu} \times \mathbf{w}] \cdot \overline{\mu_+^{-1} \nabla \times \mathbf{v}_+} \; \mathrm{d}s(\mathbf{y})$,

which leads to the variational problem of finding $\mathbf{E}^s \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathbf{E} \in \mathbf{H}(curl, \Omega)$ that satisfy,

$$a_{vol}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) + b_{\Gamma_1}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) = \mathcal{L}_{\Gamma_1} \mathbf{v}, \tag{4.95}$$

for all $\mathbf{v}$ such that $\mathbf{v}|_{\mathbb{R}^3 \setminus \overline{\Omega}} \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathbf{v}|_\Omega \in \mathbf{H}(curl, \Omega)$, where the volume terms are

$$\begin{aligned} a_{vol}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) \;\; &:= \;\; \int_{B_R \setminus \overline{\Omega}} \nabla \times \mathbf{E}^s \cdot \nabla \times \overline{\mathbf{v}} - k^2 \mathbf{E}^s \cdot \overline{\mathbf{v}} \; \mathrm{d}\,\mathbf{y} \\ &+ \;\; \int_\Omega \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{v}} - k^2 \epsilon \mathbf{E} \cdot \overline{\mathbf{v}} \; \mathrm{d}\,\mathbf{y} \\ &+ \;\; ik \left\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}^s), \mathbf{v} \right\rangle_{S_R}, \end{aligned}$$

the transmission terms on Γ_1 correspomd to

$$\begin{aligned} b_{\Gamma_1}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) \;\; &:= \;\; \int_{\Gamma_1} \mu^{-1} \nabla \times \mathbf{E} \cdot \overline{[\boldsymbol{\nu} \times \mathbf{v}]} \; \mathrm{d}s(\mathbf{y}) + \int_{\Gamma_1} [\boldsymbol{\nu} \times \mathbf{E}] \cdot \overline{\mu_+^{-1} \nabla \times \mathbf{v}_+} \; \mathrm{d}s(\mathbf{y}) \\ &+ \;\; \frac{\gamma}{h_{max}} \int_{\Gamma_1} [\mathbf{E}]_T \cdot \overline{[\mathbf{v}]_T} \; \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.96}$$

where $\mu_+^{-1} \nabla \times \mathbf{v}_+$ is the trace taken from the interior of Ω, and the right-hand-side linear form is given by

$$\begin{aligned} \mathcal{L}_{\Gamma_1} \mathbf{v} \;\; &:= \;\; \frac{1}{2} \int_{\Gamma_1} \nabla \times \mathbf{E}^i \cdot \overline{[\boldsymbol{\nu} \times \mathbf{v}]} \; \mathrm{d}s(\mathbf{y}) + \int_{\Gamma_1} \nabla \times \mathbf{E}^i \cdot \overline{\langle \boldsymbol{\nu} \times \mathbf{v} \rangle} \; \mathrm{d}s(\mathbf{y}) \\ &- \;\; \int_{\Gamma_1} \boldsymbol{\nu} \times \mathbf{E}^i \cdot \overline{\mu_+^{-1} \nabla \times \mathbf{v}_+} \; \mathrm{d}s(\mathbf{y}) - \frac{\gamma}{h_{max}} \int_{\Gamma_1} \mathbf{E}^i_T \cdot \overline{[\mathbf{v}]_T} \; \mathrm{d}s(\mathbf{y}). \end{aligned} \tag{4.97}$$

The ATCs model can be written as the problem where we seek the fields $\mathbf{E}^s \in$

$\mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathbf{E} \in \mathcal{H}_0(\Omega \setminus \overline{\Omega_\delta})$, where $\mathcal{H}_0(\Omega \setminus \overline{\Omega_\delta})$ is the space of functions in $\mathcal{H}_0$ restricted to $\Omega \setminus \overline{\Omega_\delta}$, satisfying:

$$\nabla \times \nabla \times \mathbf{E}^s - k^2 \mathbf{E}^s = \mathbf{0} \quad \text{in} \quad \Omega_{ext}, \tag{4.98}$$

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{E}\right) - k^2 \epsilon \mathbf{E} = \mathbf{0} \quad \text{in} \quad \Omega_+ \cup \Omega_-, \tag{4.99}$$

$$[\![\boldsymbol{\nu} \times \mathbf{E}]\!] = \delta\alpha_1 \langle\!\langle \mu^{-1} \nabla \times \mathbf{E} \rangle\!\rangle_T - \delta\beta_1 \overrightarrow{curl}_\Gamma curl_\Gamma \langle\!\langle \mu^{-1} \nabla \times \mathbf{E} \rangle\!\rangle_T \text{ on } \Gamma_0, \tag{4.100}$$

$$[\![\boldsymbol{\nu} \times \left(\mu^{-1} \nabla \times \mathbf{E}\right)]\!] = \delta\alpha_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle - \delta\beta_2 \overrightarrow{curl}_\Gamma \; curl_\Gamma \langle\!\langle \mathbf{E}_T \rangle\!\rangle \text{ on } \Gamma_0, \tag{4.101}$$

$$\mathbf{n} \times \left(\mu^{-1} \nabla \times \mathbf{E}\right) = \mathbf{0} \text{ on } \mathscr{S}, \tag{4.102}$$

and in addition $\mathbf{E}^s$ satisfies the Silver-Müller radiation condition (4.24). Therefore, multiplying the differential equations by a test function $\mathbf{v}$ and integrating by parts, a variational formulation of this problem is: seek $\mathbf{E}^s \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathbf{E} \in \mathcal{H}_0(\Omega \setminus \overline{\Omega_\delta})$ such that

$$a^\delta_{vol}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) + b_{\Gamma_1}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) + b_{\Gamma_0}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) = \mathcal{L}_{\Gamma_1} \mathbf{v}, \tag{4.103}$$

for all $\mathbf{v}$ such that $\mathbf{v}|_{\mathbb{R}^3 \setminus \overline{\Omega}} \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathbf{v}|_\Omega \in \mathcal{H}_0(\Omega \setminus \overline{\Omega_\delta})$, where b_{Γ_1} and $\mathcal{L}_{\Gamma_1}$ are defined by (4.96) and (4.97), respectively, and

$$\begin{aligned} a^\delta_{vol}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) \;&:=\; \int_{B_R \setminus \overline{\Omega}} \nabla \times \mathbf{E}^s \cdot \nabla \times \overline{\mathbf{v}} - k^2 \mathbf{E}^s \cdot \overline{\mathbf{v}} \; \mathrm{d}\,\mathbf{y} \\ &+\; \int_{\Omega \setminus \overline{\Omega_\delta}} \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{v}} - k^2 \epsilon \mathbf{E} \cdot \overline{\mathbf{v}} \; \mathrm{d}\,\mathbf{y} \\ &+\; ik \left\langle G_e(\widehat{\mathbf{x}} \times \mathbf{E}^s), \mathbf{v} \right\rangle_{S_R}, \end{aligned}$$

which is almost the same as (4.96), except for the integration domain of the second integral, and

$$\begin{aligned} b_{\Gamma_0}((\mathbf{E}, \mathbf{E}^s), \mathbf{v}) \;&:=\; \int_{\Gamma_0} \delta\beta_2 curl_\Gamma \langle\!\langle \mathbf{E}_T \rangle\!\rangle \, \overline{curl_\Gamma \langle\!\langle \mathbf{v}_T \rangle\!\rangle} \; \mathrm{d}s(\mathbf{y}) \qquad (4.104) \\ &-\; \int_{\Gamma_0} \delta\alpha_2 \langle\!\langle \mathbf{E}_T \rangle\!\rangle \cdot \langle\!\langle \overline{\mathbf{v}} \rangle\!\rangle_T \; \mathrm{d}s(\mathbf{y}) \\ &+\; \frac{1}{\delta} \int_{\Gamma_0} \mathcal{A}_1^{-1} [\![\boldsymbol{\nu} \times \mathbf{E}]\!] \cdot \overline{[\![\boldsymbol{\nu} \times \mathbf{v}]\!]} \; \mathrm{d}s(\mathbf{y}), \qquad (4.105) \end{aligned}$$

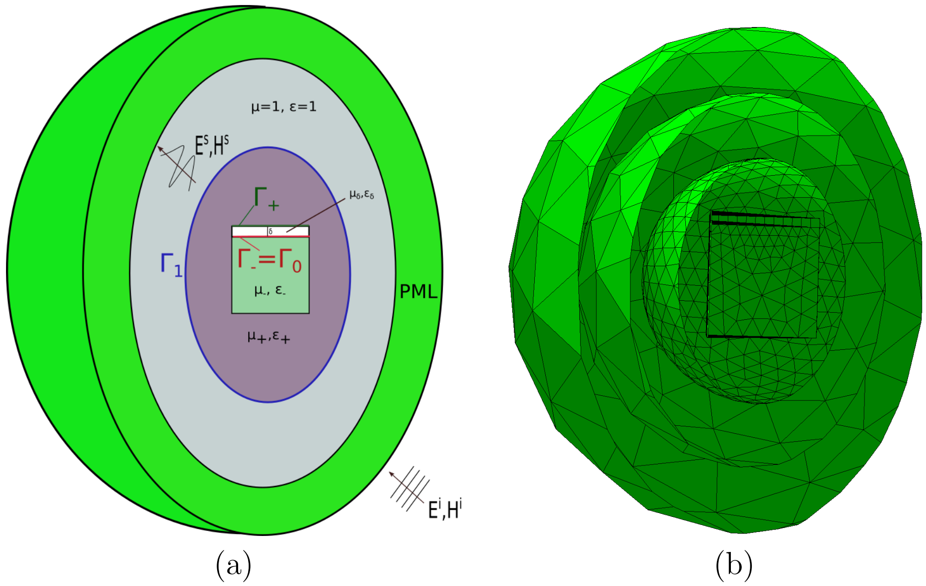


Figure 4.5: Panel (a) Geometrical setting for the numerical experiments for the validation of the ATC model. Panel (b) Mesh generated using Netgen/Ngsolve when $\delta = 0.1$.

are the terms associated with the ATCs.

In summary, the respective finite element solutions to the variational formulations (4.95) and (4.103) will be used to compare in the example of the following subsection the full and the approximate models, in order to validate our approximate model.

4.5.1 The numerical error analysis

In this subsection we describe the numerical experiments that we did to validate our approximate model, as explained in the previous part.

We considered a spherical obstacle Ω as shown in Figure 4.5, where the internal layer Ω_- is a cube. The separation or delamination Ω_δ has constant thickness and is present on one face of the inner layer Ω_-. The radius of the exterior boundary Γ_1 is $r_1 = 1.3$, whereas each face of the cube Γ has side length $r_0 = 1.2$. We chose in this case to set $\Gamma_- = \Gamma$, so that $f^- = 0$ and $f^+ = 1$. The material properties in this experiments were $\mu_+ = 1$, $\mu_- = 1$, $\mu_\delta = 1$, and $\epsilon_+ = 1 + 0.001i$, $\epsilon_- = 1 + 0.001i$, $\epsilon_\delta = 3.5 + 0.001i$.

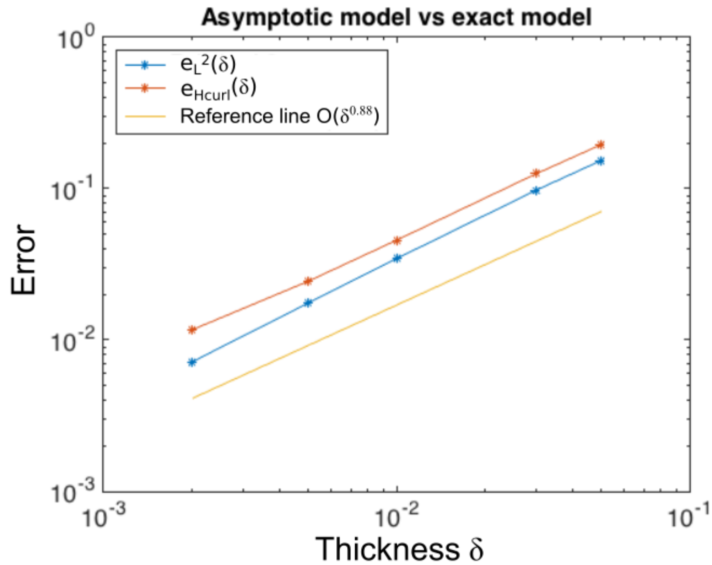


Figure 4.6: $L^2(B_R)^3$ and $\mathbf{H}(curl,\, B_R)$ relative errors, respectively, of the total fields resulting from different values of δ. In both cases the approximate rate of convergence is $O(\delta^{0.9})$.

As an incident field, we chose a plane wave $\mathbf{E}^i = \mathbf{p}e^{ik\widehat{\mathbf{d}}\cdot\mathbf{x}}$, where the wave number is $k = 3$, the direction of propagation is $\widehat{\mathbf{d}} = (0, 0, 1)$, and the polarization vector $\mathbf{p} = (1, 0, 0)$.

Finally, a spherical Perfectly Matched Layer (PML) was used in the annular region $\{\mathbf{x} : 2 < |\mathbf{x}| < 2.7\}$, with absorbing parameter $\alpha = 0.6$. It is important to mention here that for the numerical implementation of the model, it was impossible for us to compute the pointwise difference and sum of the traces of $\mathbf{E}^+$ and $\mathbf{E}^-$, in the two different boundaries Γ_+ and Γ_-. Thus instead of considering the jump and average values $[\![\cdot]\!]$ and $\langle\!\langle\cdot\rangle\!\rangle$ in (4.104), we computed the traces on the same boundary, Γ_-, i.e. we substituted the jump and mean-value by:

$$\langle u \rangle_{\Gamma_-} = \frac{1}{2}(u^+|_{\Gamma_-} + u^-|_{\Gamma_-}), \text{ and } [u]_{\Gamma_-} = u^+|_{\Gamma_-} - u^-|_{\Gamma_-}.$$

In Figure 4.6, the two relative $L^2(B_R)^3$ and $\mathbf{H}(curl, B_R)$ errors, respectively defined by:

$$e_{L^2}(\delta) = \frac{||\mathbf{E}_{ATC} - \mathbf{E}_{full}||_{L^2(B_R)}}{||\mathbf{E}_{full}||_{L^2(B_R)}} \quad \text{and} \quad e_{Hcurl}(\delta) = \frac{||\mathbf{E}_{ATC} - \mathbf{E}_{full}||_{\mathbf{H}(curl,B_R)}}{||\mathbf{E}_{full}||_{\mathbf{H}(curl,B_R)}}, \tag{4.106}$$

were computed.

It is shown that both errors have decaying behavior and an approximate rate of convergence of order $O(\delta^{0.9})$. Although by definition of the second order ATCs we expect an error of order $O(\delta^2)$, the observed reduced rate of convergence may be due to the fact that we changed the jump and average value computations on Γ_0. However, at least in this case we can observe that the corresponding ATCs model is increasingly accurate as δ decreases and may be used to develop inverse scattering results.

An alternative possibility to explain the drop in the convergence rate of the approximate model may be that the assumption $\mathbf{n} \times (\mu^{-1}\nabla \times \mathbf{E}) = \mathbf{0}$ on $\mathscr{S}$, is not of the appropriate order of accuracy. This is a line of research that should be further investigated.

4.6 Inverse problem

4.6.1 Reciprocity and mixed reciprocity principles

For the upcoming development of our nondestructive test for the detection of the delamination Γ_0, it will be important to introduce some concepts. It is well known (see for example [32] or [23]) that if $(\mathbf{E}, \mathbf{H})$ are radiating solutions of the homogeneous Maxwell equations

$$\nabla \times \mathbf{E} - ik\mathbf{H} = \mathbf{0} \tag{4.107}$$

$$\nabla \times \mathbf{H} + ik\mathbf{E} = \mathbf{0}. \tag{4.108}$$

in $\mathbb{R}^3 \setminus \overline{\Omega}$, where the boundary of Γ_1 is $\mathcal{C}^2$, then there exist analytic functions $(\mathbf{E}^\infty, \mathbf{H}^\infty)$ defined on the sphere $\mathbb{S}^2$, such that the following asymptotic expressions hold:

$$\mathbf{E}(\mathbf{x}) = \frac{e^{ikr}}{r}\mathbf{E}^\infty(\widehat{\mathbf{x}}) + \mathcal{O}\left(\frac{1}{r^2}\right) \text{ when } r \to \infty, \tag{4.109}$$

$$\mathbf{H}(\mathbf{x}) = \frac{e^{ikr}}{r}\mathbf{H}^\infty(\widehat{\mathbf{x}}) + \mathcal{O}\left(\frac{1}{r^2}\right) \text{ when } r \to \infty, \tag{4.110}$$

where $r = |\mathbf{x}|$, $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$, and the convergence is uniform in $\widehat{\mathbf{x}}$. The functions $(\mathbf{E}^\infty, \mathbf{H}^\infty)$ are called the electric and magnetic far field patterns, respectively, and it can be proved that (see Theorem 1.4 in [23]) they have the following expressions:

$$\begin{aligned} \mathbf{E}^\infty(\widehat{\mathbf{x}}) &= \frac{ik}{4\pi}\widehat{\mathbf{x}} \times \int_{\Gamma_1} \{(\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}(\mathbf{y})) + (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}(\mathbf{y}) \times \widehat{\mathbf{x}}) e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}\} \ \mathrm{d}s(\mathbf{y}), \\ \mathbf{H}^\infty(\widehat{\mathbf{x}}) &= \frac{ik}{4\pi}\widehat{\mathbf{x}} \times \int_{\Gamma_1} \{(\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})) - (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}(\mathbf{y}) \times \widehat{\mathbf{x}}) e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}\} \ \mathrm{d}s(\mathbf{y}). \end{aligned}$$

We will define the solution of the so called *background* problem to be the unique solution $\mathbf{E}_b$ in $\mathbf{H}_{loc}(curl, \mathbb{R}^3)$ such that:

$$\nabla \times \mathbf{E}_b - ik\mu\mathbf{H}_b = \mathbf{0} \quad \text{in} \quad \mathbb{R}^3, \tag{4.111}$$

$$\nabla \times \mathbf{H}_b + ik\epsilon\mathbf{E}_b = \mathbf{0} \quad \text{in} \quad \mathbb{R}^3, \tag{4.112}$$

where, again, $\mathbf{E}_b = \mathbf{E}_b^s + \mathbf{E}^i$ in Ω_{ext}, $\mathbf{E}^i$ is the incident field, and $\mathbf{E}_b^s$ is the scattered field that satisfies the Silver-Müller radiation condition (4.24).

The background solutions are the electric and magnetic fields associated with the healthy material (i.e. when the delamination is not present). Notice that in the definition of $\mathcal{P}_B$ it is implicit that the tangential components of the fields are continuous across all interfaces.

We will now define two important well-known families of solutions to the homogeneous Maxwell equations (4.107)-(4.108).

First, for a given vector $\mathbf{p} \in \mathbb{R}^3$, we define the electromagnetic field generated by an *electric dipole* with polarization $\mathbf{p}$, $(\mathbf{E}^i_{edp}(\cdot, \cdot, \mathbf{p}), \mathbf{H}^i_{edp}(\cdot, \cdot, \mathbf{p}))$, is given by:

$$\begin{cases} \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{p}) = -\frac{1}{ik}\nabla_y \times \nabla_y \times (\mathbf{p}\phi(\mathbf{y}, \mathbf{z}))\,, \\ \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{p}) = \nabla_y \times (\mathbf{p}\phi(\mathbf{y}, \mathbf{z}))\,, \end{cases}$$

where $\phi(\mathbf{y}, \mathbf{z}) = \frac{e^{ik|\mathbf{y}-\mathbf{z}|}}{4\pi|\mathbf{y}-\mathbf{z}|}$ is the radiating fundamental solution of the Helmholtz equation. It is well known that the electromagnetic pair $(\mathbf{E}, \mathbf{H}) = (\mathbf{E}^i_{edp}(\cdot, \mathbf{z}, \mathbf{p}), \mathbf{H}^i_{edp}(\cdot, \mathbf{z}, \mathbf{p}))$ is the radiating fundamental solution of the homogeneous Maxwell equations (see [23]) satisfying,

$$\nabla \times \mathbf{E} - ik\mathbf{H} = \mathbf{0} \text{ in } \mathbb{R}^3, \tag{4.113}$$

$$\nabla \times \mathbf{H} + ik\mathbf{E} = \mathbf{p}\delta(\cdot - \mathbf{z}) \text{ in } \mathbb{R}^3. \tag{4.114}$$

Second, given a direction vector $\widehat{\mathbf{d}} \in \mathbb{S}^2$ and a polarization vector $\mathbf{p} \in \mathbb{R}^3$ the corresponding electromagnetic plane wave $(\mathbf{E}^i_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}), \mathbf{H}^i_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}))$ is defined by:

$$\begin{cases} \mathbf{E}^i_{pl}(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{p}) = ik((\widehat{\mathbf{d}} \times \mathbf{p}) \times \widehat{\mathbf{d}})e^{ik\widehat{\mathbf{d}}\cdot\mathbf{y}}, \\ \mathbf{H}^i_{pl}(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{p}) = ik(\widehat{\mathbf{d}} \times \mathbf{p})e^{ik\widehat{\mathbf{d}}\cdot\mathbf{y}}. \end{cases}$$

The pair $(\mathbf{E}, \mathbf{H}) = (\mathbf{E}^i_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}), \mathbf{H}^i_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}))$ is an entire solution of the homogeneous Maxwell equations (4.107)-(4.108) in $\mathbb{R}^3$.

In the particular case when the incident field is a plane wave $\mathbf{E}^i = \mathbf{E}^i_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$, the solution to the background problem (4.111)-(4.112) will be denoted by $\mathbf{E}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$, and the corresponding scattered field by $\mathbf{E}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) \in \mathbf{H}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$.
Notice that if we define

$$\mathbf{H}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) := \frac{1}{ik}\nabla \times \mathbf{E}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}),$$

then $(\mathbf{E}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}), \mathbf{H}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}))$ solves (4.113)-(4.114) in $\mathbb{R}^3\setminus\overline{\Omega}$. The corresponding total field $\mathbf{H}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$ is defined as usual by $\mathbf{H}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) = \mathbf{H}^i_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) + \mathbf{H}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p})$. Analogously, when the incident field is an electric dipole $\mathbf{E}^i = \mathbf{E}^i_{edp}(\cdot, \cdot, \mathbf{p})$, the solution to the background problem (4.111)-(4.112) will be denoted by $\mathbf{E}_{edp}(\cdot, \cdot, \mathbf{p})$, and the corresponding scattered field by $\mathbf{E}^s_{edp}(\cdot, \cdot, \mathbf{p}) \in \mathbf{H}(curl, \mathbb{R}^3 \setminus \overline{\Omega})$.
Again, if $\mathbf{H}^s_{edp}(\cdot, \cdot, \mathbf{p}) := \frac{1}{ik}\nabla\times\mathbf{E}^s_{edp}(\cdot, \cdot, \mathbf{p})$, then $(\mathbf{E}^s_{edp}(\cdot, \cdot, \mathbf{p}), \mathbf{H}^s_{edp}(\cdot, \cdot, \mathbf{p}))$ solves (4.113)-(4.114) in $\mathbb{R}^3\setminus\overline{\Omega}$. and the corresponding total field $\mathbf{H}_{edp}(\cdot, \cdot, \mathbf{p})$ is, as usual, $\mathbf{H}_{edp}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) = \mathbf{H}^i_{edp}(\cdot, \cdot, \mathbf{p}) + \mathbf{H}^s_{edp}(\cdot, \cdot, \mathbf{p})$ in Ω_{ext}.
Finally, the radiating electromagnetic Green's tensor associated with the background medium is the generalized electric dipole, defined as the pair of second order tensors $(\mathbb{G}^E, \mathbb{G}^H)$ that for any constant vector $\mathbf{p} \in \mathbb{R}^3$ and $\mathbf{z} \in \mathbb{R}^3$, the corresponding fields

$(\mathbb{G}^E(\cdot,\mathbf{z})\mathbf{p}, \mathbb{G}^H(\cdot,\mathbf{z})\mathbf{p}) \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \{\mathbf{z}\}) \times \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \{\mathbf{z}\})$ solve the following problem:

$$\begin{cases} \nabla_y \times (\mathbb{G}^E(\cdot,\mathbf{z})\mathbf{p}) - ik\mu\mathbb{G}^H(\cdot,\mathbf{z})\mathbf{p} = \mathbf{0} \text{ in } \mathbb{R}^3, \\ \nabla_y \times (\mathbb{G}^H(\cdot,\mathbf{z})\mathbf{p}) + ik\epsilon\mathbb{G}^E(\cdot,\mathbf{z})\mathbf{p} = \mathbf{p}\delta(\cdot - \mathbf{z}) \text{ in } \mathbb{R}^3, \\ \lim_{r\to\infty} r((\mathbb{G}^H(\mathbf{x},\mathbf{z})\mathbf{p}) \times \widehat{\mathbf{x}} - ik\mathbb{G}^E(\mathbf{x},\mathbf{z})\mathbf{p}) = 0. \end{cases}$$

We are ready to prove a mixed reciprocity principle, similar to those presented in [75] for the electromagnetic case, and in [46],[17],[27] or [24] in the acoustic case.

Theorem 4.6.1. *(Mixed reciprocity principle) For all $\widehat{\mathbf{x}} \in \mathbb{S}^2$ and all $\mathbf{z} \in \mathbb{R}^3 \setminus (\Gamma \cup \Gamma_1)$,*

$$4\pi\mathbf{p} \cdot \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}},\mathbf{z})\mathbf{q} = \mathbf{q} \cdot \mathbf{E}_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}), \tag{4.115}$$

for all $\mathbf{q},\mathbf{p} \in \mathbb{R}^3$. Moreover, for $\mathbf{z} \in \Gamma \cup \Gamma_1$, then the identity (4.115) is true if $\mathbf{q} \cdot \boldsymbol{\nu}(\mathbf{z}) = 0$ and $\mathbf{p} \cdot \boldsymbol{\nu}(\mathbf{z}) = 0$.

Proof. Consider $\mathbf{p},\mathbf{q} \in \mathbb{R}^3$, and the following cases:

Case 1. $\mathbf{z} \in \mathbb{R}^3 \setminus \overline{\Omega}$.

Since in this case the pair

$$(\mathbf{E},\mathbf{H}) = \left(\mathbb{G}^E(\cdot,\mathbf{z})\mathbf{q} - \mathbf{E}^i_{edp}(\cdot,\mathbf{z},\mathbf{q}),\ \mathbb{G}^H(\cdot,\mathbf{z})\mathbf{q} - \mathbf{H}^i_{edp}(\cdot,\mathbf{z},\mathbf{q})\right)$$

is a radiating and non-singular solution of the homogeneous Maxwell equations (4.107)-(4.108) in $\mathbb{R}^3 \setminus \overline{\Omega}$, we can use the Stratton-Chu formula for radiating fields (see [65], Theorem 9.4):

$$\begin{aligned} \mathbb{G}^E(\mathbf{x},\mathbf{z})\mathbf{q} - \mathbf{E}^i_{edp}(\mathbf{x},\mathbf{z},\mathbf{q}) \quad &= \quad \nabla_x \times \int_{\Gamma_1} \boldsymbol{\nu}(\mathbf{y}) \times (\mathbb{G}^E(\mathbf{y},\mathbf{z})\mathbf{q} - \mathbf{E}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q}))\phi(\mathbf{x},\mathbf{y})\ \mathrm{d}s(\mathbf{y}) \\ &- \frac{1}{ik}\nabla_x \times \nabla_x \times \int_{\Gamma_1} \boldsymbol{\nu}(\mathbf{y}) \times (\mathbb{G}^H(\mathbf{y},\mathbf{z})\mathbf{q} - \mathbf{H}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q}))\phi(\mathbf{x},\mathbf{y})\ \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.116}$$

for all $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$.

On the other hand, for any constant $\mathbf{p} \in \mathbb{R}^3$, taking the dot product of $\mathbf{p}$ with the terms in the right-hand-side of the Stratton-Chu formula (4.116) involving the electric dipole fields $\mathbf{E}^i_{edp}$ and $\mathbf{H}^i_{edp}$, we get:

$$\begin{aligned}
& \mathbf{p} \cdot \nabla_x \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q}))\phi(\mathbf{x}, \mathbf{y}) \ \mathrm{d}s(\mathbf{y}) \\
- \ & \frac{1}{ik}\mathbf{p} \cdot \nabla_x \times \nabla_x \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q}))\phi(\mathbf{x}, \mathbf{y}) \ \mathrm{d}s(\mathbf{y}) = \\
- \ & \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \nabla_y \times (\mathbf{p}\phi(\mathbf{x}, \mathbf{y})) \ \mathrm{d}s(\mathbf{y}) \\
- \ & \frac{1}{ik}\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \nabla_y \times \nabla_y \times (\mathbf{p}\phi(\mathbf{x}, \mathbf{y})) \ \mathrm{d}s(\mathbf{y}) \\
= \ & \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \mathbf{H}^i_{edp}(\mathbf{x}, \mathbf{y}, \mathbf{p}) \ \mathrm{d}s(\mathbf{y}) \\
+ \ & \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \mathbf{E}^i_{edp}(\mathbf{x}, \mathbf{y}, \mathbf{p}) \ \mathrm{d}s(\mathbf{y}),
\end{aligned} \tag{4.117}$$

thus,

$$\begin{aligned}
& \mathbf{p} \cdot \nabla_x \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q}))\phi(\mathbf{x}, \mathbf{y}) \ \mathrm{d}s(\mathbf{y}) \\
- \ & \frac{1}{ik}\mathbf{p} \cdot \nabla_x \times \nabla_x \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q}))\phi(\mathbf{x}, \mathbf{y}) \ \mathrm{d}s(\mathbf{y}) = \\
- \ & \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \nabla_y \times (\mathbf{p}\phi(\mathbf{x}, \mathbf{y})) \ \mathrm{d}s(\mathbf{y}) \\
- \ & \frac{1}{ik}\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \nabla_y \times \nabla_y \times (\mathbf{p}\phi(\mathbf{x}, \mathbf{y})) \ \mathrm{d}s(\mathbf{y}) \\
= \ & \int_{\Omega} \{\nabla \times \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \mathbf{H}^i_{edp}(\mathbf{x}, \mathbf{y}, \mathbf{p})\} \ \ \mathrm{d}\,\mathbf{y} \\
- \ & \int_{\Omega} \{(\nabla \times \mathbf{H}^i_{edp}(\mathbf{x}, \mathbf{y}, \mathbf{p})) \cdot \mathbf{E}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})\} \ \ \mathrm{d}\,\mathbf{y} \\
+ \ & \int_{\Omega} \{\nabla \times \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})) \cdot \mathbf{E}^i_{edp}(\mathbf{x}, \mathbf{y}, \mathbf{p})\} \ \ \mathrm{d}\,\mathbf{y} \\
- \ & \int_{\Omega} \{(\nabla \times \mathbf{E}^i_{edp}(\mathbf{x}, \mathbf{y}, \mathbf{p})) \cdot \mathbf{H}^i_{edp}(\mathbf{y}, \mathbf{z}, \mathbf{q})\} \ \ \mathrm{d}\,\mathbf{y},
\end{aligned} \tag{4.118}$$

and therefore,

$$
\begin{aligned}
& \mathbf{p}\cdot\nabla_x\times\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbf{E}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q}))\phi(\mathbf{x},\mathbf{y})\ \mathrm{d}s(\mathbf{y}) \\
- & \frac{1}{ik}\mathbf{p}\cdot\nabla_x\times\nabla_x\times\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbf{H}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q}))\phi(\mathbf{x},\mathbf{y})\ \mathrm{d}s(\mathbf{y}) = \\
- & \int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbf{E}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q}))\cdot\nabla_y\times(\mathbf{p}\phi(\mathbf{x},\mathbf{y}))\ \mathrm{d}s(\mathbf{y}) \\
- & \frac{1}{ik}\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbf{H}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q}))\cdot\nabla_y\times\nabla_y\times(\mathbf{p}\phi(\mathbf{x},\mathbf{y}))\ \mathrm{d}s(\mathbf{y}) \\
= & \int_{\Omega}\{ik\mathbf{H}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q})\cdot\mathbf{H}^i_{edp}(\mathbf{x},\mathbf{y},\mathbf{p})\}\ \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Omega}\{ik\mathbf{E}^i_{edp}(\mathbf{x},\mathbf{y},\mathbf{p})\cdot\mathbf{E}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q})\}\ \mathrm{d}\,\mathbf{y} \\
+ & \int_{\Omega}\{-ik\mathbf{E}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q})\cdot\mathbf{E}^i_{edp}(\mathbf{x},\mathbf{y},\mathbf{p})\}\ \mathrm{d}\,\mathbf{y} \\
- & \int_{\Omega}\{ik\mathbf{H}^i_{edp}(\mathbf{x},\mathbf{y},\mathbf{p})\cdot\mathbf{H}^i_{edp}(\mathbf{y},\mathbf{z},\mathbf{q})\}\ \mathrm{d}\,\mathbf{y}=0,
\end{aligned}
\tag{4.119}
$$

which implies that (4.116) simplifies to

$$
\begin{aligned}
\mathbb{G}^E(\mathbf{x},\mathbf{z})\mathbf{q}-\mathbf{E}^i_{edp}(\mathbf{x},\mathbf{z},\mathbf{q}) &= \nabla_x\times\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^E(\mathbf{y},\mathbf{z})\mathbf{q})\phi(\mathbf{x},\mathbf{y})\ \mathrm{d}s(\mathbf{y}) \\
& -\frac{1}{ik}\nabla_x\times\nabla_x\times\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^H(\mathbf{y},\mathbf{z})\mathbf{q})\phi(\mathbf{x},\mathbf{y})\ \mathrm{d}s(\mathbf{y}),
\end{aligned}
\tag{4.120}
$$

for all $\mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega}$. Therefore, by taking the dot product of a constant vector $\mathbf{p}\in\mathbb{R}^3$ with (4.120), we know that the far field patterns satisfy

$$
\begin{aligned}
& \mathbf{p}\cdot(\mathbb{G}^{E,\infty}(\widehat{\mathbf{x}},\mathbf{z})\mathbf{q})-\mathbf{p}\cdot\mathbf{E}^{i,\infty}_{edp}(\widehat{\mathbf{x}},\mathbf{z},\mathbf{q}) \\
= & \frac{ik}{2\pi}\mathbf{p}\cdot\{\widehat{\mathbf{x}}\times\int_{\Gamma_1}\{(\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^E(\mathbf{y},\mathbf{z})\mathbf{q})+(\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^H(\mathbf{y},\mathbf{z})\mathbf{q})\times\widehat{\mathbf{x}}\}e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}\ \mathrm{d}s(\mathbf{y})\} \\
= & \frac{ik}{2\pi}\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^E(\mathbf{y},\mathbf{z})\mathbf{q})\cdot(-\widehat{\mathbf{x}}\times\mathbf{p})e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}} \\
+ & (\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^H(\mathbf{y},\mathbf{z})\mathbf{q})\cdot((-\widehat{\mathbf{x}})\times(\mathbf{p}\times(-\widehat{\mathbf{x}}))e^{-ik\widehat{\mathbf{x}}\cdot\mathbf{y}}\ \mathrm{d}s(\mathbf{y}) \\
= & \frac{1}{2\pi}\int_{\Gamma_1}(\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^E(\mathbf{y},\mathbf{z})\mathbf{q})\cdot\mathbf{H}^i_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{p}) \\
+ & (\boldsymbol{\nu}(\mathbf{y})\times\mathbb{G}^H(\mathbf{y},\mathbf{z})\mathbf{q})\cdot\mathbf{E}^i_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{p})\ \mathrm{d}s(\mathbf{y}),
\end{aligned}
\tag{4.121}
$$

for all $\widehat{\mathbf{x}} \in \mathbb{S}^2$. On the other hand,

$$\begin{aligned}
&\mathbf{p} \cdot \mathbf{E}^{i,\infty}_{edp}(\widehat{\mathbf{x}}, \mathbf{z}, \mathbf{q}) \\
&\quad = -\frac{1}{i4\pi k}\mathbf{p} \cdot \nabla_z \times \nabla_z \times (\mathbf{q}e^{-ik\widehat{\mathbf{x}}\cdot z}) \\
&\quad = \frac{ik}{4\pi}\mathbf{p} \cdot (-\widehat{\mathbf{x}} \times (-\widehat{\mathbf{x}} \times \mathbf{q}))e^{-ik\widehat{\mathbf{x}}\cdot z} \\
&\quad = \frac{ik}{4\pi}\mathbf{q} \cdot ((-\widehat{\mathbf{x}} \times \mathbf{p}) \times (-\widehat{\mathbf{x}}))e^{-ik\widehat{\mathbf{x}}\cdot z} \\
&\quad = \frac{1}{4\pi}\mathbf{q} \cdot \mathbf{E}^i_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}), \qquad (4.122)
\end{aligned}$$

$$(4.123)$$

and

$$\begin{aligned}
&\mathbf{p} \cdot \mathbf{H}^{i,\infty}_{edp}(\widehat{\mathbf{x}}, \mathbf{z}, \mathbf{q}) \\
&\quad = -\frac{1}{4\pi}\mathbf{p} \cdot \nabla_z \times (\mathbf{q}e^{-ik\widehat{\mathbf{x}}\cdot z}) \\
&\quad = -\frac{ik}{4\pi}\mathbf{p} \cdot (-\widehat{\mathbf{x}} \times \mathbf{q})e^{-ik\widehat{\mathbf{x}}\cdot z} \\
&\quad = \frac{ik}{4\pi}\mathbf{q} \cdot (-\widehat{\mathbf{x}} \times \mathbf{p})e^{-ik\widehat{\mathbf{x}}\cdot z} \\
&\quad = \frac{1}{4\pi}\mathbf{q} \cdot \mathbf{H}^i_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}). \qquad (4.124)
\end{aligned}$$

Hence, (4.121) can be written as

$$\begin{aligned}
&\mathbf{p} \cdot (\mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{z})\mathbf{q}) - \frac{ik}{4\pi}\mathbf{q} \cdot \mathbf{E}^i_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}) \\
&\quad = \frac{1}{4\pi}\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{H}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \;\mathrm{d}s(\mathbf{y}) \\
&\quad + \frac{1}{4\pi}\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{E}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \;\mathrm{d}s(\mathbf{y}), \qquad (4.125)
\end{aligned}$$

for all $\widehat{\mathbf{x}} \in \mathbb{S}^2$. We will now show that the right hand side of (4.125) is exactly $\frac{1}{4\pi}\mathbf{q} \cdot \mathbf{E}^s_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p})$.

Observe that on one hand, by Green's formula, for any two given solutions $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ to the homogeneous Maxwell's equations (4.107)-(4.108) in Ω,

$$\begin{aligned}
&\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_1(\mathbf{y})) \cdot \mathbf{H}_2(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) + \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_1(\mathbf{y})) \cdot \mathbf{E}_2(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) \\
&= \int_{\Omega} \{\mu^{-1} \nabla_y \times \mathbf{E}_1(\mathbf{y}) \cdot \mathbf{H}_2(\mathbf{y}) - \nabla_y \times \mathbf{H}_2(\mathbf{y}) \cdot \mathbf{E}_1(\mathbf{y})\} \; \mathrm{d}\,\mathbf{y} + \\
&\quad \int_{\Omega} \{\nabla_y \times \mathbf{H}_1(\mathbf{y}) \cdot \mathbf{E}_2(\mathbf{y}) - \nabla_y \times \mathbf{E}_2(\mathbf{y}) \cdot \mathbf{H}_1(\mathbf{y})\} \; \mathrm{d}\,\mathbf{y} \\
&= \mathbf{0}, \qquad (4.126)
\end{aligned}$$

while on the other hand, if both $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ satisfy the background problem (4.111)-(4.112) in Ω,

$$\begin{aligned}
&\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_1(\mathbf{y}) \cdot \mathbf{H}_2(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) + \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_1(\mathbf{y}) \cdot \mathbf{E}_2(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) \\
&= \int_{\Omega} \{\nabla_y \times \mathbf{E}_1(\mathbf{y}) \cdot \mathbf{H}_2(\mathbf{y}) - \nabla_y \times \mathbf{H}_2(\mathbf{y}) \cdot \mathbf{E}_1(\mathbf{y})\} \; \mathrm{d}\,\mathbf{y} + \\
&\quad \int_{\Omega} \{\nabla_y \times \mathbf{H}_1(\mathbf{y}) \cdot \mathbf{E}_2(\mathbf{y}) - \nabla_y \times \mathbf{E}_2(\mathbf{y}) \cdot \mathbf{H}_1(\mathbf{y})\} \; \mathrm{d}\,\mathbf{y} = \mathbf{0}. \qquad (4.127)
\end{aligned}$$

Therefore, for any $\mathbf{q} \in \mathbb{R}^3$ constant, by the second Stratton-Chu formula,

$$\begin{aligned}
&\mathbf{q} \cdot \mathbf{E}^s_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}) \\
&= \mathbf{q} \cdot \nabla_z \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^s_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}))\phi(\mathbf{z}, \mathbf{y}) \; \mathrm{d}s(\mathbf{y}) \\
&- \frac{1}{ik} \mathbf{q} \cdot \nabla_z \times \nabla_z \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^s_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}))\phi(\mathbf{z}, \mathbf{y}) \; \mathrm{d}s(\mathbf{y}) \\
&= \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^s_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbf{H}^i_{edp}(\mathbf{z}, \mathbf{y}, \mathbf{q}) \; \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^s_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbf{E}^i_{edp}(\mathbf{z}, \mathbf{y}, \mathbf{q}) \; \mathrm{d}s(\mathbf{y}),
\end{aligned}$$

and if we use (4.126) with $(\mathbf{E}_1, \mathbf{H}_1) = (\mathbf{E}^i_{pl}, \mathbf{H}^i_{pl})$ and $(\mathbf{E}_2, \mathbf{H}_2) = (\mathbf{E}^i_{edp}, \mathbf{H}^i_{edp})$, and (4.127) with $\mathbf{E}_1 = \mathbb{G}^E(\cdot, \mathbf{z})\mathbf{p} - \mathbf{E}^i_{edp}(\mathbf{z}, \cdot, \mathbf{p})$, , $\mathbf{H}_1 = \mathbb{G}^H(\cdot, \mathbf{z})\mathbf{p} - \mathbf{H}^i_{edp}(\mathbf{z}, \cdot, \mathbf{p})$ and $(\mathbf{E}_2, \mathbf{H}_2) =$

$(\mathbf{E}_{pl}, \mathbf{H}_{pl})$,

$$\begin{aligned}
&\mathbf{q} \cdot \mathbf{E}^s_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}) \\
&= \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbf{H}^i_{edp}(\mathbf{z}, \mathbf{y}, \mathbf{q}) \;\, \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbf{E}^i_{edp}(\mathbf{z}, \mathbf{y}, \mathbf{q}) \;\, \mathrm{d}s(\mathbf{y}) \\
&- \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbf{H}^i_{edp}(\mathbf{z}, \mathbf{y}, \mathbf{q}) \;\, \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbf{E}^i_{edp}(\mathbf{z}, \mathbf{y}, \mathbf{q}) \;\, \mathrm{d}s(\mathbf{y}) \\
&= \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot (\mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \;\, \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot (\mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \;\, \mathrm{d}s(\mathbf{y}) \\
&= \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{E}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \;\, \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{H}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \;\, \mathrm{d}s(\mathbf{y}), \qquad (4.128)
\end{aligned}$$

Therefore, combining (4.125) and (4.128),

$$4\pi \mathbf{p} \cdot (\mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{z})\mathbf{q}) = \mathbf{q} \cdot \mathbf{E}^i_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}) + \mathbf{q} \cdot \mathbf{E}^s_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}) = \mathbf{q} \cdot \mathbf{E}_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}), \quad (4.129)$$

for all $\mathbf{z} \in \mathbb{R}^3 \setminus \overline{\Omega}$. Thus completes the proof for case 1.

Case 2. Let $\mathbf{z} \in \Omega$.

Then the field $(\mathbb{G}^E(\cdot, \mathbf{z})\mathbf{q}, \mathbb{G}^H(\cdot, \mathbf{z})\mathbf{q})$ is a non-singular radiating solution of the homogeneous Maxwell equations (4.107)-(4.108) in $\mathbb{R}^3 \setminus \overline{\Omega}$, and then taking the dot product of $\mathbf{p} \in \mathbb{R}^3$ with the Stratton-Chu formula of $\mathbb{G}^E(\mathbf{x}, \mathbf{z})\mathbf{q}$ for any $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$,

$$\begin{aligned}
\mathbf{p} \cdot \mathbb{G}^E(\mathbf{x}, \mathbf{z})\mathbf{q} \;&=\; \mathbf{p} \cdot \nabla_x \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q})\phi(\mathbf{x}, \mathbf{y}) \;\, \mathrm{d}s(\mathbf{y}) \\
&\quad -\frac{1}{ik}\mathbf{p} \cdot \nabla_x \times \nabla_x \times \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q})\phi(\mathbf{x}, \mathbf{y}) \;\, \mathrm{d}s(\mathbf{y}) \\
&\quad = -\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \nabla_y \times (\phi(\mathbf{x}, \mathbf{y})\mathbf{p}) \;\, \mathrm{d}s(\mathbf{y}) \\
&\quad -\frac{1}{ik}\int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \nabla_y \times \nabla_y \times (\phi(\mathbf{x}, \mathbf{y})\mathbf{p}) \;\, \mathrm{d}s(\mathbf{y}),
\end{aligned}$$

so the far field pattern satisfies

$$
\begin{aligned}
& \mathbf{p} \cdot \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{z})\mathbf{q} \\
= & \ \frac{1}{4\pi} \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{H}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \mathrm{d}s(\mathbf{y}) \\
+ & \ \frac{1}{4\pi} \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{E}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \mathrm{d}s(\mathbf{y}) \\
= & \ \frac{1}{4\pi} \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \mathrm{d}s(\mathbf{y}) \\
+ & \ \frac{1}{4\pi} \int_{\Gamma_1} (\boldsymbol{\nu}(\mathbf{y}) \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \mathrm{d}s(\mathbf{y}) \\
= & \ \frac{1}{4\pi} \int_{\Omega} (\nabla_y \times \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \ \mathrm{d}\,\mathbf{y} \\
- & \ \frac{1}{4\pi} \int_{\Omega} (\nabla_y \times \mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q} \ \ \mathrm{d}\,\mathbf{y} \\
+ & \ \frac{1}{4\pi} \int_{\Omega} (\nabla_y \times \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \ \mathrm{d}\,\mathbf{y} \\
- & \ \frac{1}{4\pi} \int_{\Omega} (\nabla_y \times \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p})) \cdot \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q} \ \ \mathrm{d}\,\mathbf{y},
\end{aligned}
$$

where in the second equality we have used the fact that for every two radiating solutions of (4.107)-(4.108) in $\mathbb{R}^3 \setminus \overline{\Omega}$, $(\mathbf{E}^s_1, \mathbf{H}^s_1)$ and $(\mathbf{E}^s_2, \mathbf{H}^s_2)$, it is true that

$$
0 \ = \ \int_{\Gamma_1} \{(\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^s_1) \cdot \mathbf{H}^s_2 + (\boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^s_1) \cdot \mathbf{E}^s_2\} \ \mathrm{d}s(\mathbf{y}). \tag{4.130}
$$

Therefore, from (4.130),

$$
\begin{aligned}
& \mathbf{p} \cdot \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{z})\mathbf{q} \\
= & \ \frac{1}{4\pi} \int_{\Omega} ik\mu(\mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \ \mathrm{d}\,\mathbf{y} \\
+ & \ \frac{1}{4\pi} \int_{\Omega} ik\epsilon\mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \cdot \mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q} \ \ \mathrm{d}\,\mathbf{y} \\
+ & \ \frac{1}{4\pi} \int_{\Omega} (-ik\epsilon\mathbb{G}^E(\mathbf{y}, \mathbf{z})\mathbf{q} + \mathbf{q}\delta(\mathbf{y} - \mathbf{z})) \cdot \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \ \ \mathrm{d}\,\mathbf{y} \\
- & \ \frac{1}{4\pi} \int_{\Omega} ik\mu\mathbf{H}_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{p}) \cdot \mathbb{G}^H(\mathbf{y}, \mathbf{z})\mathbf{q} \ \ \mathrm{d}\,\mathbf{y} \\
= & \ \frac{1}{4\pi}\mathbf{q} \cdot \mathbf{E}_{pl}(\mathbf{z}, -\widehat{\mathbf{x}}, \mathbf{p}),
\end{aligned} \tag{4.131}
$$

as wanted.

Case 3. $\mathbf{z} \in \Gamma_1 \cup \Gamma$.

Notice that by continuity of the tangential traces of $\mathbf{E}_{pl}(\cdot, -\widehat{\mathbf{x}}, \mathbf{p})$ and $\mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \cdot)\mathbf{p}$ on $\Gamma_1 \cup \Gamma$, then the identity (4.115) is true at $\mathbf{z} \in \Gamma_1 \cup \Gamma$ as long as $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ satisfy $\boldsymbol{\nu}(\mathbf{z}) \cdot \mathbf{p} = \boldsymbol{\nu}(\mathbf{z}) \cdot \mathbf{q} = 0$, and the proof is complete. □

We now turn our attention to the approximate problem (4.20)-(4.24) for the delaminated configuration. We will next pove that the reciprocity relation satisfied by far field patters of solutions to the full model (4.1)-(4.3) (see Theorem 6.30 in [32]) is still satisfied by far field patterns of radiating solutions to the approximate model. For the sake of simplicity, in the following theorem we denote by

$$\mathbf{E}^{\infty}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) \quad \text{and} \quad \mathbf{H}^{\infty}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) \tag{4.132}$$

the far field patterns of the radiating solutions to the problem (4.20)-(4.24):

$$\mathbf{E}^{s}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}) \quad \text{and} \quad \mathbf{H}^{s}_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}), \tag{4.133}$$

where of course we have set $\mathbf{H}^s_{pl} := -ik\nabla \times \mathbf{E}^s_{pl}$ in Ω_{ext}.

Theorem 4.6.2. *(The reciprocity principle for the ATCs model) For all $\widehat{\mathbf{x}}, d \in \mathbb{S}^2$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$,*

$$\mathbf{q} \cdot \mathbf{E}^{\infty}_{pl}(\widehat{\mathbf{x}}, \widehat{\mathbf{d}}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{E}^{\infty}_{pl}(-\widehat{\mathbf{d}}, -\widehat{\mathbf{x}}, \mathbf{q}).$$

Proof. Following the arguments of the proof of Theorem 6.30 in [32], if $\widehat{\mathbf{x}}, \widehat{\mathbf{d}} \in \mathbb{S}^2$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, then from the divergence theorem in Ω,

$$\begin{aligned} 0 &= \int_{(\Gamma\setminus\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^i_{pl}(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{p}) \cdot \mathbf{H}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{q}) \ \mathrm{d}s(\mathbf{y}) \\ &= \int_{(\Gamma\setminus\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^i_{pl}(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{p}) \cdot \mathbf{E}^i_{pl}(\mathbf{y}, -\widehat{\mathbf{x}}, \mathbf{q}) \ \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.134}$$

and from the radiation condition,

$$(\mathbf{E}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p}), \mathbf{H}^s_{pl}(\cdot, \widehat{\mathbf{d}}, \mathbf{p})) \text{ and } (\mathbf{E}^s_{pl}(\cdot, -\widehat{\mathbf{x}}, \mathbf{q}), \mathbf{H}^s_{pl}(\cdot, -\widehat{\mathbf{x}}, q))$$

satisfy:

$$\begin{aligned} 0 &= \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^s_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \cdot \mathbf{H}^s_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \\ &+ \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^s_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \cdot \mathbf{E}^s_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \; \mathrm{d}s(\mathbf{y}). \end{aligned} \tag{4.135}$$

Moreover,

$$\begin{aligned} 4\pi\mathbf{q}\cdot\mathbf{E}^\infty_{pl}(\widehat{\mathbf{x}},\widehat{\mathbf{d}},\mathbf{p}) &= \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^s_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \cdot \mathbf{H}^i_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \\ &+ \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^s_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \cdot \mathbf{E}^i_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \; \mathrm{d}s(\mathbf{y}) \end{aligned} \tag{4.136}$$

and

$$\begin{aligned} 4\pi\mathbf{p}\cdot\mathbf{E}^\infty_{pl}(-\widehat{\mathbf{d}},-\widehat{\mathbf{x}},\mathbf{q}) &= \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}^s_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{H}^i_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \\ &+ \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}^s_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{E}^i_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.137}$$

and therefore considering the sum (4.134) + (4.135) + (4.136) - (4.137) we get,

$$\begin{aligned} & 4\pi(\mathbf{q}\cdot\mathbf{E}^\infty_{pl}(\widehat{\mathbf{x}},\widehat{\mathbf{d}},\mathbf{p}) - \mathbf{p}\cdot\mathbf{E}^\infty_{pl}(-\widehat{\mathbf{d}},-\widehat{\mathbf{x}},\mathbf{q})) \\ &= \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \\ &+ \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.138}$$

and thus

$$\begin{aligned} & 4\pi(\mathbf{q}\cdot\mathbf{E}^\infty_{pl}(\widehat{\mathbf{x}},\widehat{\mathbf{d}},\mathbf{p}) - \mathbf{p}\cdot\mathbf{E}^\infty_{pl}(-\widehat{\mathbf{d}},-\widehat{\mathbf{x}},\mathbf{q})) \\ &= \int_{\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \\ &+ \int_{\Gamma_+} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \\ &- \int_{\Gamma_-} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \\ &+ \int_{\Gamma_-} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \\ &+ \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_-} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \\ &+ \int_{(\Gamma\backslash\Gamma_0)\cup\Gamma_-} \boldsymbol{\nu}(\mathbf{y}) \times \mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q}) \cdot \mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p}) \; \mathrm{d}s(\mathbf{y}) \end{aligned} \tag{4.139}$$

and thus,

$$
\begin{aligned}
& 4\pi(\mathbf{q}\cdot\mathbf{E}_{pl}^{\infty}(\widehat{\mathbf{x}},\widehat{\mathbf{d}},\mathbf{p}) - \mathbf{p}\cdot\mathbf{E}_{pl}^{\infty}(-\widehat{\mathbf{d}},-\widehat{\mathbf{x}},\mathbf{q})) \\
= & \int_{\Gamma_0} [\![\boldsymbol{\nu}(\mathbf{y})\times\mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})]\!]\cdot\langle\!\langle\mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})\rangle\!\rangle \ \mathrm{d}s(\mathbf{y}) \\
+ & \int_{\Gamma_0} [\![\boldsymbol{\nu}(\mathbf{y})\times\mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})]\!]\cdot\langle\!\langle\mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})\rangle\!\rangle \ \mathrm{d}s(\mathbf{y}) \\
- & \int_{\Gamma_0} \langle\!\langle\mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})\rangle\!\rangle\cdot[\![\boldsymbol{\nu}(\mathbf{y})\times\mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})]\!] \ \mathrm{d}s(\mathbf{y}) \\
+ & \int_{\Gamma_0} \langle\!\langle\mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})\rangle\!\rangle\cdot[\![\boldsymbol{\nu}(\mathbf{y})\times\mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})]\!] \ \mathrm{d}s(\mathbf{y}) \qquad (4.140)
\end{aligned}
$$

which implies that, substituting the expression (4.17) for the ATCs of our approximate model:

$$
\begin{aligned}
& 4\pi(\mathbf{q}\cdot\mathbf{E}_{pl}^{\infty}(\widehat{\mathbf{x}},\widehat{\mathbf{d}},\mathbf{p}) - \mathbf{p}\cdot\mathbf{E}_{pl}^{\infty}(-\widehat{\mathbf{d}},-\widehat{\mathbf{x}},\mathbf{q})) \\
= & \int_{\Gamma_0} \widetilde{\mathcal{A}_1}(\langle\!\langle\mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})\rangle\!\rangle_T)\cdot\langle\!\langle\mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})\rangle\!\rangle_T \ \mathrm{d}s(\mathbf{y}) \\
+ & \int_{\Gamma_0} \widetilde{\mathcal{A}_2}(\langle\!\langle\mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})\rangle\!\rangle_T)\cdot\langle\!\langle\mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})\rangle\!\rangle \ \mathrm{d}s(\mathbf{y}) \\
- & \int_{\Gamma_0} \langle\!\langle\mathbf{E}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})\rangle\!\rangle\cdot\widetilde{\mathcal{A}_2}(\langle\!\langle\mathbf{E}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})\rangle\!\rangle_T) \ \mathrm{d}s(\mathbf{y}) \\
+ & \int_{\Gamma_0} \langle\!\langle\mathbf{H}_{pl}(\mathbf{y},-\widehat{\mathbf{x}},\mathbf{q})\rangle\!\rangle\cdot\widetilde{\mathcal{A}_1}(\langle\!\langle\mathbf{H}_{pl}(\mathbf{y},\widehat{\mathbf{d}},\mathbf{p})\rangle\!\rangle_T) \ \mathrm{d}s(\mathbf{y}) \\
= & \ 0, \qquad (4.141)
\end{aligned}
$$

where the last line is a consequence of the fact that the operators $\widetilde{\mathcal{A}_1}$ and $\widetilde{\mathcal{A}_2}$, defined by (4.18)-(4.19), are symmetric. $\square$

Remark 4.6.1. *About the representation formula for electromagnetism:*
Let $D\subset\mathbb{R}^3$ be a bounded, simply connected domain with Lipschitz boundary ∂D. If v is a function such that $v = v^{out}$ in $\mathbb{R}^3\setminus\overline{D}$ and $v = v^{\delta}$ in D, and $\boldsymbol{\nu}$ denotes the unit normal vector on ∂D pointing outwards from D, then the jump of v across ∂D is defined by:

$$[v]_{\partial D} := v^{out}|_{\partial D} - v^{\delta}|_{\partial D}. \qquad (4.142)$$

Let $\mathbf{E}^s$ in $\mathbf{H}_{loc}(curl, \mathbb{R}^3 \backslash \overline{D})$ be a scattered electric field (satisfying (4.24)) such that for a given incident field $\mathbf{E}^i$, the corresponding total field $\mathbf{E} := \mathbf{E}^i + \mathbf{E}^s$ in $\mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \partial D)$ satisfies the inhomogeneous problem:

$$\nabla \times (\nabla \times \mathbf{E}) - k^2 \mathbf{E} = \mathbf{0} \text{ in } \mathbb{R}^3 \setminus \overline{D}, \tag{4.143}$$

$$\nabla \times (\mu_D^{-1} \nabla \times \mathbf{E}) - k^2 \epsilon_D \mathbf{E} = \mathbf{0} \text{ in } D, \tag{4.144}$$

where μ_D and ϵ_D are smooth material properties. Then $\mathbf{E}^s$ holds the following representation formula:

$$\begin{aligned} \mathbf{E}^s(\mathbf{x}) &= \frac{1}{ik} \int_{\partial D} \Big\{ -\mu^{-1} \nabla \times \mathbb{G}_D^E(\mathbf{x}, \mathbf{y}) \, [\boldsymbol{\nu} \times \mathbf{E}^s]_{\partial D} (\mathbf{y}) \\ &+ \mathbb{G}_D^E(\mathbf{x}, \mathbf{y}) \left[\boldsymbol{\nu} \times (\mu^{-1} \nabla \times \mathbf{E}^s)\right]_{\partial D} (\mathbf{y}) \Big\} \, ds(\mathbf{y}), \end{aligned} \tag{4.145}$$

where $\mathbb{G}_D^E$ denotes the corresponding electric Green's tensor.

4.6.2 The linear sampling method

The inverse problem that we want to solve is to determine the location of the delaminated part of the interface Γ_0, from the knowledge of all the *far field measurments* of the electromagnetic field, when the incident fields are plane waves in all directions of incidence, and with all possible polarizations.

In order to develop our method for the solution of this inverse problem, we will introduce the necessary notation for the construction and analysis of the linear sampling method.

Let $\mathbf{g} \in \mathbf{L}_t^2(\mathbb{S}^2)$, then the *electric Hergoltz wave* $\mathbf{E}_g$ is defined by

$$\mathbf{E}_g(\mathbf{x}) = \int_{\mathbb{S}^2} \mathbf{g}(\widehat{\mathbf{d}}) e^{ik\mathbf{x}\cdot\widehat{\mathbf{d}}} \, ds_{\widehat{\mathbf{d}}}. \tag{4.146}$$

The far field operator $\mathcal{F} : \mathbf{L}_t^2(\mathbb{S}^2) \to \mathbf{L}_t^2(\mathbb{S}^2)$ associated with the medium with the defect is defined by

$$(\mathcal{F}\mathbf{g})(\widehat{\mathbf{x}}) = \int_{\mathbb{S}^2} \mathbf{E}^\infty(\widehat{\mathbf{x}}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}})) \, ds_{\widehat{\mathbf{d}}}, \tag{4.147}$$

where $\mathbf{E}^\infty(\cdot,\widehat{\mathbf{d}},\mathbf{g}(\widehat{\mathbf{d}}))$ is the far field pattern of the radiating field $\mathbf{E}^s(\cdot,\widehat{\mathbf{d}},\mathbf{g}(\widehat{\mathbf{d}}))$ associated with the solution of (4.20)-(4.24) when the incident field is the plane wave $\mathbf{E}^i(\cdot,\widehat{\mathbf{d}},\mathbf{g}(\widehat{\mathbf{d}}))$. By linearity, $\mathcal{F}\mathbf{g}$ is the far field pattern of the radiating solution of (4.20)-(4.24) when $\mathbf{E}^i = \mathbf{E}_g$.

In an analogous manner, we can define the far field operator $\mathcal{F}_B : \mathbf{L}^2_t(\mathbb{S}^2) \to \mathbf{L}^2_t(\mathbb{S}^2)$ associated with the background problem by

$$(\mathcal{F}_B\mathbf{g})(\widehat{\mathbf{x}}) = \int_{\mathbb{S}^2} \mathbf{E}^\infty_{pl}(\widehat{\mathbf{x}},\widehat{\mathbf{d}},\mathbf{g}(\widehat{\mathbf{d}}))\, ds_{\widehat{\mathbf{d}}}, \tag{4.148}$$

where $\mathbf{E}^\infty_{pl}(\cdot,\widehat{\mathbf{d}},\mathbf{g}(\widehat{\mathbf{d}}))$ is the far field pattern of the radiating field $\mathbf{E}^s_{pl}(\cdot,\widehat{\mathbf{d}},\mathbf{g}(\widehat{\mathbf{d}}))$ defined in the previous subsection. We will denote by $\mathbf{E}_{b,\mathbf{g}}$ and $\mathbf{E}^s_{b,\mathbf{g}}$ the total and radiating field solutions to the background problem (4.111)-(4.112), respectively, when the incident field is $\mathbf{E}^i = \mathbf{E}_g$. Then, again by linearity, $\mathcal{F}_B\mathbf{g}$ is the far field pattern of $\mathbf{E}^s_{b,\mathbf{g}}$.

The far-field operator associated with the defect is defined by the difference

$$\mathcal{F}_D := \mathcal{F} - \mathcal{F}_B, \tag{4.149}$$

which will provide the informaton associated with the delamination alone. If we assume that $\mathcal{F}_B$ is well known and that $\mathcal{F}$ can be measured, then we will develop a method to detect the existence and location of Γ_0 based of $\mathcal{F}_D$. Therefore, we will study the range of this operator.

Define now

$$\mathcal{H} := \left\{ \mathbf{u} \in \mathbf{H}(curl, B^\delta_R) \,\middle|\, \mu^{-1}\nabla\times\mathbf{u} \in \mathbf{L}^2(B^\delta_R) \text{ and } \langle\!\langle \mathbf{u}_T \rangle\!\rangle \in \mathbf{H}(curl_\Gamma, \Gamma_0) \right\}, \tag{4.150}$$

endowed with its graph norm,

$$\|\mathbf{u}\|^2_{\mathcal{H}} = ||\mathbf{u}||^2_{\mathbf{H}(curl,B^\delta_R)} + ||\mu^{-1}\nabla\times\mathbf{u}||^2_{L^2(B^\delta_R)} + ||\langle\!\langle \mathbf{u}_T \rangle\!\rangle||^2_{\mathbf{H}(curl_\Gamma,\Gamma_0)}. \tag{4.151}$$

A solution to the *defective* problem is given by $\mathbf{E} \in \mathbf{H}_{loc}(curl, \mathbb{R}^3 \setminus \overline{\Omega}_\delta)$ such that $\mathbf{E}|_{B_R^\delta} \in \mathcal{H}$ and that satisfies:

$$\nabla \times \left(\mu^{-1}\nabla \times \mathbf{E}\right) - k^2\epsilon\mathbf{E} = \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}_\delta, \tag{4.152}$$

$$\mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \mathbf{E}]\!] = \delta\langle\!\langle\left(\mu^{-1}\nabla \times \mathbf{E}\right)_T\rangle\!\rangle + \mathbf{h}_1, \tag{4.153}$$

$$[\![\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{E}\right)]\!] = \delta\mathcal{A}_2\langle\!\langle\mathbf{E}_T\rangle\!\rangle + \mathbf{h}_2, \tag{4.154}$$

$$\mathbf{n} \times \left(\mu^{-1}\nabla \times \mathbf{E}\right) = \mathbf{h}_3 \text{ on } \mathscr{S}, \tag{4.155}$$

and $\mathbf{E}$ satisfies the Silver-Müller radiation condition (4.24), where

$$\begin{cases} \mathbf{h}_1 & := & \delta\,\langle\!\langle\mu^{-1}\nabla \times \mathbf{v}\rangle\!\rangle_T \; - \; \mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \mathbf{v}]\!], \\ \mathbf{h}_2 & := & \delta\mathcal{A}_2\langle\!\langle\mathbf{v}\rangle\!\rangle_T - [\![\boldsymbol{\nu} \times \mu^{-1}\nabla \times \mathbf{v}]\!] \\ \mathbf{h}_3 & := & -\mathbf{n} \times \left(\mu^{-1}\nabla \times \mathbf{v}\right)\Big|_{\mathscr{S}}, \end{cases}, \tag{4.156}$$

for some $\mathbf{v} \in \mathcal{H}$.

Define the *Hergoltz operator* $\mathscr{H} : \mathbf{L}_t^2(\mathbb{S}^2) \to \mathbf{H}(curl_\Gamma, \Gamma_0) \times \mathcal{H}_0(\Gamma_0)^* \times \mathbf{H}^{-1/2}(div_{\mathscr{S}}, \mathscr{S})$ by

$$\begin{aligned} \mathscr{H}\mathbf{g} \;\; = \;\; & \Big(\delta\langle\!\langle\mu^{-1}\nabla \times \mathbf{E}_{b,\mathbf{g}}\rangle\!\rangle_T - \mathcal{A}_1^{-1}[\![\boldsymbol{\nu} \times \mathbf{E}_{b,\mathbf{g}}]\!], \\ & \delta\mathcal{A}_2(\langle\!\langle\mathbf{E}_{b,\mathbf{g}}\rangle\!\rangle_T) - [\![\boldsymbol{\nu} \times \mu^{-1}\nabla \times \mathbf{E}_{b,\mathbf{g}}]\!], , \\ - \;\; & \mathbf{n} \times \left(\mu^{-1}\nabla \times \mathbf{E}_{b,\mathbf{g}}\right)\Big|_{\mathscr{S}}\Big), \end{aligned} \tag{4.157}$$

where $\mathcal{A}_i : \mathcal{H}_0(\Gamma_0) \to \mathcal{H}_0(\Gamma_0)^*$, $i = 1, 2$, are the boundary operators defined by (4.37).

Remark 4.6.2. *Notice that $\mathcal{F}_D\mathbf{g}$ is, by linearity, the far-field pattern associated with the solution to the defective problem (4.152)-(4.155) when the incident field is $\mathbf{E}^i = \mathbf{E}_{b,\mathbf{g}}$, i.e., when the the boundary source terms are $\mathscr{H}\mathbf{g}$.*

Define $\mathscr{G} : \mathbf{H}(curl_\Gamma, \Gamma_0) \times \mathcal{H}_0(\Gamma_0)^* \times \mathbf{H}^{-1/2}(div_{\mathscr{S}}, \mathscr{S}) \to \mathbf{L}_t^2(\mathbb{S}^2)$ by

$$\mathscr{G}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = \mathbf{E}^\infty,$$

where $\mathbf{E}^\infty$ is the far field pattern of the scattered field $\mathbf{E}$ that solves the defective problem (4.152)-(4.155). From Theorem 4.4.1, we know that the operator $\mathscr{G}$ is well defined and bounded.

Then it is clear that the following factorization of the far field operator $\mathcal{F}_D$ holds:

$$\mathcal{F}_D = \mathscr{G}\mathscr{H},$$

and we will prove the linear sampling method based on the properties of $\mathscr{G}$ and $\mathscr{H}$.

Proposition 4.6.1. *In addition to the assumptions of Theorem 4.4.1, assume that the function* $|\nabla\mu|$ *is bounded in a neighbourhood of* Γ *and* $\mathbf{n}\times(\mu^{-1}\nabla\times\mathbf{E}_{b,\mathbf{g}})\Big|_{\mathscr{S}} = \mathbf{0}$ *if and only if* $\mathbf{g}=\mathbf{0}$*. Then the operator* $\mathscr{H}$ *is injective with dense range.*

Proof. Observe first that by linearity,

$$\mathbf{E}_{b,\mathbf{g}}(\mathbf{x}) = \int_{\mathbb{S}^2} \mathbf{E}_{pl}(\mathbf{x}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}}))\, ds_{\widehat{\mathbf{d}}}, \tag{4.158}$$

and thanks to Theorem 4.6.1, for all $\mathbf{p}\in\mathbb{R}^3$,

$$\mathbf{p}\cdot\mathbf{E}_{b,\mathbf{g}}(\mathbf{x}) = \int_{\mathbb{S}^2} 4\pi\mathbf{g}(\widehat{\mathbf{d}})\cdot\mathbb{G}^{E,\infty}(-\widehat{\mathbf{d}},\mathbf{x})\mathbf{p}\, ds_{\widehat{\mathbf{d}}}. \tag{4.159}$$

Observe that

$$\mathbf{H}(curl_\Gamma,\Gamma_0) = \left\{\nabla_\Gamma p + \nabla_\Gamma\times q \,\middle|\, \nabla_\Gamma\times q\in\mathbf{H}^1_t(\Gamma_0),\ \nabla_\Gamma p\in\mathbf{L}^2_t(\Gamma_0)\right\} \tag{4.160}$$

and thus

$$\begin{aligned}\mathbf{H}(curl_\Gamma,\Gamma_0)^* &= \left\{\nabla_\Gamma p + \nabla_\Gamma\times q \,\middle|\, \nabla_\Gamma\times q\in\mathbf{H}^{-1}_0(\Gamma_0),\ \nabla_\Gamma p\in\mathbf{L}^2_t(\Gamma_0)\right\}, && (4.161)\\ &= \mathbf{H}^{-1}_0(div_\Gamma,\Gamma_0). && (4.162)\end{aligned}$$

Given $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\sigma}) \in \mathbf{H}_0^{-1}(div_\Gamma, \Gamma_0) \times \mathcal{H}_0(\Gamma_0) \times \tilde{\mathbf{H}}^{-1/2}(curl_{\mathscr{S}}, \mathscr{S})$, if $\langle \cdot, \cdot \rangle$ is the corresponding duality pairing, then by always using as pivoting space $\mathbf{L}_t^2$ (in either Γ_0 or $\mathscr{S}$):

$$\begin{aligned}
\langle \mathscr{H}\mathbf{g},\, (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\sigma}) \rangle \;&=\; \int_{\Gamma_0} \left\{ \delta \langle\!\langle \mu^{-1} \nabla \times \mathbf{E}_{b,\mathbf{g}} \rangle\!\rangle_T \cdot \overline{\boldsymbol{\xi}} - \mathcal{A}_1^{-1} [\![\boldsymbol{\nu} \times \mathbf{E}_{b,\mathbf{g}}]\!] \cdot \overline{\boldsymbol{\xi}} \right\} \;\mathrm{d}s(\mathbf{y}) \\
&+\; \int_{\Gamma_0} \left\{ \delta \mathcal{A}_2 \langle\!\langle \mathbf{E}_{b,\mathbf{g}} \rangle\!\rangle_T \cdot \overline{\boldsymbol{\eta}} - [\![\boldsymbol{\nu} \times \mu^{-1} \nabla \times \mathbf{E}_{b,\mathbf{g}}]\!] \cdot \overline{\boldsymbol{\eta}} \right\} \;\mathrm{d}s(\mathbf{y}) \\
&-\; \int_{\mathscr{S}} \mathbf{n} \times \left(\mu^{-1} \nabla \times \mathbf{E}_{b,\mathbf{g}} \right) \cdot \overline{\boldsymbol{\sigma}} \;\mathrm{d}s(\mathbf{y}) \\
&=\; \int_{\Gamma_0} \int_{\mathbb{S}^2} \left\{ \delta \langle\!\langle \mu^{-1} \nabla \times \mathbf{E}_b(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}})) \rangle\!\rangle_T \cdot \overline{\boldsymbol{\xi}} \right. \\
&-\; \left. \mathcal{A}_1^{-1} [\![\boldsymbol{\nu} \times \mathbf{E}_b(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}}))]\!] \cdot \overline{\boldsymbol{\xi}} \right\} \;\mathrm{d}s(\widehat{\mathbf{d}}) \;\mathrm{d}s(\mathbf{y}) \\
&+\; \int_{\Gamma_0} \int_{\mathbb{S}^2} \left\{ \delta \mathcal{A}_2 \langle\!\langle \mathbf{E}_b(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}})) \rangle\!\rangle_T \cdot \overline{\boldsymbol{\eta}} \right. \\
&-\; \left. [\![\boldsymbol{\nu} \times \mu^{-1} \nabla \mathbf{E}_b(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}}))]\!] \cdot \overline{\boldsymbol{\eta}} \right\} \;\mathrm{d}s(\widehat{\mathbf{d}}) \;\mathrm{d}s(\mathbf{y}) \\
&-\; \int_{\mathscr{S}} \int_{\mathbb{S}^2} \mathbf{n} \times \mu^{-1} \nabla \times \mathbf{E}_b(\mathbf{y}, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}})) \cdot \overline{\boldsymbol{\sigma}} \;\mathrm{d}s(\widehat{\mathbf{d}}) \;\mathrm{d}s(\mathbf{y}), \quad (4.163)
\end{aligned}$$

and by the mixed reciprocity principle,

$$\begin{aligned}
\frac{1}{4\pi} \langle \mathscr{H}\mathbf{g},\, (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\sigma}) \rangle \;&=\; \int_{\mathbb{S}^2} \mathbf{g}(\widehat{\mathbf{d}}) \cdot \left\{ \int_{\Gamma_0} \left\{ \delta \langle\!\langle \mu^{-1} \nabla \times \mathbb{G}^{E,\infty}(-\widehat{\mathbf{d}}, \mathbf{y})) \rangle\!\rangle_T \overline{\boldsymbol{\xi}(\mathbf{y})} \right. \right. \\
&\quad \left. + [\![\mathbb{G}^{E,\infty}(-\widehat{\mathbf{d}}, \mathbf{y})]\!] (\boldsymbol{\nu} \times \mathcal{A}_1^{-1} \overline{\boldsymbol{\xi}(\mathbf{y})}) \right\} \;\mathrm{d}s(\mathbf{y}) \\
&\quad + \int_{\Gamma_0} \left\{ \delta \langle\!\langle \mathbb{G}^{E,\infty}(-\widehat{\mathbf{d}}, \mathbf{y}) \rangle\!\rangle_T \mathcal{A}_2 \overline{\boldsymbol{\eta}(\mathbf{y})} \right. \\
&\quad \left. + [\![\mu^{-1} \nabla \times \mathbb{G}^{E,\infty}(-\widehat{\mathbf{d}}, \mathbf{y})]\!]_T (\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}(\mathbf{y})}) \right\} \;\mathrm{d}s(\mathbf{y}) \\
&\quad \left. + \int_{\mathscr{S}} \mu^{-1} \nabla \times \mathbb{G}^{E,\infty}(-\widehat{\mathbf{d}}, \mathbf{y}) (\mathbf{n} \times \overline{\boldsymbol{\sigma}(\mathbf{y})}) \;\mathrm{d}s(\mathbf{y}) \right\} \;\mathrm{d}s(\widehat{\mathbf{d}}), (4.164)
\end{aligned}$$

Notice that in general $\mathcal{A}_1^{-1}$ and $\mathcal{A}_2$ are not self-adjoint operators because $\Im(\epsilon_\delta) > 0$. Thus if we define

$$\overline{\mathcal{A}_1} \boldsymbol{\eta} \;:=\; \overline{\alpha_1} \boldsymbol{\eta} - \overline{\beta_1} \overrightarrow{curl}_\Gamma \, curl_\Gamma \, \boldsymbol{\eta}, \quad \text{and} \tag{4.165}$$

$$\overline{\mathcal{A}_2} \boldsymbol{\eta} \;:=\; \overline{\alpha_2} \boldsymbol{\eta} - \overline{\beta_2} \overrightarrow{curl}_\Gamma \, curl_\Gamma \, \boldsymbol{\eta}, \tag{4.166}$$

then we conclude that the conjugate transpose operator $\mathscr{H}^* : \mathbf{H}^{-1/2}(curl_\Gamma, \Gamma_0) \times \mathbf{H}^{-1/2}(curl_\Gamma, \Gamma_0) \to \mathbf{L}^2_t(\mathbb{S}^2)$ of $\mathscr{H}$, is given by:

$$\begin{aligned}
\frac{1}{4\pi}\mathscr{H}^*(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\sigma}) &= \int_{\Gamma_0}\Big\{\delta\overline{\langle\!\langle \mu^{-1}\nabla\times\mathbb{G}^{E,\infty}(-\cdot,\mathbf{y}))\rangle\!\rangle_T}\,\boldsymbol{\xi}(\mathbf{y}) \\
&+ \overline{[\![\mathbb{G}^{E,\infty}(-\cdot,\mathbf{y})]\!]}(\boldsymbol{\nu}\times\overline{\mathcal{A}_1}^{-1}\boldsymbol{\xi}(\mathbf{y}))\Big\}\,\mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_0}\Big\{\delta\overline{\langle\!\langle \mathbb{G}^{E,\infty}(-\cdot,\mathbf{y})\rangle\!\rangle_T}\,\overline{\mathcal{A}_2}\boldsymbol{\eta}(\mathbf{y}) \\
&+ \overline{[\![\mu^{-1}\nabla\times\mathbb{G}^{E,\infty}(-\cdot,\mathbf{y})]\!]_T}(\boldsymbol{\nu}\times\boldsymbol{\eta}(\mathbf{y}))\Big\}\,\mathrm{d}s(\mathbf{y}) \\
&+ \int_{\mathscr{S}}\overline{\mu^{-1}\nabla\times\mathbb{G}^{E,\infty}(-\cdot,\mathbf{y})}(\mathbf{n}\times\boldsymbol{\sigma}(\mathbf{y}))\,\mathrm{d}s(\mathbf{y}).
\end{aligned}\tag{4.167}$$

Thus $\mathbf{E}^\infty(\widehat{\mathbf{x}}) = \frac{1}{4\pi}\overline{\mathscr{H}^*(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\sigma})}(-\widehat{\mathbf{x}})$ is the far field pattern of the following potential:

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \int_{\Gamma_0}\Big\{\delta\langle\!\langle \mu^{-1}\nabla\times\mathbb{G}^{E}(\mathbf{x},\mathbf{y}))\rangle\!\rangle_T\,\overline{\boldsymbol{\xi}(\mathbf{y})} \\
&+ [\![\mathbb{G}^{E}(\mathbf{x},\mathbf{y})]\!](\boldsymbol{\nu}\times\overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}(\mathbf{y})})\Big\}\,\mathrm{d}s(\mathbf{y}) \\
&+ \int_{\Gamma_0}\Big\{\delta\langle\!\langle \mathbb{G}^{E}(\mathbf{x},\mathbf{y})\rangle\!\rangle_T\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}(\mathbf{y})} \\
&+ [\![\mu^{-1}\nabla\times\mathbb{G}^{E}(\mathbf{x},\mathbf{y})]\!]_T(\boldsymbol{\nu}\times\overline{\boldsymbol{\eta}(\mathbf{y})})\Big\}\,\mathrm{d}s(\mathbf{y}) \\
&+ \int_{\mathscr{S}}\mu^{-1}\nabla\times\mathbb{G}^{E}(\mathbf{x},\mathbf{y})(\mathbf{n}\times\overline{\boldsymbol{\sigma}(\mathbf{y})})\,\mathrm{d}s(\mathbf{y}).
\end{aligned}\tag{4.168}$$

Observe that $\mathbf{E} = (\mathbf{E}_{\Gamma_+})_T + (\mathbf{E}_{\Gamma_-})_T + \mathbf{E}_{\mathscr{S}}$, where we define

$$
\begin{aligned}
\mathbf{E}_{\Gamma_+}(\mathbf{x}) \;&:=\; \int_{\Gamma_0} \left\{ \frac{\delta}{2}\mu_+^{-1}\nabla \times \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu})\, \overline{\boldsymbol{\xi}(\mathbf{y}_\Gamma)} \right. \\
&+\; \left. \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu})(\boldsymbol{\nu} \times \overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}(\mathbf{y}_\Gamma)}) \right\} \, \mathrm{d}\,\mathbf{y}_\Gamma \\
&+\; \int_{\Gamma_0} \left\{ \frac{\delta}{2}\mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu})\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}(\mathbf{y}_\Gamma)} \right. \\
&+\; \left. \mu_+^{-1}\nabla \times \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu})(\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}(\mathbf{y}_\Gamma)}) \right\} \, \mathrm{d}\,\mathbf{y}_\Gamma, \qquad (4.169)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}_{\Gamma_-}(\mathbf{x}) \;&:=\; \int_{\Gamma_0} \left\{ \frac{\delta}{2}\mu_-^{-1}\nabla \times \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma - \delta f^- \boldsymbol{\nu})\, \overline{\boldsymbol{\xi}(\mathbf{y}_\Gamma)} \right. \\
&-\; \left. \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma - \delta f^- \boldsymbol{\nu})(\boldsymbol{\nu} \times \overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}(\mathbf{y}_\Gamma)}) \right\} \, \mathrm{d}\,\mathbf{y}_\Gamma \\
&+\; \int_{\Gamma_0} \left\{ \frac{\delta}{2}\mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma - \delta f^- \boldsymbol{\nu})\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}(\mathbf{y}_\Gamma)} \right. \\
&-\; \left. \mu_-^{-1}\nabla \times \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma - \delta f^- \boldsymbol{\nu})(\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}(\mathbf{y}_\Gamma)}) \right\} \, \mathrm{d}\,\mathbf{y}_\Gamma, \qquad (4.170)
\end{aligned}
$$

$$
\mathbf{E}_{\mathscr{S}}(\mathbf{x}) \;:=\; \int_{\mathscr{S}} \mu^{-1}\nabla \times \mathbb{G}^E(\mathbf{x}, \mathbf{y})(\mathbf{n} \times \overline{\boldsymbol{\sigma}(\mathbf{y})}) \;\, \mathrm{d}s(\mathbf{y}). \qquad (4.171)
$$

Moreover, using the notation and the representation formula in Remark 4.145, we know that the following jump conditions on Γ_-, Γ_+, and $\mathscr{S}$ are satisfied by $\mathbf{E}$:

$$
\begin{aligned}
[\boldsymbol{\nu} \times \mathbf{E}]_{\Gamma_+} &= \left[\boldsymbol{\nu} \times \mathbf{E}_{\Gamma_+}\right]_{\Gamma_+} \\
&= -ik\frac{\delta}{2}\overline{\boldsymbol{\xi}} - ik(\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}}), \qquad (4.172) \\
[\boldsymbol{\nu} \times \mathbf{E}]_{\Gamma_-} &= \left[\boldsymbol{\nu} \times \mathbf{E}_{\Gamma_-}\right]_{\Gamma_-} \\
&= -ik\frac{\delta}{2}\overline{\boldsymbol{\xi}} + ik(\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}}), \qquad (4.173) \\
[\boldsymbol{\nu} \times \mathbf{E}]_{\mathscr{S}} &= [\boldsymbol{\nu} \times \mathbf{E}_{\mathscr{S}}]_{\mathscr{S}} = -ik(\mathbf{n} \times \overline{\boldsymbol{\sigma}}), \qquad (4.174)
\end{aligned}
$$

and

$$
\begin{aligned}
\big[\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}\big)\big]_{\Gamma_+} &= \big[\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}_{\Gamma_+}\big)\big]_{\Gamma_+} \\
&= ik(\boldsymbol{\nu} \times \overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}}) + ik\frac{\delta}{2}\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}}, \qquad (4.175) \\
\big[\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}\big)\big]_{\Gamma_-} &= \big[\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}_{\Gamma_-}\big)\big]_{\Gamma_-} \\
&= -ik(\boldsymbol{\nu} \times \overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}}) + ik\frac{\delta}{2}\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}}, \qquad (4.176) \\
\big[\mathbf{n} \times \big(\mu^{-1}\nabla \times \mathbf{E}\big)\big]_{\mathscr{S}} &= \big[\mathbf{n} \times \big(\mu^{-1}\nabla \times \mathbf{E}_{\mathscr{S}}\big)\big]_{\mathscr{S}} \\
&= \mathbf{0}. \qquad (4.177)
\end{aligned}
$$

Now, suppose then that $\mathbf{E}^\infty = \mathbf{0}$, then by the Rellich Lemma (Lemma 9.28 in [65]), we know that $\mathbf{E} = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, from the assumptions of $|\nabla\mu|$ and the unique continuation principle for isotropic time harmonic Maxwell equations (Theorem 2.3 in [70]), we ensure that $\mathbf{E} = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{\Omega}_\delta$. Then:

$$
[\boldsymbol{\nu} \times \mathbf{E}]_{\Gamma_+} = \boldsymbol{\nu} \times \mathbf{E}^{out}\Big|_{\Gamma_+} - \boldsymbol{\nu} \times \mathbf{E}^{\delta}\Big|_{\Gamma_+} = -\boldsymbol{\nu} \times \mathbf{E}^{\delta}\Big|_{\Gamma_+}, \qquad (4.178)
$$

$$
[\boldsymbol{\nu} \times \mathbf{E}]_{\Gamma_-} = \boldsymbol{\nu} \times \mathbf{E}^{\delta}\Big|_{\Gamma_-} - \boldsymbol{\nu} \times \mathbf{E}^{out}\Big|_{\Gamma_-} = \boldsymbol{\nu} \times \mathbf{E}^{\delta}\Big|_{\Gamma_-}, \qquad (4.179)
$$

$$
[\boldsymbol{\nu} \times \mathbf{E}]_{\mathscr{S}} = \mathbf{n} \times \mathbf{E}^{out}\Big|_{\mathscr{S}} - \mathbf{n} \times \mathbf{E}^{\delta}\Big|_{\mathscr{S}} = -\mathbf{n} \times \mathbf{E}^{\delta}\Big|_{\mathscr{S}}, \qquad (4.180)
$$

and

$$
\begin{aligned}
\big[\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}\big)\big]_{\Gamma_+} &= \boldsymbol{\nu} \times \mu^{-1}\nabla \times \mathbf{E}^{out}\Big|_{\Gamma_+} \\
&\quad -\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\big)\Big|_{\Gamma_+} \\
&= -\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\big)\Big|_{\Gamma_+}, \qquad (4.181) \\
\big[\boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}\big)\big]_{\Gamma_-} &= \boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\big)\Big|_{\Gamma_-} \\
&\quad -\boldsymbol{\nu} \times \mu^{-1}\nabla \times \mathbf{E}^{out}\Big|_{\Gamma_-} \\
&= \boldsymbol{\nu} \times \big(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\big)\Big|_{\Gamma_-}, \qquad (4.182) \\
\big[\mathbf{n} \times \big(\mu^{-1}\nabla \times \mathbf{E}\big)\big]_{\mathscr{S}} &= -\mathbf{n} \times \big(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\big)\Big|_{\mathscr{S}}. \qquad (4.183)
\end{aligned}
$$

thus combining (4.172)-(4.177) with (4.178)-(4.183),

$$\boldsymbol{\nu} \times \mathbf{E}^{\delta}\Big|_{\Gamma_+} = ik\frac{\delta}{2}\overline{\boldsymbol{\xi}} + ik(\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}}), \tag{4.184}$$

$$\boldsymbol{\nu} \times \mathbf{E}^{\delta}\Big|_{\Gamma_-} = -ik\frac{\delta}{2}\overline{\boldsymbol{\xi}} + ik(\boldsymbol{\nu} \times \overline{\boldsymbol{\eta}}), \tag{4.185}$$

$$\mathbf{n} \times \mathbf{E}^{\delta}\Big|_{\mathscr{S}} = ik(\mathbf{n} \times \overline{\boldsymbol{\sigma}}), \tag{4.186}$$

and

$$\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\right)\Big|_{\Gamma_+} = -ik(\boldsymbol{\nu} \times \overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}}) - ik\frac{\delta}{2}\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}}, \tag{4.187}$$

$$\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\right)\Big|_{\Gamma_-} = -ik(\boldsymbol{\nu} \times \overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}}) + ik\frac{\delta}{2}\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}}, \tag{4.188}$$

$$\mathbf{n} \times \left(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\right)\Big|_{\mathscr{S}} = \mathbf{0}. \tag{4.189}$$

Introducing the following notation for the *internal jump* and the *internal mean value*:

$$[\![u^{\delta}]\!]_{\Omega_\delta} := u^{\delta}\Big|_{\Gamma_+} - u^{\delta}\Big|_{\Gamma_-}, \quad \langle\!\langle u^{\delta}\rangle\!\rangle_{\Omega_\delta} := \frac{1}{2}(u^{\delta}\Big|_{\Gamma_+} + u^{\delta}\Big|_{\Gamma_-})$$

for every u^{δ} (scalar or vectorial field) defined in Ω_δ, we know that,

$$[\![\boldsymbol{\nu} \times \mathbf{E}^{\delta}]\!]_{\Omega_\delta} = ik\delta\overline{\boldsymbol{\xi}}, \tag{4.190}$$

$$\langle\!\langle \mathbf{E}^{\delta}_T\rangle\!\rangle_{\Omega_\delta} = ik\overline{\boldsymbol{\eta}}, \tag{4.191}$$

and

$$[\![\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\right)]\!]_{\Omega_\delta} = -ik\delta\overline{\mathcal{A}_2}\overline{\boldsymbol{\eta}}, \tag{4.192}$$

$$\langle\!\langle (\mu^{-1}\nabla \times \mathbf{E}^{\delta})_T\rangle\!\rangle_{\Omega_\delta} = -ik\overline{\mathcal{A}_1}^{-1}\overline{\boldsymbol{\xi}}. \tag{4.193}$$

Therefore, $\mathbf{E}^{\delta} = \mathbf{E}|_{\Omega_\delta} \in \mathbf{H}(curl, \Omega_\delta)$ satisfies

$$\nabla \times \mu_{\pm}^{-1}\nabla \times \mathbf{E}^{\delta} - k^2\epsilon_{\pm}\mathbf{E} = \mathbf{0} \quad \text{in} \quad \Omega_\delta^{\pm}, \tag{4.194}$$

$$\overline{\mathcal{A}_1}^{-1}[\![\boldsymbol{\nu} \times \mathbf{E}^{\delta}]\!]_{\Omega_\delta} = -\delta\langle\!\langle (\mu^{-1}\nabla \times \mathbf{E}^{\delta})_T\rangle\!\rangle_{\Omega_\delta}, \tag{4.195}$$

$$[\![\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\right)]\!]_{\Omega_\delta} = -\delta\overline{\mathcal{A}_2}\langle\!\langle \mathbf{E}^{\delta}_T\rangle\!\rangle_{\Omega_\delta}, \tag{4.196}$$

$$\mathbf{n} \times \left(\mu^{-1}\nabla \times \mathbf{E}^{\delta}\right)\Big|_{\mathscr{S}} = \mathbf{0}, \tag{4.197}$$

$$\mathbf{n} \times \mathbf{E}^{\delta}\Big|_{\mathscr{S}} = ik(\mathbf{n} \times \overline{\boldsymbol{\sigma}}). \tag{4.198}$$

Notice that (4.194)-(4.198) is an over-determined system which in general will not have a solution. However, to investigate further, multiply by a test function and integrating by parts we get that $\mathbf{E}^\delta$ necessarily satisfies,

$$a_{\Omega_\delta}(\mathbf{E}^\delta, \mathbf{v}^\delta) = 0, \quad \text{for all} \quad \mathbf{v}^\delta \in \mathcal{H}_0^\delta, \tag{4.199}$$

where

$$\begin{aligned} a_{\Omega_\delta}(\mathbf{E}^\delta, \mathbf{v}^\delta) &= \int_{\Omega_\delta} \left\{ \mu_\pm^{-1} \nabla \times \mathbf{E}^\delta \cdot \nabla \times \overline{\mathbf{v}}^\delta - k^2 \epsilon_\pm \mathbf{E}^\delta \cdot \overline{\mathbf{v}}^\delta \right\} \, \mathrm{d}\,\mathbf{y} & (4.200) \\ &+ \frac{1}{\delta} \int_{\Gamma_0} \overline{\mathcal{A}_1}^{-1} [\![\boldsymbol{\nu} \times \mathbf{E}^\delta]\!]_{\Omega_\delta} \cdot [\![\boldsymbol{\nu} \times \overline{\mathbf{v}}^\delta]\!]_{\Omega_\delta} \, \mathrm{d}s(\mathbf{y}) \\ &- \int_{\Gamma_0} \delta \overline{\mathcal{A}_2} \langle\!\langle \mathbf{E}_T^\delta \rangle\!\rangle_{\Omega_\delta} \cdot \langle\!\langle \overline{\mathbf{v}}^\delta \rangle\!\rangle_{\Omega_\delta} \, \mathrm{d}s(\mathbf{y}), & (4.201) \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_0^\delta := & \left\{ \mathbf{u}^\delta \in \mathbf{H}(curl, \Omega_\delta) \,\middle|\, \langle\!\langle \mathbf{u}_T^\delta \rangle\!\rangle_{\Omega_\delta} \in \mathbf{H}(curl_\Gamma, \Gamma_0) \text{ and} \right. \\ & \left. \mathbf{n} \times \left(\mu^{-1} \nabla \times \mathbf{u}^\delta \right) \Big|_{\mathscr{S}} = \mathbf{0} \right\}, \end{aligned} \tag{4.202}$$

with the graph norm

$$\left\| \mathbf{u}^\delta \right\|^2_{\mathcal{H}_0^\delta} = \left\| \mathbf{u}^\delta \right\|^2_{\mathbf{H}(curl, \Omega_\delta)} + \left\| \langle\!\langle \mathbf{u}_T^\delta \rangle\!\rangle_{\Omega_\delta} \right\|^2_{\mathbf{H}(curl_\Gamma, \Gamma_0)}. \tag{4.203}$$

Observe that $a_{\Omega_\delta}(\cdot,\cdot)$ has the same structure as the sesquilinear form $a^+(\cdot,\cdot) + b(\cdot,\cdot)$, where $a^+(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are respectively defined in (4.50) and (4.51). Hence, repeating the arguments given in the proofs of Proposition 4.3.2 and Theorem 4.4.1, it is possible to prove that the problem (4.199) is well posed. Therefore, its unique solution is $\mathbf{E}^\delta = \mathbf{0}$. We then deduce that $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\sigma}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, and thus $\mathscr{H}^*$ is injective, implying that $\mathscr{H}$ has dense range.

Now, to show that $\mathscr{H}$ is injective, lets observe that if $\mathscr{H}(\mathbf{g}) = \mathbf{0}$, then in particular $\mathbf{n} \times \mu^{-1} \nabla \times \mathbf{E}_{b,\mathbf{g}} \Big|_{\mathscr{S}} = \mathbf{0}$, which by the assumptions is only possible if $\mathbf{g} = \mathbf{0}$. □

Given a regular surface $L \subset \Gamma$, we define

$$\mathcal{C}_{0,t}^\infty(L) := \left\{ \mathbf{u} \in (\mathcal{C}_0^\infty(L))^3 \,|\, \boldsymbol{\nu} \cdot \mathbf{u} = 0 \text{ on } L \right\}.$$

For any density $\mathbf{a}_L \in \mathcal{C}^\infty_{0,t}(L)$, we define ϕ^∞_L by

$$\boldsymbol{\phi}^\infty_L(\widehat{\mathbf{x}}) \;\; := \;\; \int_L \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}) \mathbf{a}_L(\mathbf{y}_\Gamma) \; \mathrm{d}\, \mathbf{y}_\Gamma. \tag{4.204}$$

The collection of functions $\boldsymbol{\phi}^\infty_L$ will be called *test functions*, and with them we will characterize the range of $\mathscr{G}$:

Lemma 4.6.1. *(Characterization of the range of $\mathscr{G}$) Let $L \subset \Gamma$, and $\mathbf{a}_L \in \mathcal{C}^\infty_{0,t}(L)$ such that $\mathbf{a}_L$ does not vanish in any open subset of L. Then $L \subset \Gamma_0$ if and only if $\boldsymbol{\phi}^\infty_L \in Range(\mathscr{G})$.*

Proof. Let $L \subset \Gamma_0$ and $\mathbf{a}_L \in \mathcal{C}^\infty_{0,t}(L)$. Then its extension by zero $\widetilde{\mathbf{a}}_L$ in Γ_0 belongs to $\mathcal{C}^\infty_t(\Gamma_0)$, and the corresponding test function

$$\boldsymbol{\phi}^\infty_L(\widehat{\mathbf{x}}) \;\; := \;\; \int_L \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{y}) \mathbf{a}_L(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) = \int_{\Gamma_0} \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}, \mathbf{y}) \widetilde{\mathbf{a}}_L(\mathbf{y}) \; \mathrm{d}s(\mathbf{y}) \tag{4.205}$$

is the far field pattern of $P\widetilde{\mathbf{a}}_L$, defined by:

$$(P\widetilde{\mathbf{a}}_L)(\mathbf{x}) := \begin{cases} \int_{\Gamma_0} \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}) \widetilde{\mathbf{a}}_L(\mathbf{y}) \; \mathrm{d}\, \mathbf{y}_\Gamma & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\delta \cup \Omega_-}, \\ \\ \int_{\Gamma_0} \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma - \delta f^- \boldsymbol{\nu}) \widetilde{\mathbf{a}}_L(\mathbf{y}_\Gamma) \; \mathrm{d}\, \mathbf{y}_\Gamma & \text{in } \Omega_-. \end{cases}$$

Due to well-known properties of the single- and double-layer potentials (see Remark 4.145),

$$\nabla \times \mu^{-1} \nabla \times P\widetilde{\mathbf{a}}_L - k^2 \epsilon P\widetilde{\mathbf{a}}_L = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\delta}, \tag{4.206}$$

$$[\![\boldsymbol{\nu} \times P\widetilde{\mathbf{a}}_L]\!] = \mathbf{0}, \tag{4.207}$$

$$[\![\boldsymbol{\nu} \times \left(\mu^{-1} \nabla \times P\widetilde{\mathbf{a}}_L\right)]\!] = ik\widetilde{\mathbf{a}}_L, \tag{4.208}$$

and $P\widetilde{\mathbf{a}}_L$ is a radiating field. Therefore, $P\widetilde{\mathbf{a}}_L$ is the solution to (4.152)-(4.155), for $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ defined by (4.156), for $\mathbf{v} = -P\widetilde{\mathbf{a}}_L$, and $\mathscr{G}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = \boldsymbol{\phi}^\infty_L$.

To prove the other direction, let $\mathbf{a}_L \in \mathcal{C}^\infty_{0,t}(L)$ such that $\boldsymbol{\phi}^\infty_L \in Range(\mathscr{G})$. Then there is $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) \in \mathbf{H}(curl_\Gamma, \Gamma_0) \times \mathcal{H}_0(\Gamma_0)^* \times \mathbf{H}^{-1/2}(div_\mathscr{S}, \mathscr{S})$ such that $\mathbf{E}$ is a solution to (4.152)-(4.155) and its far field pattern satisfies $\mathbf{E}^\infty = \boldsymbol{\phi}^\infty_L$.

On the other hand, $\boldsymbol{\phi}^\infty_L$ is also the far field pattern of the radiating field $\boldsymbol{\phi}_L$ defined by

$$\boldsymbol{\phi}_L(\mathbf{x}) \;\; := \;\; 4\pi \int_L \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\gamma + \delta \mathbf{f}^+ \boldsymbol{\nu}) \mathbf{a}_L(\mathbf{y}_\Gamma) \; \mathrm{d}\, \mathbf{y}_\Gamma.$$

By Rellich's lemma, $\mathbf{E}$ and $\boldsymbol{\phi}_L$ are identical in $\mathbb{R}^3 \setminus \overline{\Gamma_0 \cup L}$. Suppose $L \setminus \overline{\Gamma_0} \neq \emptyset$, then since $\mathbf{a}_L$ does not vanish in open sets of L there is $\mathbf{x} \in L \setminus \overline{\Gamma}_0$ such that $\mathbf{a}_L(\mathbf{x}) \neq 0$. Then $\boldsymbol{\nu} \times \mu^{-1} \nabla \times \mathbf{E}$ would be continuous at $\mathbf{x}$ while $\boldsymbol{\nu} \times \mu^{-1} \nabla \times \boldsymbol{\phi}_L$ would have a jump at that same point, which is a contradiction. □

Proposition 4.6.2. *Under the same hypothesis of Proposition 4.6.1, $\mathcal{F}_D : \mathbf{L}^2_t(\mathbb{S}^2) \to \mathbf{L}^2_t(\mathbb{S}^2)$ is injective with dense range.*

Proof. The fact that $\mathcal{F}_D = \mathscr{G}\mathscr{H}$ is injective is an immediate consequence of Proposition 4.6.1 and from the injectivity of $\mathscr{G}$, which follows from the well-posedness of (4.152)-(4.155).

To see that $\mathcal{F}_D$ has dense range, take $P : \mathbf{L}^2_t(\Gamma_0) \to \mathbf{L}^2_t(\mathbb{S}^2)$ defined by

$$(P\mathbf{a})(\widehat{\mathbf{d}}) \;=\; \frac{1}{4\pi} \int_{\Gamma_0} \mathbb{G}^{E,\infty}(\widehat{\mathbf{d}}, \mathbf{y})\mathbf{a}(\mathbf{y}) \;\mathrm{d}s(\mathbf{y}) \tag{4.209}$$

and then

$$\begin{aligned} \langle P\mathbf{a}, \mathbf{g} \rangle_{\mathbf{L}^2_t(\mathbb{S}^2)} &= 4\pi \int_{\mathbb{S}^2} \overline{\mathbf{g}(\widehat{\mathbf{d}})} \cdot \Big\{ \int_{\Gamma_0} \mathbb{G}^{E,\infty}(\widehat{\mathbf{d}}, \mathbf{y})\mathbf{a}(\mathbf{y}) \;\mathrm{d}s(\mathbf{y}) \Big\} \;\mathrm{d}s(\widehat{\mathbf{d}}) \\ &= \int_{\Gamma_0} \mathbf{a}(\mathbf{y}) \cdot \Big\{ \int_{\mathbb{S}^2} \mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{d}}, \overline{\mathbf{g}(\widehat{\mathbf{d}})}) \;\mathrm{d}s(\widehat{\mathbf{d}}) \Big\} \;\mathrm{d}s(\mathbf{y}), \end{aligned} \tag{4.210}$$

thus

$$(P^*\mathbf{g})(\mathbf{y}) \;=\; \int_{\mathbb{S}^2} \overline{\mathbf{E}_{pl}(\mathbf{y}, -\widehat{\mathbf{d}}, \overline{\mathbf{g}(\widehat{\mathbf{d}})})} \;\mathrm{d}s(\widehat{\mathbf{d}}) = \overline{\mathbf{E}_{b,\widetilde{\mathbf{g}}}(\mathbf{y})}, \tag{4.211}$$

where $\widetilde{\mathbf{g}} = \mathbf{g}(-\widehat{\mathbf{x}})$. Therefore, $P^*\mathbf{g} = \mathbf{0}$ if and only if $\mathbf{g} = \mathbf{0}$ and thus P has dense range. Since $Range(P) \subset Range(\mathcal{F}_D)$, the proof is complete. □

Now we can prove the standard theorem to justify the linear sampling method.

Theorem 4.6.3. *(linear sampling method) Let $\mathcal{F}_D : \mathbf{L}^2_t(\mathbb{S}^2) \to \mathbf{L}^2_t(\mathbb{S}^2)$ be the far field operator given by (4.149). Then the following hold:*

1. *For any arbitrary open surface $L \subset \Gamma_0$ and $\varepsilon > 0$, there exists a function $\mathbf{g}_\varepsilon \in \mathbf{L}^2_t(\mathbb{S}^2)$ such that*
$$\|\mathcal{F}_D \mathbf{g}_\varepsilon - \boldsymbol{\phi}^\infty_L\|_{\mathbf{L}^2_t(\mathbb{S}^2)} < \varepsilon,$$
and, as $\varepsilon \to 0$, the corresponding solution $\mathbf{E}_{b,\mathbf{g}_\varepsilon}$ to the background problem (4.111)-(4.112) converges in $\mathcal{H}_0$ to the unique solution $\mathbf{E}_L$ of (4.152)–(4.155) with $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ are given by (4.156), with $\mathbf{v} = \boldsymbol{\phi}^\infty_L$.

2. *For $L \not\subset \Gamma_0$ and $\varepsilon > 0$, every function $\mathbf{g}_\varepsilon \in \mathbf{L}^2_t(\mathbb{S}^2)$ such that*

$$\|\mathcal{F}_D \mathbf{g}_\varepsilon - \boldsymbol{\phi}^\infty_L\|_{\mathbf{L}^2_t(\mathbb{S}^2)} \leq \varepsilon$$

is such that the corresponding solution $\mathbf{E}_{b,\mathbf{g}_\varepsilon}$ to the background problem (4.111)-(4.112) satisfies

$$\lim_{\varepsilon \to 0} \|\mathbf{E}_{b,\mathbf{g}_\varepsilon}\|_{\mathbf{H}_{loc}(curl,\mathbb{R}^3)} = \infty, \quad \text{and} \quad \lim_{\varepsilon \to 0} \|\mathbf{g}_\varepsilon\|_{\mathbf{L}^2_t(\mathbb{S}^2)} = \infty.$$

Remark 4.6.3. *Theorem 4.6.3 is the basis for the NDT for the detection of the delaminated region $\Gamma_0 \subset \Gamma$, but it is worth noticing that from the definition (4.204), the test functions correspond to far field patterns of potentials given by*

$$\boldsymbol{\phi}_L(\mathbf{x}) := \int_L \mathbb{G}^E(\mathbf{x}, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}) \mathbf{a}_L(\mathbf{y}_\Gamma) \;\; d\mathbf{y}_\Gamma, \tag{4.212}$$

which are discontinuous on the shifted segment

$$L_+ := \{\mathbf{y} = \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}(\mathbf{y}_\gamma) d \,|\, \mathbf{y}_\gamma \in L\}, \tag{4.213}$$

that in principle we do not know, since δf^+ is an unknowns quantity. However, let τ is a tangential vector to $L \subset \Gamma$ at $\mathbf{y}_\Gamma$, then by the mixed reciprocity principle Theorem 4.6.1,

$$\begin{aligned} &4\pi\tau \cdot \mathbb{G}^{E,\infty}(\cdot, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}(\mathbf{y}_\Gamma)) \mathbf{a}_L(\mathbf{y}_\Gamma) \\ &\qquad = \mathbf{a}_L(\mathbf{y}_\Gamma) \cdot \mathbf{E}_{pl}(\mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}(\mathbf{y}_\Gamma), -\cdot, \tau), \end{aligned} \tag{4.214}$$

and since the tangential traces of $\mathbf{E}_{pl}(\cdot, -\widehat{\mathbf{x}}, \tau)$ are continuous, then if δ is small enough,

$$\begin{aligned} &4\pi\tau \cdot \mathbb{G}^{E,\infty}(\cdot, \mathbf{y}_\Gamma + \delta f^+ \boldsymbol{\nu}(\mathbf{y}_\Gamma)) \mathbf{a}_L(\mathbf{y}_\Gamma) \\ &\qquad \sim \mathbf{a}_L(\mathbf{y}_\Gamma) \cdot \mathbf{E}_{pl}(\mathbf{y}_\Gamma, -\cdot, \tau), \end{aligned} \tag{4.215}$$

which can be computed because it is defined on the known surface Γ.

4.7 Numerical experiments

In this part of the chapter we present some numerical results, using a numerical algorithm reconstruction based on the linear sampling method Theorem 4.6.3. In what follows we explain how the discretized far field operator and test functions were constructed. This is based on [23], p.47.

Let $\{\widehat{\mathbf{x}}_j\}_{j=1}^N \subset \mathbb{S}^2$ be the nodes and $\boldsymbol{\omega} = (\omega_1, ..., \omega_N)^T \in \mathbb{R}_+^N$ be the weights vector associated with a given quadrature rule on the unit sphere $\mathbb{S}^2$. We will set both the incidence and the observation directions to coincide with $\{\widehat{\mathbf{x}}_j\}_{j=1}^N$.

Therefore, the far-field operator acting on $\mathbf{g} \in \mathbf{L}_t^2(\mathbb{S}^2)^3$, satisfies

$$\begin{aligned}(\mathcal{F}\mathbf{g})(\widehat{\mathbf{x}}_i) &= \int_{\mathbb{S}^2} \mathbf{E}^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{d}}, \mathbf{g}(\widehat{\mathbf{d}}))\ \mathrm{d}s(\widehat{\mathbf{d}}) \\ &\sim \sum_{j=1}^N \omega_j \mathbf{E}_{comp}^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{x}}_j, \mathbf{g}(\widehat{\mathbf{x}}_j)). \end{aligned} \tag{4.216}$$

If we define for every $j = 1, ..., N$, given $\widehat{\mathbf{x}}_j \times \mathbf{p}_j \neq 0$,

$$\widehat{\mathbf{p}}_j^\theta := \frac{\mathbf{p}_j \times \widehat{\mathbf{x}}_j}{|\mathbf{p}_j \times \widehat{\mathbf{x}}_j|} \quad \text{and} \quad \widehat{\mathbf{p}}_j^\phi := \frac{\mathbf{p}_j \times (\widehat{\mathbf{x}}_j \times \mathbf{p}_j)}{|\mathbf{p}_j \times (\widehat{\mathbf{x}}_j \times \mathbf{p}_j)|}, \tag{4.217}$$

and for $\ell \in \{\theta, \phi\}$ and $j \in \{1, ..., N\}$

$$g_j^\ell := \mathbf{g}(\widehat{\mathbf{x}}_j) \cdot \widehat{\mathbf{p}}_j^\ell,$$

then by linearity,

$$\mathbf{E}_{comp}^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{x}}_j, \mathbf{g}(\widehat{\mathbf{x}}_j)) = \mathbf{E}_{comp}^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{x}}_j, \widehat{\mathbf{p}}_j^\theta) g_j^\theta + \mathbf{E}_{comp}^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{x}}_j, \widehat{\mathbf{p}}_j^\phi) g_j^\phi. \tag{4.218}$$

Thus denoting

$$\mathbf{A}_{i,j}^\ell = \mathbf{E}_{comp}^\infty(\widehat{\mathbf{x}}_i, \widehat{\mathbf{x}}_j, \widehat{\mathbf{p}}_j^\ell) \quad \text{for } \ell \in \{\theta, \phi\}, \tag{4.219}$$

at the discrete level the associated far field equation becomes

$$\sum_{j=1}^N \omega_j \mathbf{A}_{i,j}^\theta g_j^\theta + \omega_j \mathbf{A}_{i,j}^\phi g_j^\phi = \boldsymbol{\phi}_{\mathbf{z}}^\infty(\widehat{\mathbf{x}}_i). \tag{4.220}$$

However, the discrete far-field equation (4.220) is in tensor form. To get a standard matrix equation we take the dot product of both sides of equation (4.220) with $\widehat{\mathbf{p}}_j^\beta$, for $\beta \in \{\theta, \phi\}$ and thus

$$\sum_{j=1}^N \omega_j \mathbf{A}_{i,j}^{\beta,\theta} g_j^\theta + \omega_j \mathbf{A}_{i,j}^{\beta,\phi} g_j^\phi = f_{\mathbf{z}}^\beta(\widehat{\mathbf{x}}_i), \tag{4.221}$$

where

$$\mathbf{A}_{i,j}^{\beta,\ell} := \widehat{\mathbf{p}}_i^{\beta} \cdot \mathbf{A}_{i,j}^{\ell} = \widehat{\mathbf{p}}_i^{\beta} \cdot \mathbf{E}_{comp}^{\infty}(\widehat{\mathbf{x}}_i, \widehat{\mathbf{x}}_j, \widehat{\mathbf{p}}_j^{\ell}) \quad \text{and} \quad f_{\mathbf{z}}^{\beta}(\widehat{\mathbf{x}}_i) := \widehat{\mathbf{p}}_i^{\beta} \cdot \boldsymbol{\phi}_{\mathbf{z}}^{\infty}(\widehat{\mathbf{x}}_i). \tag{4.222}$$

In matrix form, if $\mathbf{M}^{\ell,\beta} \in \mathbb{C}^{N\times N}$ is defined by

$$\mathbf{M}^{\ell,\beta} = \mathbf{A}^{\ell,\beta}\mathbf{D}, \tag{4.223}$$

where $\mathbf{D} = diag\{\boldsymbol{\omega}\}$ is the diagonal matrix whose principal diagonal is the weights vector $\boldsymbol{\omega}$.

The far-field equation becomes

$$\left[\begin{array}{c|c} \mathbf{M}^{\theta,\theta} & \mathbf{M}^{\theta,\phi} \\ \hline \mathbf{M}^{\phi,\theta} & \mathbf{M}^{\phi,\phi} \end{array}\right]_{2N\times 2N} \left[\begin{array}{c} \mathbf{g}^{\theta} \\ \hline \mathbf{g}^{\phi} \end{array}\right]_{2N\times 1} = \left[\begin{array}{c} f_{\mathbf{z}}^{\theta} \\ \hline f_{\mathbf{z}}^{\phi} \end{array}\right]_{2N\times 1}.$$

As the discrete version of the ill-posedness of the far-field equation, this linear equation is also severely ill-posed but can be approximately solved using, for example, the standard Tikhonov regularization method. Given the solution $\mathbf{g}_{\mathbf{z},\eta}$ to the regularized problem associaled to the regularization parameter $0 < \eta \ll 1$, the indicator function that we would compute is given by

$$G_{\eta}(\mathbf{z}) = 1/||\mathbf{g}_{\mathbf{z},\eta}||. \tag{4.224}$$

In our case, for the practical examples, the test function will be considered when the surface $L \subset \Gamma$ shrinks to the point $\mathbf{z}$ and the density $\mathbf{a}_L$ tends to a delta function $\delta(\cdot - \mathbf{z})\boldsymbol{\tau}_{\ell}$, where $\boldsymbol{\tau}_{\ell}$ for $\ell = 1, 2$ are the basis of the tangential plane to Γ at $\mathbf{z}$. In such a case,

$$\boldsymbol{\phi}_{\mathbf{z},\ell}^{\infty}(\widehat{\mathbf{x}}_i) \quad = \quad \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}_i, \mathbf{z} + \delta f^{+}\boldsymbol{\nu})\boldsymbol{\tau}_{\ell} \quad \text{for} \quad \ell = 1, 2. \tag{4.225}$$

And then the right-hand-side terms for the discrete far field equation are:

$$f^{\beta}_{\mathbf{z},\ell}(\widehat{\mathbf{x}}_i) = \widehat{\mathbf{p}}^{\beta}_i \cdot \boldsymbol{\phi}^{\infty}_{\mathbf{z},\ell}(\widehat{\mathbf{x}}_i) \tag{4.226}$$

$$= \widehat{\mathbf{p}}^{\beta}_i \cdot \mathbb{G}^{E,\infty}(\widehat{\mathbf{x}}_i, \mathbf{z} + \delta f^{+}\boldsymbol{\nu})\boldsymbol{\tau}_\ell \tag{4.227}$$

$$= \frac{1}{4\pi}\boldsymbol{\tau}_\ell \cdot \mathbf{E}_b(\mathbf{z} + \delta f^{+}\boldsymbol{\nu}, -\widehat{\mathbf{x}}_i, \widehat{\mathbf{p}}^{\beta}_i)$$

$$\sim \frac{1}{4\pi}\boldsymbol{\tau}_\ell \cdot \mathbf{E}_b(\mathbf{z}, -\widehat{\mathbf{x}}_i, \widehat{\mathbf{p}}^{\beta}_i). \tag{4.228}$$

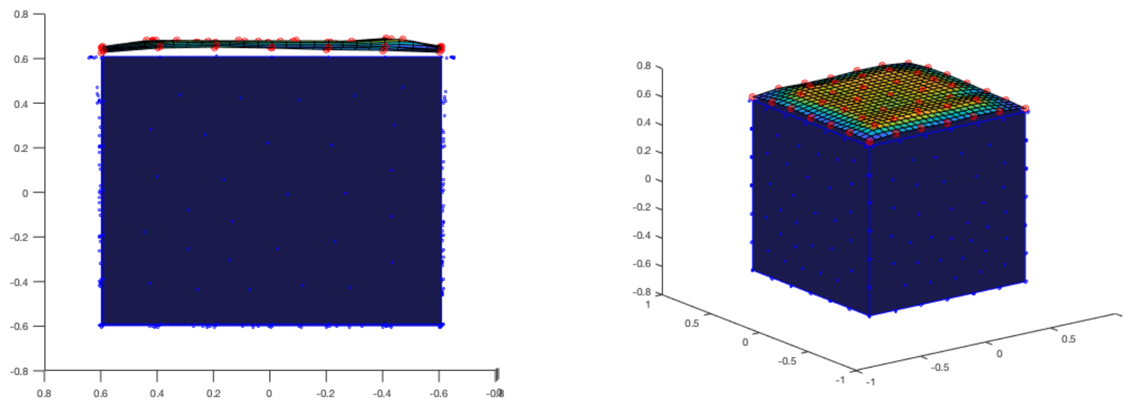
4.7.1 Numerical reconstruction example

The example we presented here was done for the configuration shown in Fig. (4.5), were Ω_- is the interior of a cube, and the delamination is located on one side of the cube.

The material physical and geometrical parameters in this particular example were: $k = 3$, $\delta = 0.01$, $\mu_- = \mu_+ = \mu_\delta = 1$, $\epsilon_+ = 2 + 0.001i$, $\epsilon_- = 4 + 0.001i$, $\epsilon_\delta = 3.5 + 0.001i$. In this example, the PML is spherical, with external boundary the sphere of radius $R_{pml} = 2.7$, and the internal boundary of radius $R = 2$. On the other hand Γ_1 is a sphere of radius $R_+ = 1.3$, and Γ is the surface of the cube centered at the origin and with side-length $l = 1.2$.

In total, the number of incident directions $\widehat{\mathbf{d}} \in \mathbb{S}^2$ was 93 in this example, and they were generated as the nodes of a uniform mesh on the unit sphere $\mathbb{S}^2$ constructed by Netgen/Ngsolve. The sampling points $\{\mathbf{z}_\ell\}_{\ell=1}^{N_s}$, $N_s = 152$, were constructed in a similar way, by defining a surface mesh on the cube. The regularization parameter for the Tikhonov regularization method in this example was chosen as $\eta = 10^{-10}$.

Both the far field data and the right-hand-side (4.228) were computed by solving the full problem (4.1)-(4.2) and (4.3), and using a finite element method for the variational formulation (4.95). The Nitsche's parameter was set as $\gamma = 10^3$, and the mesh refinement level $h_{max} = 0.2$. Some noise in the data was added in order to avoid numerical crimes. We consider $\tilde{A}_{ij} = A_{ij}(1 + \varepsilon\zeta_{ij})$, where $\{\zeta_{ij}\}$ is a collection of independent random variables with uniform distribution over the interval $[-0.5, 0.5]$, and $\varepsilon > 0$ is a constant. The level of noise is defined by $\rho := ||A - \tilde{A}||_2/||A||_2$.



(a) (b)

Figure 4.7: Panel Reconstruction of the delamination on one side of the cube, under $\rho = 5.5\%$ noise.

Conclusion

A Chun's-type ATCs model for the scattering of electromagnetic waves in the presence of planar delaminations with constant thickness was proposed, based on a particular case of the more general ATCs model whose complete derivation can be found in Proposition C.3.1.

After a well-posedness result was successfully established when the absortion in the whole obstacle is large, i.e. when $\Im(\epsilon_\pm)$, $\Im(\epsilon_\delta) \geq \epsilon_{min} > 0$, the ATCs model was used to develop a NDT of delamination by adapting the LSM to this case. Although the LSM proven analitically works only under the assumption that one of the boundaries of the delamination is known (Γ_+), the practical implementation still works when suitable approximations to the test function are considered and this extra impractical assumption is eliminated. The reconstruction algorithm has been successfully tested in a numerical example.

Chapter 5

OPEN PROBLEMS AND FUTURE WORK

As described before, this thesis combines asymptotic techniques and qualitative methods for the development of NDT to detect delaminations and cracks. The mathematical methodologies that were applied to solve the inverse scattering problems discussed in this thesis are diverse, and different lines of research may still be explored. Here we point out some open problems that arise from the analysis presented throughout the thesis.

Regarding the problem of crack detection in elastic materials discussed in Chapter 2, it would be interesting, for example, to consider the following two generalizations:

• Crack detection in layered anisotropic elastic materials.

• More general interfacial conditions on the crack.

Concerning the interfacial conditions, it would be a possibility, for example, to explore an ATCs model for elastic materials in the fashion of the asymptotic model that was described in Chapter 3 for the acoustic wave scattering. These models would presumably be more accurate to describe the so-called *delamination cracks* in elastic media.

In the case of inverse acoustic scattering for the detection of delaminated interfaces considered in Chapter 3, it would be interesting to

• Investigate the performance of the numerical algorithm for limited aperture data.

• Develop a mathematically more rigorous test via the the Generalized Linear Sampling Method (GLSM) [12, 11].

Moreover, due to the geometry of the defect, the singuarity of the solution at the edges of the delamination Ω_δ plays a role in the stability analysis of the asymptotic model as the thickness of the delamination goes to zero, and this has still to be theoretically understood (see Remark 3.2.1).

Regarding the asymptotic model for the scattering of electromagnetic waves in the presence of thin domains (delaminations), whose analysis constitutes the first part of Chapter 4, the problem of considering more general geometries is still the main issue. More specifically, it would be desirable to study the following two cases: **(a)** delaminations of constant thickness but sitting on curved interfaces, and **(b)** delaminations of variable thickness.

Although the second order ATCs model derived in Proposition C.3.1, takes into consideration the two general cases **(a)** and **(b)**, the complications arise in the integration-by-parts (4.25), which is the previous step in the derivation of a variational formulation of the problem:

$$\int_{\Gamma_-} \boldsymbol{\nu}^- \times (\mu_-^{-1}\nabla \times \mathbf{E}^-) \cdot \mathbf{v}^- \ \mathrm{d}s(\mathbf{y}) - \int_{\Gamma_+} \boldsymbol{\nu}^+ \times (\mu_+^{-1}\nabla \times \mathbf{E}^+) \cdot \mathbf{v}^+ \ \mathrm{d}s(\mathbf{y}). \tag{5.1}$$

Below we describe in more detail the complications that the cases **(a)** and **(b)** described before lead to.

• In case **(a)**, where the functions $f^\pm$ are constant (and thus $\boldsymbol{\nu} = \boldsymbol{\nu}^- = \boldsymbol{\nu}^+$), but the curvature of Γ_0 is not identically vanishing, then (5.1) becomes:

$$- \textstyle\int_{\Gamma_0} D^+\boldsymbol{\nu} \times (\mu_+^{-1}\nabla \times \mathbf{E}^+) \cdot \mathbf{v}^+ \ \mathrm{d}s(\mathbf{y}) \tag{5.2}$$
$$- \textstyle\int_{\Gamma_0} [\![\sqrt{D}\boldsymbol{\nu} \times (\mu^{-1}\nabla \times \mathbf{E})]\!] \cdot \langle\!\langle \sqrt{D}\mathbf{v} \rangle\!\rangle \ \mathrm{d}s(\mathbf{y})$$
$$- \textstyle\int_{\Gamma_0} \langle\!\langle \sqrt{D}\boldsymbol{\nu} \times (\mu^{-1}\nabla \times \mathbf{E}) \rangle\!\rangle \cdot [\![\sqrt{D}\mathbf{v}]\!] \ \mathrm{d}s(\mathbf{y}), \tag{5.3}$$

where $D^\pm = 1 \pm \delta(c_1 + c_2)f^\pm + \delta^2 c_1 c_2 (f^\pm)^2$ are the determinants of the Jacobians associated with the change of variables $\mathbf{x}_\Gamma \mapsto \mathbf{x}_\Gamma \pm \delta f^\pm \boldsymbol{\nu}(\mathbf{x}_\Gamma)$.

In [30], the authors have successfully addressed the case when $f^- = f^+$, noticing that the expression (5.3) simplifies in terms of the non-weighted jumps and average values $[\![\boldsymbol{\nu} \times (\mu^{-1}\nabla \times \mathbf{E})]\!]$ and $\langle\!\langle \boldsymbol{\nu} \times (\mu^{-1}\nabla \times \mathbf{E}) \rangle\!\rangle$, and thus the ATCs model can be used in such case. However, the extra hypothesis $f^- = f^+$ is not considered in our work since our main interest is the inverse problem of the detection of Γ_0, where $f^- = f^+$ cannot be *a priori* assumed.

• In case **(b)**, where the functions $f^{\pm} = f^{\pm}(\mathbf{x}_\Gamma)$ are non-constant but the principal curvatures c_1 and c_2 vanish identically on Γ_0, the determinants $D^{\pm}$ of the transformations $\mathbf{x}_\Gamma \mapsto \mathbf{x}_\Gamma \pm \delta f^{\pm} \boldsymbol{\nu}(\mathbf{x}_\Gamma)$ are $D^{\pm} = 1$ identically. In this case, (5.1) becomes:

$$\begin{aligned}\int_{\Gamma_-} \boldsymbol{\nu} \times (\mu_-^{-1} \nabla \times \mathbf{E}^-) \cdot \mathbf{v}^- \ \mathrm{d}s(\mathbf{y}) - \int_{\Gamma_+} \boldsymbol{\nu}^+ \times (\mu_+^{-1} \nabla \times \mathbf{E}^+) \cdot \mathbf{v}^+ \ \mathrm{d}s(\mathbf{y}) \\ = - \int_{\Gamma_0} [\![\boldsymbol{\nu}^{\pm} \times (\mu^{-1} \nabla \times \mathbf{E})]\!] \cdot \langle\!\langle \mathbf{v} \rangle\!\rangle \ \mathrm{d}s(\mathbf{y}) \\ - \int_{\Gamma_0} \langle\!\langle \boldsymbol{\nu}^{\pm} \times (\mu^{-1} \nabla \times \mathbf{E}) \rangle\!\rangle \cdot [\![\mathbf{v}]\!] \ \mathrm{d}s(\mathbf{y}),\end{aligned} \tag{5.4}$$

where we have included the superindices $\pm$ on the normal vectors defined on the boundary of the delamination, $\boldsymbol{\nu}^+$ and $\boldsymbol{\nu}^-$, to emphasize the fact that they do not coincide with the unit normal vector on Γ_0, $\boldsymbol{\nu}$. Therefore, in this case it is difficult to relate the terms in (5.4) to the ATCs expressions in Proposition C.3.1.

Appendix A

AUXILIARY LEMMAS FOR ELASTICITY

This Appendix contains some auxiliary technical results of linear elasticity, used in Chapter 2. The notation here, unless otherwise stated, corresponds to that chapter. We start by deriving an analogue of Green's representation formula.

Lemma A.0.1. *Let $D \subset \mathbb{R}^3$ be a bounded, connected and open domain with Lipschitz boundary ∂D. Let $\mathbf{u}^{sc} \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})^3$ be a radiating solution to $\Delta^*_{\lambda_0,\mu_0}\mathbf{u}^{sc} + \omega^2\mathbf{u}^{sc} = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. Then, if we denote by $\boldsymbol{\nu}_D$ the unit normal vector on ∂D that points to the exterior of D,*

$$\mathbf{u}^{sc}(\mathbf{x}) = \int_{\partial D} \left\{ \partial^*_{\boldsymbol{\nu}_D(y)} \boldsymbol{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y}) - \boldsymbol{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}_D}\mathbf{u}^{sc}(\mathbf{y}) \right\} \, ds(\mathbf{y}), \tag{A.1}$$

for all $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}$.

Proof. Let $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}$ and $R > 0$ large enough so that the open ball B_R of radius

$R > 0$ is such that $\{\mathbf{x}\} \cup \overline{D} \subset B_R$. Then, integrating by parts in $\Omega_R := B_R \setminus \overline{D}$,

$$\begin{aligned}
\mathbf{u}^{sc}(\mathbf{x}) &= -\int_{\Omega_R} (\Delta^*_{\lambda_0,\mu_0}\mathbf{\Gamma}_0(\mathbf{y},\mathbf{x}) + \omega^2\mathbf{\Gamma}_0(\mathbf{y},\mathbf{x})) \cdot \mathbf{u}^{sc}(\mathbf{y}) \ \mathrm{d}\,\mathbf{y} \\
&= \int_{\Omega_R} \Big\{\nabla\mathbf{u}^{sc}(\mathbf{y}) : \mathbf{C} : \nabla_y\mathbf{\Gamma}_0(\mathbf{y},\mathbf{x}) - \omega^2\mathbf{\Gamma}_0(\mathbf{y},\mathbf{x}) \cdot \mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}\,\mathbf{y} \\
&+ \int_{\partial D} \Big\{\partial^*_{\boldsymbol{\nu}_D(y)}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}) - \int_{\partial B_R} \Big\{\partial^*_{\widehat{y}}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}) \\
&= -\int_{\Omega_R} \Big\{\mathbf{\Gamma}_0(\mathbf{y},\mathbf{x})(\Delta^*_0\mathbf{u}^{sc}(\mathbf{y}) + \omega^2\mathbf{u}^{sc}(\mathbf{y}))\Big\} \ \mathrm{d}\,\mathbf{y} \\
&+ \int_{\partial D} \Big\{\partial^*_{\boldsymbol{\nu}_D(y)}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}_D}\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\partial B_R} \Big\{\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\widehat{y}}\mathbf{u}^{sc}(\mathbf{y}) - \partial^*_{\widehat{y}}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}) \\
&= \int_{\partial D} \Big\{\partial^*_{\boldsymbol{\nu}_D(y)}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}_D}\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}) \\
&+ \int_{\partial B_R} \Big\{\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\widehat{y}}\mathbf{u}^{sc}(\mathbf{y}) - \partial^*_{\widehat{y}}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}), \qquad \text{(A.2)}
\end{aligned}$$

Observe that when $R \to \infty$, the integral on ∂B_R in the last line of (A.2) vanishes, so that

$$\mathbf{u}^{sc}(\mathbf{x}) = \int_{\partial D} \Big\{\partial^*_{\boldsymbol{\nu}_D(y)}\mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\mathbf{u}^{sc}(\mathbf{y}) - \mathbf{\Gamma}_0(\mathbf{x},\mathbf{y})\partial^*_{\boldsymbol{\nu}_D}\mathbf{u}^{sc}(\mathbf{y})\Big\} \ \mathrm{d}s(\mathbf{y}). \qquad \text{(A.3)}$$

□

Using the previous result we can prove the following representation formula for the far field pattern of the Green's matrix $\mathbb{G}_b$ defined by (2.49).

Corollary A.0.1. *For all* $\mathbf{z} \in \Omega$*,*

$$\mathbb{G}^\infty_b(\widehat{\mathbf{x}},\mathbf{z}) = \int_{\Gamma_1} \Big\{\partial^*_{\boldsymbol{\nu}_{(y)}}\mathbf{\Gamma}^\infty_0(\widehat{\mathbf{x}},\mathbf{y})\mathbb{G}_b(\mathbf{y},\mathbf{z}) - \mathbf{\Gamma}^\infty_0(\widehat{\mathbf{x}},\mathbf{y})\partial^*_{\boldsymbol{\nu}_D}\mathbb{G}_b(\mathbf{y},\mathbf{z})\Big\} \ ds(\mathbf{y}), \qquad \text{(A.4)}$$

We now prove an identity of scattered elastic fields, which is a necessary auxiliary result for the proofs of the mixed reciprocity principle Theorem 2.3.1 and the propeties of the scattering operator $\mathcal{S}_b$ *stated in Proposition 2.3.1.*

Lemma A.0.2. *Let $D \subset \mathbb{R}^3$ be a bounded, connected and open domain with Lipschitz boundary ∂D. Let $\mathbf{u}$, $\mathbf{w} \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})^3$ be two regular radiating solutions of $\Delta_0^* \mathbf{u}^{sc} + \omega^2 \mathbf{u}^{sc} = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. Then, if we denote by $\boldsymbol{\nu}_D$ the unit normal vector on ∂D that points to the exterior of D,*

$$0 = \int_{\partial D} \left\{ \partial^*_{\boldsymbol{\nu}_D} \mathbf{u}(\mathbf{y}) \cdot \mathbf{w}(\mathbf{y}) - \mathbf{u}(\mathbf{y}) \cdot \partial^*_{\boldsymbol{\nu}_D} \mathbf{w}(\mathbf{y}) \right\} \, ds(\mathbf{y}). \tag{A.5}$$

Proof. Let $R > 0$ be large enough so that $\overline{D} \subset B_R$. Then, integrating by parts in $\Omega_R := B_R \setminus \overline{D}$,

$$\begin{aligned}
0 &= -\int_{\Omega_R} (\Delta_0^* \mathbf{w}(\mathbf{y}) + \omega^2 \mathbf{w}(\mathbf{y})) \cdot \mathbf{u}(\mathbf{y}) \, \mathrm{d}\,\mathbf{y} \\
&= \int_{\Omega_R} \left\{ \nabla \mathbf{u}(\mathbf{y}) : \mathbf{C}_0 : \nabla \mathbf{w}(\mathbf{y}) - \omega^2 \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}\,\mathbf{y} \\
&\quad + \int_{\partial D} \left\{ \partial^*_{\boldsymbol{\nu}_D} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}) - \int_{\partial B_R} \left\{ \partial^*_{\widehat{y}} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}) \\
&= -\int_{\Omega_R} \left\{ \mathbf{w}(\mathbf{y}) \cdot (\Delta_0^* \mathbf{u}(\mathbf{y}) + \omega^2 \mathbf{u}(\mathbf{y})) \right\} \, \mathrm{d}\,\mathbf{y} \\
&\quad + \int_{\partial D} \left\{ \partial^*_{\boldsymbol{\nu}_D} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) - \mathbf{w}(\mathbf{y}) \cdot \partial^*_{\boldsymbol{\nu}_D} \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}) \\
&\quad + \int_{\partial B_R} \left\{ \mathbf{w}(\mathbf{y}) \cdot \partial^*_{\widehat{y}} \mathbf{u}(\mathbf{y}) - \partial^*_{\widehat{y}} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}) \\
&= \int_{\partial D} \left\{ \partial^*_{\boldsymbol{\nu}_D} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) - \mathbf{w}(\mathbf{y}) \cdot \partial^*_{\boldsymbol{\nu}_D} \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}) \\
&\quad + \int_{\partial B_R} \left\{ \mathbf{w}(\mathbf{y}) \cdot \partial^*_{\widehat{y}} \mathbf{u}(\mathbf{y}) - \partial^*_{\widehat{y}} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}).
\end{aligned} \tag{A.6}$$

Now, from the radiation conditions (2.7),

$$\begin{aligned}
&\int_{\partial B_R} \left\{ \mathbf{w}(\mathbf{y}) \cdot \partial^*_{\widehat{y}} \mathbf{u}(\mathbf{y}) - \partial^*_{\widehat{y}} \mathbf{w}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}) \\
&= R \int_{\mathbb{S}^2} \left\{ \mathbf{w}(R\widehat{\mathbf{d}}) \cdot \partial^*_{\widehat{\mathbf{d}}} \mathbf{u}(R\widehat{\mathbf{d}}) - \partial^*_{\widehat{\mathbf{d}}} \mathbf{w}(R\widehat{\mathbf{d}}) \cdot \mathbf{u}(R\widehat{\mathbf{d}}) \right\} \, \mathrm{d}s(\widehat{\mathbf{d}}) \\
&= ik\alpha \int_{\mathbb{S}^2} \left\{ \mathbf{w}^\infty(\widehat{\mathbf{d}}) \cdot \mathbf{u}^\infty(\widehat{\mathbf{d}}) - \mathbf{w}^\infty(\widehat{\mathbf{d}}) \cdot \mathbf{u}^\infty(\widehat{\mathbf{d}}) \right\} \, \mathrm{d}s(\widehat{\mathbf{d}}) + O\left(\frac{1}{R}\right) \\
&= O\left(\frac{1}{R}\right),
\end{aligned} \tag{A.7}$$

and then taking the limit in (A.6) when $R \to \infty$, we get finally that

$$0 = \int_{\partial D} \left\{ \partial^*_{\boldsymbol{\nu}_D} \mathbf{u}(\mathbf{y}) \cdot \mathbf{w}(\mathbf{y}) - \mathbf{u}(\mathbf{y}) \cdot \partial^*_{\boldsymbol{\nu}_D} \mathbf{w}(\mathbf{y}) \right\} \, \mathrm{d}s(\mathbf{y}). \tag{A.8}$$

□

Appendix B

DERIVATION OF THE APPROXIMATE TRANSMISSION CONDITIONS FOR THE ACOUSTIC CASE

The material in this appendix concerns the detailed derivation of the second order approximate transmission conditions (ATCs) in $\mathbb{R}^2$ used in Chapter 3.
At the risk of repetition, to allow an independent study of this appendix, we describe here in detail the setting of the original acoustic scattering problem addressed in Chapter 3.

Denote by $\Omega \subset \mathbb{R}^2$, the support of a layered medium which is composed of two different materials adjacent to one another with constitutive material properties μ_+, n_+ and μ_-, n_-. We denote their bounded support by Ω_- and Ω_+, respectively, and the shared interface by $\Gamma := \partial\Omega_-$ (i.e. $\Omega = \Omega_- \cup \Omega_+$). Both the outer boundary $\partial\Omega_+$ of the domain Ω_+ and the boundary $\partial\Omega_-$ of the simply connected domain Ω_- are assumed to be piece-wise smooth, unless mentioned otherwise, and $\boldsymbol{\nu}$ denotes the unit normal always oriented outwards to the region bounded by the curve. For simplicity we let $\Omega_{\text{ext}} := \mathbb{R}^2 \setminus \overline{\Omega}$. Furthermore, we assume that along a part of the interface, denoted here by $\Gamma_0 \subset \Gamma$, these two materials have detached (delaminated) and we model this fact with the appearance of an opening with support Ω_δ and material properties μ_δ, n_δ (see Fig. B.1). Note that $\Gamma_0 = \Omega_\delta \cap \Gamma$. The material properties (possibly complex valued) in each of the domains are assumed to be smooth, i.e. $\mu_+, n_+ \in \mathcal{C}^1(\Omega_+)$, $\mu_-, n_- \in \mathcal{C}^1(\Omega_-)$ and $\mu_\delta, n_\delta \in \mathcal{C}^1(\Omega_\delta)$ (however note that across the interfaces there are discontinuities in the material properties). Assuming now that the incident field and the other fields in the problem are time harmonic (i.e. the time dependent incident field is of the form $\Re\left(u^i(\mathbf{x})e^{i\omega t}\right)$ where ω is the angular frequency), then the total field $u^{ext} = u^s + u^i$ in

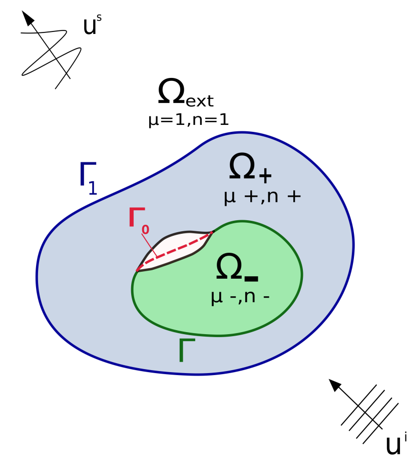


Figure B.1: Layered media with a thin delamination at the interface of two layers Ω_- and Ω_+. The opening Ω_δ, with coefficients μ_δ, n_δ is shown as the white region.

Ω_{ext}, where u^s is the scattered field, and the fields u^+, u^- and U inside Ω_+, Ω_- and Ω_δ, respectively, satisfy

$$\Delta u^{ext} + k^2 u^{ext} = 0 \qquad \text{in} \qquad \Omega_{\text{ext}} \tag{B.1}$$

$$\nabla \cdot \left(\frac{1}{\mu_+} \nabla u^+ \right) + k^2 n_+ u^+ = 0 \qquad \text{in} \qquad \Omega_+ \tag{B.2}$$

$$\nabla \cdot \left(\frac{1}{\mu_-} \nabla u^- \right) + k^2 n_- u^- = 0 \qquad \text{in} \qquad \Omega_- \tag{B.3}$$

$$\nabla \cdot \left(\frac{1}{\mu_\delta} \nabla U \right) + k^2 n_\delta U = 0 \qquad \text{in} \qquad \Omega_\delta. \tag{B.4}$$

Here the wave number $k = \omega / c_{\text{ext}}$ with c_{ext} denoting the sound speed of the homogeneous background. Across the interfaces the fields on either side and their conormal derivatives are continuous, i.e.

$$u^{ext} = u^+ \qquad \text{and} \qquad \frac{\partial u^{ext}}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_+}\frac{\partial u^+}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma_1 \tag{B.5}$$

$$u^+ = u^- \qquad \text{and} \qquad \frac{1}{\mu_+}\frac{\partial u^+}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_-}\frac{\partial u^-}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma\backslash\overline{\Gamma}_0 \tag{B.6}$$

$$U = u^+ \qquad \text{and} \qquad \frac{1}{\mu_\delta}\frac{\partial U}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_+}\frac{\partial u^+}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma_+ \tag{B.7}$$

$$U = u^- \qquad \text{and} \qquad \frac{1}{\mu_\delta}\frac{\partial U}{\partial \boldsymbol{\nu}} = \frac{1}{\mu_-}\frac{\partial u^-}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma_-. \tag{B.8}$$

Of course the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{B.9}$$

uniformly in $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, where $\mathbf{x} \in \mathbb{R}^2$ and $r = |\mathbf{x}|$. Let $\mathbf{x}_\Gamma \in \mathcal{C}^1[0, L]$ be the counter-

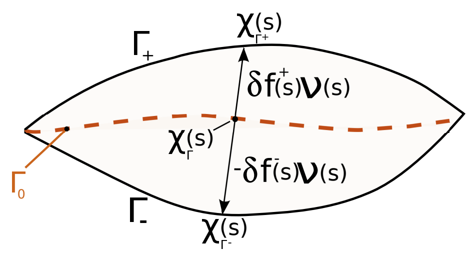


Figure B.2: Zoom of the thin delamination Ω_δ, and the parametrization of the boundaries Γ_- and Γ_+. Here δ scales the width of the delamination and is assumed small compared to other characteristic dimensions of the problem.

clockwise arc-length parametrization of Γ_0. If the curve Γ_0 is regular and $c(s)$ denotes its curvature at $\mathbf{x}_\Gamma(s)$, then $0 \leq c_m := \max\{|c(s)| \,:\, s \in [0, L]\}$ is finite. Hence, in the neighborhood of Γ_0, one can define the curvilinear coordinates $(s, \eta) \in [0, L] \times (-\frac{1}{c_m}, \frac{1}{c_m})$ by

$$\mathbf{x} = \mathbf{x}_\Gamma(s) + \eta\boldsymbol{\nu}(s),$$

where we recall that $\boldsymbol{\nu}$ is the unit normal vector on Γ_0 oriented outward to Ω_- (and taking $\frac{1}{c_m} = \infty$ if $c_m = 0$). Therefore, if the curvature of Γ_0 is small enough, both the outer and inner boundaries of Ω_δ, denoted here by Γ_+ and Γ_-, can be written in this coordinate system as

$$\Gamma_+ = \left\{ \mathbf{x}_{\Gamma_+}(s) := \mathbf{x}_\Gamma(s) + \delta f^+(s)\boldsymbol{\nu}(s), \quad s \in [0, L] \right\}$$

and

$$\Gamma_- = \left\{ \mathbf{x}_{\Gamma_-}(s) := \mathbf{x}_\Gamma(s) - \delta f^-(s)\boldsymbol{\nu}(s), \quad s \in [0, L] \right\}.$$

Note that the function $\delta(f^+ + f^-)(s)$ defined on Γ_0 describes the thickness of Ω_δ. Here $\delta > 0$ is a small parameter (compared to both the wave length and the size of the domains involved), and $\max_{s\in[0,L]} f^\pm(s) = 1$ (see Fig. B.2). In an open neighborhood of Ω_δ, we can now express the fields U, u^-, and u^+ in terms of the curvilinear variables (s, η). Ignoring small neighborhoods of the tip points $s = 0$ and $s = L$, since Ω_δ plays here the role of a boundary layer, in order to transfer the small parameter δ from the geometry to the expression of the fields we make a stretching change of variables inside Ω_δ defined by $\zeta = \frac{\eta}{\delta}$. Hence, $\zeta = \frac{\eta}{\delta}$ and s are now the new coordinates inside Ω_δ. Next, following [10] and [71], we formally make the following ansatz for the fields U and $u^\pm$ in an open neighborhood of Ω_δ:

$$U(s, \zeta) = \sum_{j=0}^{\infty} \delta^j U_j(s, \zeta) \text{ in } \Omega_\delta,$$

and

$$u^\pm(s, \eta) = \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \delta^j \frac{\eta^\ell}{\ell!} \frac{\partial^\ell}{\partial \eta^\ell} u_j^\pm(s, 0).$$

We will derive here the two sets of Dirichlet and Neumann transmission conditions for the first three terms in the asymptotic expansion, as well as the PDEs that they satisfy.

The transmission conditions

The Dirichlet transmission conditions

The original Dirichlet transmission conditions on the field u, correspond to the continuity of the total field across Ω_δ interface:

$$u^\pm = U \text{ on } \Gamma_\pm, \text{ respectively.}$$

Directly from the asymptotic expressions, on $\Gamma_\pm$ we have in particular the Dirichlet boundary conditions for the lowest order terms become:

- $j = 0$:
$$U_0(s, \pm f^\pm) = u_0^\pm(s, 0) \tag{B.10}$$

- $j = 1$:
$$U_1(s, \pm f^\pm) = \pm f^\pm \frac{\partial u_0^\pm}{\partial \eta}(s, 0) + u_1^\pm(s, 0) \tag{B.11}$$

- $j = 2$:
$$U_2(s, \pm f^\pm) = \frac{f^\pm}{2} \frac{\partial^2 u_0^\pm}{\partial \eta^2}(s, 0) \pm f^\pm \frac{\partial u_1^\pm}{\partial \eta}(s, 0) + u_2^\pm(s, 0). \tag{B.12}$$

The Neumann transmission conditions

Next, we turn our attention to the original Neumann transmission conditions of the field u, given by the continuity condition of the flux of the field at the boundary of the delamination:

$$\frac{1}{\mu_\pm} \nabla u^\pm \cdot \boldsymbol{\nu}^\pm = \frac{1}{\mu_\delta} \nabla U \cdot \boldsymbol{\nu}^\pm \text{ on } \Gamma_\pm, \text{ respectively.}$$

As derived in [10], the unit normal vectors $\boldsymbol{\nu}^\pm$ to $\Gamma_\pm$ have the following expressions:

$$\boldsymbol{\nu}^\pm = \frac{1}{|\boldsymbol{\tau}^\pm|}((1 \pm \delta f^\pm)\boldsymbol{\nu} \mp \delta f^{\pm\prime} \boldsymbol{\tau}),$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the outer unit normal vector and unit tangential vector defined on Γ_0, respectively, and $\boldsymbol{\tau}_\pm$ is the tangential vector defined on $\Gamma_\pm$ (which is not a unit vector in general).

Moreover, abusing notation and denoting by U the inner field both in Cartesian and in curvilinear coordinates, we have

$$\nabla U(\mathbf{x}) = \frac{1}{(1 + \eta c)} \frac{\partial U}{\partial s} \boldsymbol{\tau} + \frac{\partial U}{\partial \eta} \boldsymbol{\nu}.$$

where $c = c(s)$ is the curvature of Γ_0.

So we have all the ingredients to compute the Neumann transmission conditions:

$$\boldsymbol{\nu}^\pm \cdot \nabla u^\pm = \frac{1}{|\boldsymbol{\tau}^\pm|}((1 \pm \delta f^\pm) \frac{\partial u^\pm}{\partial \eta} \mp \frac{\delta f^{\pm\prime}}{(1 \pm \delta f^\pm)} \frac{\partial u^\pm}{\partial s})$$

and

$$\boldsymbol{\nu}^{\pm}\cdot\nabla U=\frac{1}{|\boldsymbol{\tau}^{\pm}|}((1\pm\delta f^{\pm})\frac{\partial U}{\partial\eta}\mp\frac{\delta f^{\pm\prime}}{(1\pm\delta f^{\pm})}\frac{\partial U}{\partial s})$$

so then

$$\begin{aligned}&\frac{1}{\mu_{\pm}}((1\pm\delta f^{\pm})\frac{\partial u^{\pm}}{\partial\eta}\mp\frac{\delta f^{\pm\prime}}{(1\pm\delta f^{\pm})}\frac{\partial u^{\pm}}{\partial s})\\ =\;&\frac{1}{\mu_{\delta}}((1\pm\delta f^{\pm})\frac{\partial U}{\partial\eta}\mp\frac{\delta f^{\pm\prime}}{(1\pm\delta f^{\pm})}\frac{\partial U}{\partial s})\end{aligned}\tag{B.13}$$

so

$$\begin{aligned}&(1\pm\delta f^{\pm})^2\Big(\frac{1}{\mu_{\pm}}\frac{\partial u^{\pm}}{\partial\eta}-\frac{1}{\delta\mu_{\delta}}\frac{\partial U}{\partial\zeta}\Big)\\ =\;&\pm\delta f^{\pm\prime}\Big(\frac{1}{\mu_{\pm}}\frac{\partial u^{\pm}}{\partial s}-\frac{1}{\mu_{\delta}}\frac{\partial U}{\partial s}\Big)\end{aligned}\tag{B.14}$$

and hence,

$$\begin{aligned}&(1\pm 2\delta f^{\pm}+\delta^2 f^{\pm^2})\Big(\tfrac{1}{\mu_{\pm}}\textstyle\sum_{j=0}^{\infty}\delta^j\sum_{\ell=0}^{j}\frac{(\pm1)^{j-\ell}(f^{\pm})^{j-\ell}}{j-\ell!}\frac{\partial^{j-\ell+1}}{\partial\eta^{j-\ell+1}}u_{\ell}^{\pm}(s,0)\\ &-\tfrac{1}{\mu_{\delta}}\textstyle\sum_{j=-1}^{\infty}\delta^j\frac{\partial}{\partial\zeta}U_{j+1}(s,\pm f^{\pm})\Big)\\ &=\pm\delta f^{\pm\prime}\Big(\tfrac{1}{\mu_{\pm}}\textstyle\sum_{j=0}^{\infty}\delta^j\sum_{\ell=0}^{j}\frac{(\pm1)^{j-\ell}(f^{\pm})^{j-\ell}}{j-\ell!}\frac{\partial^{j-\ell+1}}{\partial\eta^{j-\ell}\partial s}u_{\ell}^{\pm}(s,0)\\ &-\tfrac{1}{\mu_{\delta}}\textstyle\sum_{j=0}^{\infty}\delta^j\frac{\partial}{\partial s}U_j(s,\pm f^{\pm})\Big)\end{aligned}$$

which can be expressed as

$$\begin{aligned}&\textstyle\sum_{j=-1}^{\infty}\delta^j\left(\frac{1}{\mu_{\pm}}\sum_{\ell=0}^{j-1}\frac{(\pm1)^{j-\ell-1}(f^{\pm})^{j-\ell-1}}{j-\ell-1!}\frac{\partial^{j-\ell}}{\partial\eta^{j-\ell}}u_{\ell}^{\pm}(s,0)-\frac{1}{\mu_{\delta}}\frac{\partial}{\partial\zeta}U_{j+1}(s,\pm f^{\pm})\right)\\ &\pm f^{\pm}\textstyle\sum_{j=0}^{\infty}\delta^j\left(\frac{1}{\mu_{\pm}}\sum_{\ell=0}^{j-1}\frac{(\pm1)^{j-\ell-1}(f^{\pm})^{j-\ell-1}}{j-\ell-1!}\frac{\partial^{j-\ell}}{\partial\eta^{j-\ell}}u_{\ell}^{\pm}(s,0)-\frac{1}{\mu_{\delta}}\frac{\partial}{\partial\zeta}U_{j+1}(s,\pm f^{\pm})\right)\\ &+f^{\pm^2}\textstyle\sum_{j=1}^{\infty}\delta^j\left(\frac{1}{\mu_{\pm}}\sum_{\ell=0}^{j-1}\frac{(\pm1)^{j-\ell-1}(f^{\pm})^{j-\ell-1}}{j-\ell-1!}\frac{\partial^{j-\ell}}{\partial\eta^{j-\ell}}u_{\ell}^{\pm}(s,0)-\frac{1}{\mu_{\delta}}\frac{\partial}{\partial\zeta}U_{j+1}(s,\pm f^{\pm})\right)\\ &=\pm f^{\pm\prime}\textstyle\sum_{j=0}^{\infty}\delta^j\left(\frac{1}{\mu_{\pm}}\sum_{\ell=0}^{j}\frac{(\pm1)^{j-\ell}(f^{\pm})^{j-\ell}}{j-\ell!}\frac{\partial^{j-\ell+1}}{\partial\eta^{j-\ell}\partial s}u_{\ell}^{\pm}(s,0)-\frac{1}{\mu_{\delta}}\frac{\partial}{\partial s}U_j(s,\pm f^{\pm})\right).\end{aligned}$$

And then one can finally get the following expressions for the Neumann B.C.:

$$\begin{aligned}&\pm f^{\pm\prime}\left(\tfrac{1}{\mu_{\delta}}\tfrac{\partial U_{j-1}}{\partial s}(s,\pm f^{\pm})-\tfrac{1}{\mu_{\pm}}\textstyle\sum_{\ell=0}^{j-1}\frac{(\pm1)^{j-\ell-1}(f^{\pm})^{j-\ell-1}}{(j-\ell-1)!}\frac{\partial^{j-\ell}u_{\ell}^{\pm}}{\partial\eta^{j-\ell-1}\partial s}(s,0)\right)\\ &=\left(\tfrac{1}{\mu_{\delta}}\tfrac{\partial U_{j+1}}{\partial\zeta}(s,\pm f^{\pm})-\tfrac{1}{\mu_{\pm}}\textstyle\sum_{\ell=0}^{j}\frac{(\pm1)^{j-\ell}(f^{\pm})^{j-\ell}}{(j-\ell)!}\frac{\partial^{j-\ell+1}u_{\ell}^{\pm}}{\partial\eta^{j-\ell+1}}(s,0)\right)+\\ &\pm2f^{\pm}c\left(\tfrac{1}{\mu_{\delta}}\tfrac{\partial U_{j}}{\partial\zeta}(s,\pm f^{\pm})-\tfrac{1}{\mu_{\pm}}\textstyle\sum_{\ell=0}^{j-1}\frac{(\pm1)^{j-\ell-1}(f^{\pm})^{j-\ell-1}}{(j-\ell-1)!}\frac{\partial^{j-\ell}u_{\ell}^{\pm}}{\partial\eta^{j-\ell}}(s,0)\right)\\ &+c^2(f^{\pm})^2\left(\tfrac{1}{\mu_{\delta}}\tfrac{\partial U_{j-1}}{\partial\zeta}(s,\pm f^{\pm})-\tfrac{1}{\mu_{\pm}}\textstyle\sum_{\ell=0}^{j-2}\frac{(\pm1)^{j-\ell-2}(f^{\pm})^{j-\ell-2}}{(j-\ell-2)!}\frac{\partial^{j-\ell-1}u_{\ell}^{\pm}}{\partial\eta^{j-\ell-1}}(s,0)\right),\end{aligned}\tag{B.15}$$

for $j=-1,0,1,2,...$, for all $s\in[0,L]$, and with the convention that $u_\ell=0$ for negative ℓ.
Therefore, the three lowest order terms become:

- $j=-1$:

$$\frac{\partial U_0}{\partial \zeta}(s,\pm f^\pm)=0. \tag{B.16}$$

- $j=0$:

$$\frac{1}{\mu_\delta^\pm}\frac{\partial U_1}{\partial \zeta}(s,\pm f^\pm)=\frac{1}{\mu_\pm}\frac{\partial u_0^\pm}{\partial \eta}(s,0). \tag{B.17}$$

- $j=1$:

$$\begin{aligned}
&\pm f^\pm\left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_0}{\partial s}(s,\pm f^\pm)-\tfrac{1}{\mu_\pm}\tfrac{\partial u_0^\pm}{\partial s}(s,0)\right)\\
=&\left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_2}{\partial \zeta}(s,\pm f^\pm)-\tfrac{\pm f^\pm}{\mu_\pm}\tfrac{\partial^2 u_0^\pm}{\partial \eta^2}(s,0)-\tfrac{1}{\mu_\pm}\tfrac{\partial^2 u_1^\pm}{\partial \eta}(s,0)\right)\\
&\pm 2f^\pm\left(\tfrac{1}{\mu_\delta}\tfrac{\partial U_2}{\partial \zeta}(s,\pm f^\pm)-\tfrac{1}{\mu_\pm}\tfrac{\partial^2 u_0^\pm}{\partial \eta}(s,0)\right)\\
&+\tfrac{c^2(f^\pm)^2}{\mu_\delta}\tfrac{\partial U_0}{\partial \zeta}(s,\pm f^\pm).
\end{aligned}$$

The partial differential equations for the inner field

Considering the expression of the gradient and divergence in curvilinear coordinates:

$$\nabla\cdot\left(\frac{1}{\mu}\nabla U\right)=\frac{1}{(1+\eta c)}\frac{\partial}{\partial s}\left(\frac{1}{\mu}\frac{1}{(1+\eta c)}\frac{\partial U}{\partial s}\right)+\frac{1}{(1+\eta c)}\frac{\partial}{\partial \eta}\left(\frac{(1+\eta c)}{\mu}\frac{\partial U}{\partial \eta}\right),$$

it is possible to express the PDE of the field U by

$$\frac{1}{(1+\delta\zeta c)}\frac{\partial}{\partial s}\left(\frac{1}{\mu}\frac{1}{(1+\delta\zeta c)}\frac{\partial U}{\partial s}\right)+\frac{1}{\delta}\frac{1}{(1+\delta\zeta c)}\frac{\partial U}{\partial \zeta}\left(\frac{(1+\delta\zeta c)}{\delta\mu}\frac{\partial u}{\partial \zeta}\right)+k^2n_\delta U=0.$$

So substituting the Ansatz for U and collecting terms that correspond to same powers of δ, one gets the following equation:

$$\begin{aligned}
\frac{\partial}{\partial \zeta}\left(\frac{1}{\mu_\delta}\frac{\partial}{\partial \zeta}\right)U_j+\left(3\zeta c\frac{\partial}{\partial \zeta}\left(\frac{1}{\mu_\delta}\frac{\partial}{\partial \zeta}\right)+\frac{c}{\mu_\delta}\frac{\partial}{\partial \zeta}\right)U_{j-1}\quad &+\\
\left(\frac{\partial}{\partial s}\left(\frac{1}{\mu_\delta}\frac{\partial}{\partial s}\right)+3\zeta^2c^2\frac{\partial}{\partial \zeta}\left(\frac{1}{\mu_\delta}\frac{\partial}{\partial \zeta}\right)+\frac{2c^2\zeta}{\mu_o}\frac{\partial}{\partial \zeta}+k^2n_\delta\right)U_{j-2}\quad &+\\
\left(\zeta c\frac{\partial}{\partial s}\left(\frac{1}{\mu_\delta}\frac{\partial}{\partial s}\right)+\zeta^3c^3\frac{\partial}{\partial \zeta}\left(\frac{1}{\mu_\delta}\frac{\partial}{\partial \zeta}\right)+\frac{c^3\zeta^2}{\mu_o}\frac{\partial}{\partial \zeta}-\frac{\zeta c'}{\mu_\delta}\frac{\partial}{\partial s}+3\zeta ck^2n_\delta\right)U_{j-3}\quad &+\\
+3\zeta^2c^2k^2n_\delta U_{j-4}+\zeta^3c^3k^2n_\delta U_{j-5}=\quad &0,
\end{aligned} \tag{B.18}$$

for $j = 0, 1, 2...$, and where $c = c(s)$ is the curvature of Γ, again using the convention that $u_\ell = 0$ for negative ℓ.

B.1 Derivation of the approximate transmission conditions.

In what follows, we define two different kind of jumps:

1) **For the inner fields**. If V is a function in $H^1(\Omega_\delta)$, then the jump and mean values of V are refered to the traces of V on $\Gamma_\pm$. So they are respectively defined by:
$$[\![V]\!]_{\Omega_\delta} := V|_{\Gamma^+} - V|_{\Gamma^-} \text{ and } \langle\!\langle V\rangle\!\rangle_{\Omega_\delta} := \frac{1}{2}(V|_{\Gamma^+} + V|_{\Gamma^-}).$$

2) **For the outer fields**. if v^+ and v^- are functions in $H^1(\Omega_+ \cup \Omega_\delta)$ and $H^1(\Omega_- \cup \Omega_\delta)$, respectively, then the jump and mean values of v^+ and v^- on Γ_0 are defined as: $[v] := v^+|_{\Gamma_0} - v^-|_{\Gamma_0}$ and $\langle v\rangle := \frac{1}{2}(v^+|_{\Gamma_0} + v^-|_{\Gamma_0})$.

Using this notation, we derive the following:

- From (B.18), $j = 0$, we know that $\frac{\partial^2 U_0}{\partial \zeta^2} = 0$, so then
$$\frac{\partial U_0}{\partial \zeta}(s, \zeta) = [\![\frac{\partial U_0}{\partial \zeta}]\!]_{\Omega_\delta}(s),$$
and then from B.17, $\frac{\partial U_0}{\partial \zeta}(s, \zeta) = 0$, so
$$u_0(s, \zeta) = \langle u_0\rangle (s).$$
Now, since from B.10 we know that $u_0(s, \pm f^\pm) = u_0^\pm(s, \pm\delta f^\pm)$, then
$$[u_0] = 0. \tag{B.19}$$

- From (B.18), $j = 1$, $\frac{\partial^2 U_1}{\partial \zeta^2} = 0$, so then
$$\frac{\partial U_1}{\partial \zeta}(s, \zeta) = [\![\frac{\partial U_1}{\partial \zeta}]\!]_{\Omega_\delta}(s),$$
and hence $[\![\frac{1}{\mu_\delta}\frac{\partial U_1}{\partial \zeta}]\!]_{\Omega_\delta} = 0$. So from (B.17), $\frac{1}{\mu_\delta}\frac{\partial U_1}{\partial \zeta}(s, \pm f^\pm) = \frac{1}{\mu_\pm}\frac{\partial u_0}{\partial \eta}(s, \pm\delta f^\pm)$ and thus
$$\left[\frac{1}{\mu}\frac{\partial u_0}{\partial \eta}\right] = 0. \tag{B.20}$$

- From (B.11),
$$[u_1] = [\![U_1]\!]_{\Omega_\delta} - 2\left\langle f^\pm \frac{\partial u_0}{\partial \eta}\right\rangle,$$

and since

$$[\![U_1]\!]_{\Omega_\delta} \;=\; \int_{-f^-}^{f^+} \frac{\partial U_1}{\partial \zeta}(s,\zeta)\mathrm{d}\zeta = (f^+ + f^-)\langle\!\langle \frac{\partial U_1}{\partial \zeta}\rangle\!\rangle_{\Omega_\delta} = \mu_\delta (f^+ + f^-)\left\langle \frac{1}{\mu}\frac{\partial u_0}{\partial \eta}\right\rangle,$$

where for the last equality we used (B.17).
Therefore,

$$\begin{aligned}
[u_1] \;&=\; \mu_\delta (f^+ + f^-)\left\langle \frac{1}{\mu}\frac{\partial u_0}{\partial \eta}\right\rangle - 2\left\langle f^\pm \frac{\partial u_0}{\partial \eta}\right\rangle \\
&=\; \left(\mu_\delta \frac{(f^+ + f^-)}{2} - f^+\mu^+\right)\frac{1}{\mu_+}\frac{\partial u_0^+}{\partial \eta} \\
&+\; \left(\mu_\delta \frac{(f^+ + f^-)}{2} - f^-\mu^-\right)\frac{1}{\mu_-}\frac{\partial u_0^-}{\partial \eta} \\
&=\; \left(\mu_\delta \frac{(f^+ + f^-)}{2} - f^+\mu^+\right)\left\langle \frac{1}{\mu}\frac{\partial u_0}{\partial \eta}\right\rangle \\
&+\; \left(\mu_\delta \frac{(f^+ + f^-)}{2} - f^-\mu^-\right)\left\langle \frac{1}{\mu}\frac{\partial u_0}{\partial \eta}\right\rangle \\
&=\; 2\left\langle f(\mu_\delta - \mu)\right\rangle \left\langle \frac{1}{\mu}\frac{\partial u_0}{\partial \eta}\right\rangle. \qquad \text{(B.21)}
\end{aligned}$$

- From B.18,

$$\begin{aligned}
&2\left(\langle\!\langle \tfrac{f'}{\mu_\delta}\tfrac{\partial U_0}{\partial s}\rangle\!\rangle_{\Omega_\delta} - \left\langle \tfrac{f'}{\mu}\tfrac{\partial u_0}{\partial s}\right\rangle\right) \\
&= [\![\tfrac{1}{\mu_\delta}\tfrac{\partial U_2}{\partial \zeta}]\!]_{\Omega_\delta} - 2\left\langle \tfrac{f'}{\mu}\tfrac{\partial^2 u_0}{\partial \eta^2}\right\rangle - \left[\tfrac{1}{\mu}\tfrac{\partial u_1}{\partial \eta}\right] \\
&\quad +4c\left(\langle\!\langle \tfrac{f}{\mu_\delta}\tfrac{\partial U_0}{\partial \zeta}\rangle\!\rangle_{\Omega_\delta} - \left\langle \tfrac{f}{\mu}\tfrac{\partial u_0}{\partial \eta}\right\rangle\right),
\end{aligned}$$

so

$$\begin{aligned}
&2\left\langle f'\left(\frac{1}{\mu_\delta} - \frac{1}{\mu}\right)\right\rangle \frac{\partial}{\partial s}\left\langle u_0\right\rangle \\
&\qquad = [\![\frac{1}{\mu_\delta}\frac{\partial U_2}{\partial \zeta}]\!]_{\Omega_\delta} - 2\left\langle \frac{f'}{\mu}\frac{\partial^2 u_0}{\partial \eta^2}\right\rangle - \left[\frac{1}{\mu}\frac{\partial u_1}{\partial \eta}\right]. \qquad \text{(B.22)}
\end{aligned}$$

(i) Now, using (B.18) for $j = 2$,

$$\begin{aligned}
&[\![\tfrac{1}{\mu_\delta}\tfrac{\partial U_2}{\partial \zeta}]\!]_{\Omega_\delta} = \textstyle\int_{-f^-}^{f^+} \tfrac{\partial^2 U_2}{\partial \zeta^2}(s,\zeta)\mathrm{d}\zeta \\
&= \textstyle\int_{-f^-}^{f^+} \Big(- c\langle\!\langle \tfrac{1}{\mu_\delta}\tfrac{\partial U_1}{\partial \zeta}\rangle\!\rangle_{\Omega_\delta} - \tfrac{1}{\mu_\delta}\tfrac{\partial^2}{\partial \zeta^2}\langle\!\langle U_0\rangle\!\rangle_{\Omega_\delta} - k^2 n_\delta \langle\!\langle U_0\rangle\!\rangle_{\Omega_\delta}\Big)\mathrm{d}\zeta
\end{aligned}$$

and hence

$$
\begin{aligned}
[\![\tfrac{1}{\mu_\delta}\tfrac{\partial U_2}{\partial \zeta}]\!]_{\Omega_\delta} &= -c(f^- + f^+)\langle\!\langle \tfrac{1}{\mu_\delta}\tfrac{\partial U_1}{\partial \zeta}\rangle\!\rangle_{\Omega_\delta} \\
&\quad -\tfrac{(f^-+f^+)}{\mu_\delta}\tfrac{\partial^2}{\partial s^2}\langle\!\langle U_0\rangle\!\rangle_{\Omega_\delta} - (f^- + f^+)k^2 n_\delta \langle\!\langle U_0\rangle\!\rangle_{\Omega_\delta}
\end{aligned} \tag{B.23}
$$

(ii) From the differential equation that the outer field satisfies, one knows that:

$$
\tfrac{1}{\mu_\pm}\tfrac{\partial^2 u_0^\pm}{\partial \eta^2}(s, \pm\delta f^\pm) = \tfrac{1}{\mu_\pm}\tfrac{\partial^2 u_0^\pm}{\partial s^2}(s, \pm\delta f^\pm) + \tfrac{1}{\mu_\pm} c \tfrac{\partial u_0^\pm}{\partial \eta}(s, \pm\delta f^\pm) + k^2 n^\pm u_0^\pm,
$$

and hence:

$$
-2\left\langle \tfrac{f}{\mu}\tfrac{\partial^2 u_0}{\partial \eta^2}\right\rangle = 2\left\langle \tfrac{1}{\mu}\tfrac{\partial^2 u_0}{\partial s^2}\right\rangle + 2c\left\langle \tfrac{1}{\mu}\tfrac{\partial u_0}{\partial \eta}\right\rangle + 2k^2\langle n\, u_0\rangle . \tag{B.24}
$$

Therefore, and substituting (B.23) and (B.24) in (B.22),

$$
\begin{aligned}
& 2\left\langle f'\left(\tfrac{1}{\mu_\delta} - \tfrac{1}{\mu}\right)\right\rangle \tfrac{\partial}{\partial s}\langle u_0\rangle \\
&= -c(f^- + f^+)\langle\!\langle \tfrac{1}{\mu_\delta}\tfrac{\partial U_1}{\partial \zeta}\rangle\!\rangle_{\Omega_\delta} - \tfrac{(f^-+f^+)}{\mu_\delta}\tfrac{\partial^2}{\partial s^2}\langle\!\langle U_0\rangle\!\rangle_{\Omega_\delta} \\
&\quad -(f^- + f^+)k^2 n_\delta\langle\!\langle U_0\rangle\!\rangle_{\Omega_\delta} + 2\left\langle \tfrac{f}{\mu}\tfrac{\partial^2 u_0}{\partial s^2}\right\rangle + 2c\left\langle \tfrac{f}{\mu}\tfrac{\partial u_0}{\partial \eta}\right\rangle \\
&\quad +2k^2\langle f n u_0\rangle - \left[\tfrac{1}{\mu}\tfrac{\partial u_1}{\partial \eta}\right] \\
&= 2\left\langle f\left(\tfrac{1}{\mu} - \tfrac{1}{\mu_\delta}\right)\right\rangle \tfrac{\partial^2}{\partial s^2}\langle u_0\rangle + 2k^2\langle f(n - n_\delta)\rangle\langle u_0\rangle - \left[\tfrac{1}{\mu}\tfrac{\partial u_1}{\partial \eta}\right],
\end{aligned}
$$

and then,

$$
\left[\frac{1}{\mu}\frac{\partial u_1}{\partial \eta}\right] \;=\; 2\left\langle \left(\frac{1}{\mu} - \frac{1}{\mu_\delta}\right)\frac{\partial}{\partial s}\left(f\frac{\partial}{\partial s}\right)\right\rangle\langle u_0\rangle + 2k^2\langle f(n - n_\delta)\rangle\langle u_0\rangle .
$$

Moreover, if $\mu_\pm$ are constant along Γ_0,

$$
\left[\frac{1}{\mu}\frac{\partial u_1}{\partial \eta}\right] \;=\; 2\frac{\partial}{\partial s}\left(\left\langle f\left(\frac{1}{\mu} - \frac{1}{\mu_\delta}\right)\right\rangle\frac{\partial}{\partial s}\right)\langle u_0\rangle + 2k^2\langle f(n - n_\delta)\rangle\langle u_0\rangle .
$$

In summary, on one hand we deduce for the jump of the field u:

$$
\begin{aligned}
[u] &= [u_0] + \delta\,[u_1] + O(\delta^2) \\
&= \delta\alpha\left\langle \frac{1}{\mu}\frac{\partial u_0}{\partial \boldsymbol{\nu}}\right\rangle + O(\delta^2) \\
&= \delta\alpha\left\langle \frac{1}{\mu}\frac{\partial u}{\partial \boldsymbol{\nu}}\right\rangle + O(\delta^2),
\end{aligned} \tag{B.25}
$$

(B.26)

where in the second line we have used (B.19) and (B.21), and have defined $\alpha = 2\left\langle f(\mu_\delta - \mu)\right\rangle$.

On the other hand, for the jump on the flux:

$$\begin{aligned}
\left[\frac{1}{\mu}\frac{\partial u}{\partial \boldsymbol{\nu}}\right] &= \left[\frac{1}{\mu}\frac{\partial u_0}{\partial \boldsymbol{\nu}}\right] + \delta\left[\frac{1}{\mu}\frac{\partial u_1}{\partial \boldsymbol{\nu}}\right] + O(\delta^2) \\
&= \delta\left(-\frac{\partial}{\partial s}\left(\beta\frac{\partial}{\partial s}\right)\langle u_0\rangle + \gamma\langle u_0\rangle\right) + O(\delta^2) \\
&= \delta\left(-\frac{\partial}{\partial s}\left(\beta\frac{\partial}{\partial s}\right) + \gamma\right)\langle u\rangle + O(\delta^2), \qquad \text{(B.27)}
\end{aligned}$$

(B.28)

where in the second line we have substituted (B.20) and (B.25), and have defined $\gamma = 2k^2\left\langle f(n - n_\delta)\right\rangle$ and $\beta = 2\left\langle f\left(\frac{1}{\mu_\delta} - \frac{1}{\mu}\right)\right\rangle$.

By neglecting the terms of order $O(\delta^2)$ in (B.25) and (B.27), we finally get that the second order approximate transmission conditions (ATCs) for acoustic scattering are given by,

$$[u] = \delta\alpha\left\langle\frac{1}{\mu}\frac{\partial u}{\partial \boldsymbol{\nu}}\right\rangle, \qquad \text{(B.29)}$$

$$\left[\frac{1}{\mu}\frac{\partial u}{\partial \boldsymbol{\nu}}\right] = \delta\left(-\frac{\partial}{\partial s}\left(\beta\frac{\partial}{\partial s}\right) + \gamma\right)\langle u\rangle. \qquad \text{(B.30)}$$

Appendix C

DERIVATION OF ATC MODELS FOR ELECTROMAGNETIC SCATTERING IN THE PRESENCE OF A DELAMINATION

C.1 The full model for the scattering of electromagnetic waves in the presence of delamination

Let Ω be an inhomogeneity in free space, which consists of a bounded connected subset of $\mathbb{R}^3$. We are interested in the case where Ω is a composite material consisting of two layers, Ω_- and Ω_+, with a thin opening at their interface that we will denote by Ω_δ. Here Ω_- is the inner layer, a bounded simply connected subset of $\mathbb{R}^3$, and Ω_+ is the outer layer (see Fig. C.1).

Denote by $\partial\Omega_- \cap \partial\Omega_+$ the interface between the two layers Ω_- and Ω_+, and let Γ_0 be any smooth curve such that $\Gamma = \overline{\Gamma_0 \cup (\partial\Omega_- \cap \partial\Omega_+)}$ is a smooth surface. Assume in addition that the relative boundary of Γ_0 in Γ is a Lipschitz continuous curve. The

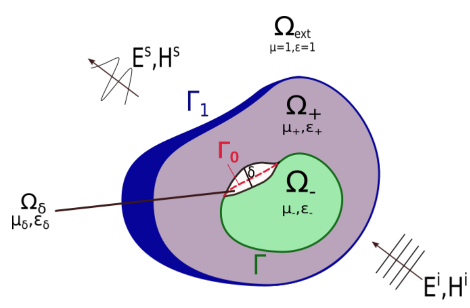


Figure C.1: Layered media with a thin delamination at the interface of two layers Ω_- and Ω_+. The opening Ω_δ is the thin domain.

aim of this section is to study how an electromagnetic wave scatters in the presence of the thin delaminated domain Ω_δ.

Let's observe that the inhomogeneity is $\Omega = \Omega_+ \cup \overline{\Omega_- \cup \Omega_\delta}$. If we define the total electric and magnetic fields as follows:

$$(\mathbf{E}, \mathbf{H}) = \begin{cases} (\mathbf{E}^{ext}, \mathbf{H}^{ext}) & \text{in } \mathbb{R}^3 \backslash \overline{\Omega}, \\ (\mathbf{E}^{\pm}, \mathbf{H}^{\pm}) & \text{in } \Omega_{\pm}, \\ (\mathbf{E}^{\delta}, \mathbf{H}^{\delta}) & \text{in } \Omega_{\delta}, \end{cases}$$

then the boundary value problem that the total fields (E, H) satisfy is

$$ik\mathbf{H}^{ext} - curl(\mathbf{E}^{ext}) = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\Omega}, \tag{C.1}$$

$$ik\mathbf{E}^{ext} + curl(\mathbf{H}^{ext}) = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\Omega}, \tag{C.2}$$

$$ik\mu_{\pm}\mathbf{H}^{\pm} - curl(\mathbf{E}^{\pm}) = 0 \quad \text{in } \Omega_{\pm}, \tag{C.3}$$

$$ik\epsilon_{\pm}\mathbf{E}^{\pm} + curl(\mathbf{H}^{\pm}) = 0 \quad \text{in } \Omega_{\pm}, \tag{C.4}$$

$$ik\mu_{\delta}\mathbf{H}^{\delta} - curl(\mathbf{E}^{\delta}) = 0 \quad \text{in } \Omega_{\delta}, \tag{C.5}$$

$$ik\epsilon_{\delta}\mathbf{E}^{\delta} + curl(\mathbf{H}^{\delta}) = 0 \quad \text{in } \Omega_{\delta}, \tag{C.6}$$

along with the transmission conditions

$$\boldsymbol{\nu}^{\pm} \times \mathbf{E}^{\pm} = \boldsymbol{\nu}^{\pm} \times \mathbf{E}^{\delta} \quad \text{on} \quad \Gamma_{\pm} \cup \Gamma_1, \tag{C.7}$$

$$\boldsymbol{\nu}^{\pm} \times \mathbf{H}^{\pm} = \boldsymbol{\nu}^{\pm} \times \mathbf{H}^{\delta} \quad \text{on} \quad \Gamma_{\pm} \cup \Gamma_1. \tag{C.8}$$

In addition, in $\mathbb{R}^3 \backslash \overline{\Omega}$, $(\mathbf{E}^{ext}, \mathbf{H}^{ext}) = (\mathbf{E}^i, \mathbf{H}^i) + (\mathbf{E}^s, \mathbf{H}^s)$, where $(\mathbf{E}^i, \mathbf{H}^i)$ is the incident field and $(\mathbf{E}^s, \mathbf{H}^s)$ is the scattered field resulting from the interaction with the inhomogeneity, and that satisfies the Silver-Müller radiation condition:

$$\lim_{r \to \infty} r \left(\mathbf{H}^s \times \widehat{\mathbf{x}} - \mathbf{E}^s \right) = 0 \tag{C.9}$$

uniformly in $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, where $r = |\mathbf{x}|$.

C.1.1 Elements of Differential Geometry

In this section we follow the notation and ideas of [47]. Let Γ_0 be a $\mathcal{C}^2$- surface, then for every point $\mathbf{x}_\Gamma \in \Gamma_0$, the unit normal vector to Γ_0 at $\mathbf{x}_\Gamma$ pointing to infinity,

$\boldsymbol{\nu}(\mathbf{x}_\Gamma)$, is well defined. Moreover, there is area number $0 < \eta_*$ such that in the open neighborhood $\mathcal{N} := \{\mathbf{x} \in \mathbb{R}^3 \mid \min_{\mathbf{x}\in\Gamma_0} |\mathbf{x} - y| < \eta_*\}$ of Γ_0, the projection $\Pi_{||} : \mathcal{N} \to \Gamma_0$ such that

$$\mathbf{x} \mapsto^{\Pi_{||}} \mathbf{x}_\Gamma := argmin\{|\mathbf{x} - y| \ : \ y \in \Gamma_0\}$$

is also well defined.

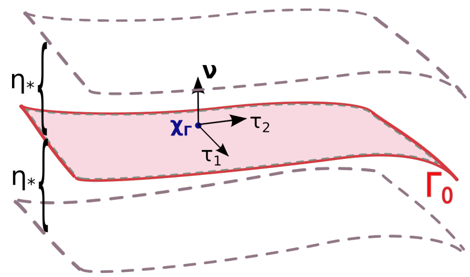


Figure C.2: Curvilinear coordnates on the neighborhood of Γ_0.

Then the mapping

$$(\mathbf{x}_\Gamma, \eta) \mapsto \mathbf{x} = \mathbf{x}_\Gamma + \eta\boldsymbol{\nu}(\mathbf{x}_\Gamma),$$

is an isomorphism from $\Gamma_0 \times (-\eta_*, \eta_*)$ to $\mathcal{N}$, and therefore it defines a valid change of coordinates around Γ_0.

Now, let's define the tangential vector field $\breve{\boldsymbol{\nu}}$ in $\mathcal{N}$ by

$$\breve{\boldsymbol{\nu}}(\mathbf{x}_\Gamma + s\boldsymbol{\nu}(\mathbf{x}_\Gamma)) := \boldsymbol{\nu}(\mathbf{x}_\Gamma),$$

then the *curvature tensor* is the symmetric linear operator $\mathcal{C}_{\mathbf{x}_\Gamma} : \mathbb{R}^3 \to T_{\mathbf{x}_\Gamma}$ defined by

$$\mathcal{C}_{\mathbf{x}_\Gamma} := \nabla_\Gamma \breve{\boldsymbol{\nu}}(\mathbf{x}_\Gamma). \tag{C.10}$$

Abusing notation, we will omit the explicit dependence on $\mathbf{x}_\Gamma$ and simply denote the curvature tensor as $\mathcal{C}$.

The (real) eigenvalues of $\mathcal{C}$, denoted by $c_1, c_2, 0$, are associated with the set of orthonormal vectors $\{\boldsymbol{\tau}_1^p, \boldsymbol{\tau}_2^p, \boldsymbol{\nu}\}$. The tangential vectors $\boldsymbol{\tau}_1^p, \boldsymbol{\tau}_2^p$ associated with the possibly non-zero eigenvalues $c_1 \leq c_2$, are called the directions of principal curvature of Γ_0.

The tangent plane to Γ_0 at $\mathbf{x}_\Gamma$ will be denoted by $T_{\mathbf{x}_\Gamma}$. If $\boldsymbol{\xi} = (\xi_1, \xi_2) \mapsto \mathbf{x}_\Gamma$ is a parametrization of a neighborhood of $\mathbf{x}_\Gamma$ in Γ_0, then we can define the *covariant* base of $T_{\mathbf{x}_\Gamma}$, consisting of the vectors $\{\boldsymbol{\tau}_\alpha := \partial_{\xi_\alpha}\mathbf{x}_\Gamma\}_{\alpha=1,2}$. The associated *contravariant* (or dual) base is $\{\boldsymbol{\tau}^\alpha\}_{\alpha=1,2} \subset T^*_{\mathbf{x}_\Gamma}$ defined by

$$\langle \boldsymbol{\tau}^\alpha, \boldsymbol{\tau}_\beta \rangle_{T_{\mathbf{x}_\Gamma}, T^*_{\mathbf{x}_\Gamma}} = \delta_{\alpha,\beta}$$

for $\alpha, \beta = 1, 2$, and where $\delta_{\alpha,\beta}$ is the Kronecker symbol and $\langle \cdot, \cdot \rangle_{T_{\mathbf{x}_\Gamma}, T^*_{\mathbf{x}_\Gamma}}$ is the duality pairing. **Surface differential operators.**

1. Given a scalar field u defined on Γ_0, one can compute its surface gradient defined by

$$\nabla_\Gamma \mathbf{u}(\mathbf{x}_\Gamma) := \nabla \check{\mathbf{u}}(\mathbf{x}_\Gamma),$$

where, in turn, the scalar field $\check{\mathbf{u}} : \mathcal{N} \to \mathbb{C}$ is defined by $\check{\mathbf{u}}(\mathbf{x}_\Gamma + \eta \boldsymbol{\nu}(\mathbf{x}_\Gamma)) := \mathbf{u}(\mathbf{x}_\Gamma)$.

In terms of the contravariant basis $\{\boldsymbol{\tau}_\alpha\}$, it can be written as

$$\nabla_\Gamma = (\partial_{\xi_1} \cdot)\boldsymbol{\tau}_1 + (\partial_{\xi_2} \cdot)\boldsymbol{\tau}_2$$

the adjoint of ∇_Γ is $-div_\Gamma$, which coincides with

$$div_\Gamma \mathbf{v} = \partial_{\xi_1}(\mathbf{v} \cdot \boldsymbol{\tau}_1) + \partial_{\xi_2}(\mathbf{v} \cdot \boldsymbol{\tau}_2),$$

for all $\mathbf{v}$ defined in $\mathcal{N}$.

Using the proof of Lemma 2.6 in [47], we know that if v is a vector field in $\mathbb{R}^3$ we can express the differential operator *curl* in these new curvilinear coordinates $(\mathbf{x}_\Gamma, \eta)$ as

$$curl(v) = (\mathcal{R}_\eta \boldsymbol{\tau}^\alpha \times \partial_{\xi_\alpha} v) + \boldsymbol{\nu} \times \partial_\eta v, \tag{C.11}$$

where for each $\mathbf{x}_\Gamma \in \Gamma_0$ the linear operator $\mathcal{R}_\eta : \mathcal{N} \to T_{\mathbf{x}_\Gamma}$ is well defined as the unique linear extension that satisfies the following two conditions:

1. Restricted to $\mathbf{x} \in T_{\mathbf{x}_\Gamma}$, $\mathcal{R}_\eta$ is the inverse of $\mathbf{x} \mapsto \mathbf{x} + \eta\mathcal{C}\mathbf{x}$.
2. $\mathcal{R}_\eta \boldsymbol{\nu} = 0$.

In other words, $\mathcal{R}_\eta(\Pi_{||} + \eta\mathcal{C}) = \Pi_{||}$ and $\mathcal{R}_\eta\boldsymbol{\nu} = 0$. Therefore if $|\eta| < 1$, we have the following expression:

$$\mathcal{R}_\eta = \sum_{l=0}^{\infty} (-\eta)^l \mathcal{C}^l, \quad \text{where} \quad \mathcal{C}^0 := \Pi_{||}. \tag{C.12}$$

Local parametrization of the surfaces $\Gamma_\pm$

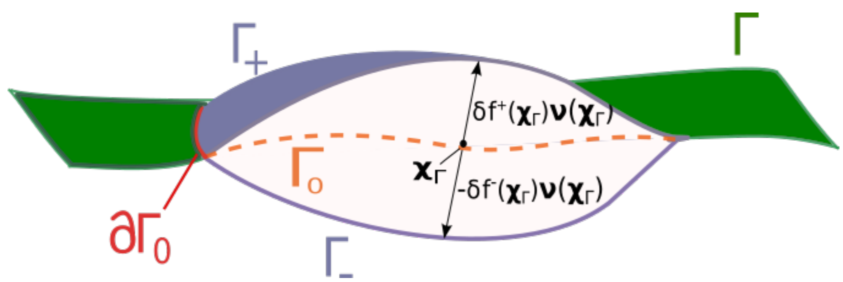


Figure C.3: Zoom on the thin domain Ω_δ.

Since the surface Γ_0 is Riemannian, then we know that for all $\mathbf{x}_\Gamma \in \Gamma_0$, there is an open neighborhood of $\mathbf{x}_\Gamma$ in Γ_0 such that the parametrization $\boldsymbol{\xi} = (\xi_1, \xi_2) \mapsto \mathbf{x}_\Gamma$ satisfies that the covariant vectors $\boldsymbol{\tau}_\alpha := \partial_{\xi_\alpha}\mathbf{x}_\Gamma$, for $\alpha = 1, 2$, i.e. the parametrization is alligned with the directions of principal curvature.

Let's now suppose that our delaminated domain Ω_δ is thin enough so that $\Omega_\delta \subset \mathcal{N}$, and then the two boundaries $\Gamma_\pm$ of Γ_0 can be writen in our new curvilinear coordinates as follows:

$$\Gamma_+ := \{\mathbf{x}_{\Gamma_+} = \mathbf{x}_\Gamma + \delta f^+(\mathbf{x}_\Gamma)\boldsymbol{\nu}(\mathbf{x}_\Gamma) \, : \, \mathbf{x}_\Gamma \in \Gamma_0\}, \tag{C.13}$$

and

$$\Gamma_- := \{\mathbf{x}_{\Gamma_-} = \mathbf{x}_\Gamma - \delta f^-(\mathbf{x}_\Gamma)\boldsymbol{\nu}(\mathbf{x}_\Gamma) \, : \, \mathbf{x}_\Gamma \in \Gamma_0\}, \tag{C.14}$$

where $\delta > 0$ is a small parameter that characterizes the scale of the thickness of the delamination, and $f^\pm : \Gamma_0 \to [0, 1]$ are the functions that define the profile of the delamination, and therefore satisfy that $f^{\pm}|_{\partial\Gamma_0} = 0$, where $\partial\Gamma_0$ denotes the relative boundary of Γ_0 in Γ.

Now, differentiating the parametrization of the boundaries $\Gamma_\pm$ (C.13) and (C.14), we get a set of two linearly independent tangential vectors on them, as follows

$$\begin{aligned}\boldsymbol{\tau}_\alpha^\pm &:= \partial_{\xi_\alpha}\mathbf{x}_{\Gamma_\pm}\\ &= \partial_{\xi_\alpha}\mathbf{x}_\Gamma \pm \partial_{\xi_\alpha}(f^\pm)\boldsymbol{\nu}(\mathbf{x}_\Gamma) \pm f^\pm\nabla\boldsymbol{\nu}\partial_{\xi_\alpha}\mathbf{x}_\Gamma\\ &= \boldsymbol{\tau}_\alpha \pm \delta\partial_{\xi_\alpha}(f^\pm)\boldsymbol{\nu} + \delta f^\pm\mathcal{C}\boldsymbol{\tau}_\alpha. \end{aligned} \tag{C.15}$$

Then the (non-unit) outward normal vector to the surfaces $\Gamma_\pm$ is $N^\pm := \boldsymbol{\tau}_1^\pm \times \boldsymbol{\tau}_2^\pm$ and has the following exact asymptotic expression

$$N^\pm = \boldsymbol{\nu} + \delta\Big\{\pm f^\pm(c_1+c_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^\pm\Big\} + \delta^2\Big\{(\pm f^\pm)^2 c_1c_2\boldsymbol{\nu} \mp f^\pm\mathcal{C}\nabla_\Gamma f^\pm\Big\} \tag{C.16}$$

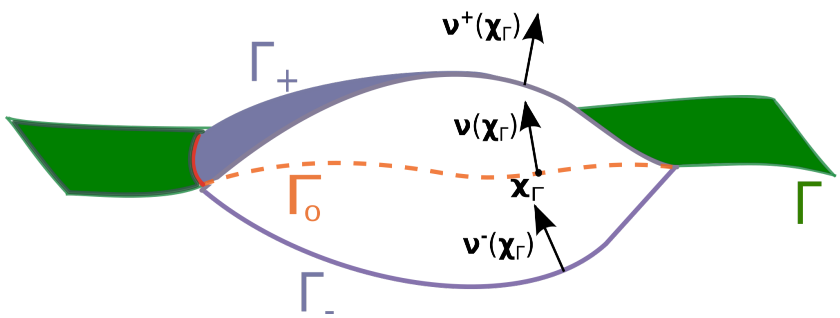


Figure C.4: Zoom on the thin domain and the normal vectors $\boldsymbol{\nu}(\mathbf{x}_\Gamma)$, $\boldsymbol{\nu}^-(\mathbf{x}_\Gamma)$, and $\boldsymbol{\nu}^+(\mathbf{x}_\Gamma)$.

C.2 Model I: Derivation of the Approximate Transmission Conditions for a crack-type model

The model we will derive in this section is the electromagnetic analogue to the one derived for acoustic scattering in the previous chapter. The setting is formally the same, i.e., we approximate the jump condition of the fields at an intermediate surface Γ_0 that crosses the delamination.

Unfortunately, in the case of electromagnetic scattering, it has been shown that this kind of model is unstable for the time-domain [30], and although we have found the same issues while attempting to prove well-posedness results in the frequency domain, we considered impotant to include this model for the sake of completeness of the thesis.

C.2.1 The formal asymptotic analysis

If the parameter δ is small enough, then the thin delamination is a subset of $\mathcal{N}$, and then, we formally assume that the following asymptotic expansions of the fields are valid in a neighbourhood $\mathcal{N}_0$ of Ω_δ, with $\mathcal{N}_0 \subset \mathcal{N}$:

$$(\mathbf{E}^\pm(\mathbf{x}_\Gamma,\eta),\mathbf{H}^\pm(\mathbf{x}_\Gamma,\eta)) = \sum_{l=0}^{\infty} \delta^l (\mathbf{E}^\pm_l(\mathbf{x}_\Gamma,\eta),\mathbf{H}^\pm_l(\mathbf{x}_\Gamma,\eta)) \quad \text{in } \Omega^\pm, \tag{C.17}$$

where each asymptotic term $(\mathbf{E}^\pm_l(\mathbf{x}_\Gamma,\eta),\mathbf{H}^\pm_l(\mathbf{x}_\Gamma,\eta))$ is assumed to be analytic and independent of δ, for all $l \geq 0$. If in addition we Taylor-expand around $\eta = 0$ (that is, around the surface Γ_0), then:

$$(\mathbf{E}^\pm(\mathbf{x}_\Gamma,\eta),\mathbf{H}^\pm(\mathbf{x}_\Gamma,\eta)) = \sum_{l=0}^{\infty} \delta^l \sum_{j=0}^{\infty} \frac{\eta^j}{j!} \frac{\partial^j(\mathbf{E}^\pm_l,\mathbf{H}^\pm_l)}{\partial\eta^j}(\mathbf{x}_\Gamma,0) \quad \text{in } \Omega^\pm, \tag{C.18}$$

and if we do the same with the derivative of (C.17) with respect to η,

$$\frac{\partial(\mathbf{E}^\pm_l,\mathbf{H}^\pm_l)}{\partial\eta}(\mathbf{x}_\Gamma,\eta) = \sum_{l=0}^{\infty} \delta^l \sum_{j=0}^{\infty} \frac{\eta^{j+1}}{j!} \frac{\partial^{j+1}(\mathbf{E}^\pm_l,\mathbf{H}^\pm_l)}{\partial\eta^j}(\mathbf{x}_\Gamma,0) \quad \text{in } \Omega^\pm. \tag{C.19}$$

Notice that expressions (C.18) and (C.19) imply that on the boundaries $\Gamma_\pm$ (i.e. for $\eta = \pm\delta f^\pm$),

$$(\mathbf{E}^\pm,\mathbf{H}^\pm)(\mathbf{x}_\Gamma,\pm\delta f^\pm) = \sum_{l=0}^{\infty} \delta^l \sum_{j=0}^{l} \frac{(\pm f^\pm)^{l-j}}{(l-j)!} \frac{\partial^{l-j}(\mathbf{E}^\pm_j,\mathbf{H}^\pm_j)}{\partial\eta^{l-j}}(\mathbf{x}_\Gamma,0) \quad \text{on } \Gamma_\pm, \tag{C.20}$$

and

$$\frac{\partial(\mathbf{E}^\pm_l,\mathbf{H}^\pm_l)}{\partial\eta}(\mathbf{x}_\Gamma,\pm\delta f^\pm) = \sum_{l=0}^{\infty} \delta^l \sum_{j=0}^{l} \frac{(\pm f^\pm)^{l-j}}{(l-j)!} \frac{\partial^{l-j+1}(\mathbf{E}^\pm_j,\mathbf{H}^\pm_j)}{\partial\eta^{l-j+1}}(\mathbf{x}_\Gamma,0) \quad \text{on } \Gamma_\pm. \tag{C.21}$$

On the other hand, the ansatz for the asymptotic expansion inside the delamination Ω_δ is slightly different because since here Ω_δ plays the role of a boundary layer, and we expect rapid changes of the fields in the thin domain. We then regularize the sigular asymptotic problem by the usual stretching of the normal variable (see for example [10],[47],[71]) $\zeta = \frac{\eta}{\delta}$, leading to

$$(\mathbf{E}^\delta, \mathbf{H}^\delta)(\mathbf{x}_\Gamma, \zeta) = \sum_{l=0}^{\infty} \delta^l (\mathbf{E}_l(\mathbf{x}_\Gamma, \xi), \mathbf{H}_l(\mathbf{x}_\Gamma, \zeta) \quad \text{in } \Omega_\delta, \tag{C.22}$$

where again, none of the terms $(\mathbf{E}_l, \mathbf{H}_l)$ depend on δ any longer.

Remark C.2.1. *(On the notation) From now on, the sub-indices T and N will indicate the tangential and normal component of a vector with respect to the surface Γ, respectively. That is, for a vector $\mathbf{v}$ defined in $\mathcal{N}$, $\Pi_{||}\mathbf{v} = \mathbf{v}_T$ and $\Pi_\perp \mathbf{v} = \mathbf{v}_N$.*

C.2.2 An identity for the outer fields

Lemma C.2.1. *The lowest order terms $(\mathbf{E}_0^\pm, \mathbf{H}_0^\pm)$ in the ansatz (C.18) satisfy the following identities:*

$$\begin{aligned} \boldsymbol{\nu} \times \partial_\eta \mathbf{E}_0^\pm(\mathbf{x}_\Gamma, 0) &= ik\mu_\pm \mathbf{H}_0^\pm(\mathbf{x}_\Gamma, 0) - \overrightarrow{curl}_\Gamma(\mathbf{E}_0^\pm)_N(\mathbf{x}_\Gamma, 0) \\ &- curl_\Gamma(\mathbf{E}_0^\pm)_T(\mathbf{x}_\Gamma, 0)\boldsymbol{\nu}, \end{aligned} \tag{C.23}$$

$$\begin{aligned} -\boldsymbol{\nu} \times \partial_\eta \mathbf{H}_0^\pm(\mathbf{x}_\Gamma, 0) &= ik\epsilon_\pm \mathbf{E}_0^\pm(\mathbf{x}_\Gamma, 0) + \overrightarrow{curl}_\Gamma(\mathbf{H}_0^\pm)_N(\mathbf{x}_\Gamma, 0) \\ &+ curl_\Gamma(\mathbf{H}_0^\pm)_T(\mathbf{x}_\Gamma, 0)\boldsymbol{\nu}. \end{aligned} \tag{C.24}$$

Proof. From the differential equations (C.3)-(C.4), expressions (C.20)-(C.21), and the ansatz for the operator $\mathcal{R}_\eta$ (C.12) for $\eta = \pm\delta f^\pm$, we get that on the boundaries $\Gamma_\pm$:

$$\begin{gathered} 0 = \textstyle\sum_{l=0}^\infty \delta^l ik\mu_\pm \sum_{j=0}^l \frac{(\pm f^\pm)^{l-j}}{(l-j)!} \partial_\eta^{l-j} \mathbf{H}_j^\pm(\mathbf{x}_\Gamma, 0) \\ + \textstyle\sum_{l=0}^\infty \delta^l \sum_{j=0}^l \frac{(\pm f^\pm)^{l-j}}{(l-j)!} \boldsymbol{\nu} \times \partial_\eta^{l-j} \mathbf{E}_j^\pm(\mathbf{x}_\Gamma, 0) \\ - \textstyle\sum_{l=0}^\infty \delta^l (\pm f^\pm)^l \left(\mathcal{C}^l \boldsymbol{\tau}^\alpha\right) \\ \times \textstyle\sum_{l=0}^\infty \delta^l \sum_{j=0}^l \left\{ \frac{(\pm f^\pm)^{l-j}}{(l-j)!} \partial_{\xi_\alpha} \partial_\eta^{l-j} \mathbf{E}_j^\pm(\mathbf{x}_\Gamma, 0) \right. \\ \left. + \delta_{l-j,0} \partial_{\xi_\alpha} \left(\frac{(\pm f^\pm)^{l-j}}{(l-j)!} \right) \partial_\eta^{l-j} \mathbf{E}_j^\pm(\mathbf{x}_\Gamma, 0) \right\}, \end{gathered} \tag{C.25}$$

where $\delta_{l-j,0} = 1$ if $l - j = 0$ and $\delta_{l-j,0} = 0$ otherwise. Collecting the terms of order $O(1)$ and $O(\delta)$,

$$
\begin{aligned}
ik\mu_{\pm}\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) + ik\mu_{\pm}\delta\left(\pm f^{\pm}\partial_\eta \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) + \mathbf{H}_1^{\pm}(\mathbf{x}_\Gamma, 0)\right)\partial_{\xi_\alpha}\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) \\
-(\boldsymbol{\tau}^\alpha \times \partial_{\xi_\alpha}\mathbf{E}_0^{\pm}) - \delta\boldsymbol{\tau}^\alpha \times \left\{ \pm f^{\pm}\partial_{\xi_\alpha}\partial_\eta \mathbf{E}_0^{\pm} + \partial_{\xi_\alpha}(\pm f^{\pm})\partial_\eta \mathbf{E}_0^{\pm} + \partial_{\xi_\alpha}\mathbf{E}_1^{\pm}\right\} \\
-\delta(\pm f^{\pm})(\mathcal{C}\boldsymbol{\tau}^\alpha) \times (\partial_{\xi_\alpha}\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0)) - \boldsymbol{\nu} \times \partial_\eta \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) \\
-\delta\left\{ \pm f^{\pm}\boldsymbol{\nu} \times \partial_\eta^2 \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) + \boldsymbol{\nu} \times \partial_\eta \mathbf{E}_1^{\pm}(\mathbf{x}_\Gamma, 0)\right\} = O(\delta^2).
\end{aligned}
\tag{C.26}
$$

Thus by identifying the terms of the same order (and repeating the same procedure for equation (C.4))

$$
\begin{aligned}
ik\mu_{\pm}\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \boldsymbol{\tau}^\alpha \times \partial_{\xi_\alpha}\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \boldsymbol{\nu} \times \partial_\eta \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) &= 0, && \text{(C.27)} \\
ik\epsilon_{\pm}\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) + \boldsymbol{\tau}^\alpha \times \partial_{\xi_\alpha}\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) + \boldsymbol{\nu} \times \partial_\eta \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) &= 0. && \text{(C.28)}
\end{aligned}
$$

Finally, recalling that $\boldsymbol{\tau}^\alpha \times \partial_{\xi_\alpha} v = \overrightarrow{curl}_\Gamma(v)_N + curl_\Gamma(v)_T\boldsymbol{\nu}$, the lemma is proven. □

C.2.3 The boundary value problem for the inner fields

Lemma C.2.2. *The lowest order terms* $(\mathbf{E}_0, \mathbf{H}_0)$ *and* $(\mathbf{E}_1, \mathbf{H}_1)$ *in the ansatz of* $(\mathbf{E}^\delta, \mathbf{H}^\delta)$ *satisfy the following equations:*

$$
\begin{aligned}
\boldsymbol{\nu} \times \partial_\xi \mathbf{E}_0 &= 0, && \text{(C.29)} \\
\boldsymbol{\nu} \times \partial_\xi \mathbf{H}_0 &= 0. && \text{(C.30)}
\end{aligned}
$$

and

$$
\begin{aligned}
ik\mu_\delta \mathbf{H}_0 - \overrightarrow{curl}_\Gamma(\mathbf{E}_0)_N - curl_\Gamma(\mathbf{E}_0)_T\boldsymbol{\nu} &= \boldsymbol{\nu} \times \partial_\xi \mathbf{E}_1, && \text{(C.31)} \\
-ik\epsilon_\delta \mathbf{E}_0 - \overrightarrow{curl}_\Gamma(\mathbf{H}_0)_N - curl_\Gamma(\mathbf{H}_0)_T\boldsymbol{\nu} &= \boldsymbol{\nu} \times \partial_\xi \mathbf{H}_1, && \text{(C.32)}
\end{aligned}
$$

for all $\mathbf{x}_\Gamma \in \Gamma_0$ *and* $\zeta \in (-f^-(\mathbf{x}_\Gamma), f^+(\mathbf{x}_\Gamma))$*, in addition to the following boundary conditions:*

$$
\begin{aligned}
\boldsymbol{\nu} \times \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{E}_0(\mathbf{x}_\Gamma, \pm f^{\pm}), && \text{(C.33)} \\
\boldsymbol{\nu} \times \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{H}_0(\mathbf{x}_\Gamma, \pm f^{\pm}), && \text{(C.34)}
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{\nu} \times \mathbf{E}_1^{\pm}(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{E}_1(\mathbf{x}_\Gamma, \pm f^{\pm}) \mp f^{\pm}\boldsymbol{\nu} \times \partial_\eta \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) \\
&\pm \langle (\mathbf{E}_0)_N \rangle \left(\frac{\epsilon_\delta}{\epsilon_\pm} - 1 \right) \overrightarrow{curl}_\Gamma(f^{\pm}), \qquad \text{(C.35)} \\
\boldsymbol{\nu} \times \mathbf{H}_1^{\pm}(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{H}_1(\mathbf{x}_\Gamma, \pm f^{\pm}) \mp f^{\pm}\boldsymbol{\nu} \times \partial_\eta \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) \\
&\pm \langle (\mathbf{H}_0)_N \rangle \left(\frac{\mu_\delta}{\mu_\pm} - 1 \right) \overrightarrow{curl}_\Gamma(f^{\pm}). \qquad \text{(C.36)}
\end{aligned}
$$

Proof. Consider first the *differential equation.* Substituting in equation (C.5) the ansatz (C.22), the expression (C.11) of the *curl* operator, and the ansatz (C.12) of $\mathcal{R}_\eta$ when $\eta = \pm\delta f^{\pm}$, we get:

$$
\begin{aligned}
&\sum_{l=0}^{\infty} \delta^l ik\mu_\delta \mathbf{H}_l(\mathbf{x}_\Gamma, \zeta) - \sum_{l=0}^{\infty} \delta^{l-1} \boldsymbol{\nu} \times \partial_\xi \mathbf{E}_l(\mathbf{x}_\Gamma, \zeta) \\
&\quad - \sum_{l=0}^{\infty} \delta^l (-\zeta)^l \left(\mathcal{C}^l \boldsymbol{\tau}^\alpha \right) \times \sum_{l=0}^{\infty} \delta^l \partial_{\xi_\alpha} \mathbf{E}_l(\mathbf{x}_\Gamma, \zeta) = 0, \qquad \text{(C.37)}
\end{aligned}
$$

in Ω_δ. Therefore, collecting terms of order $O(\delta^{-1})$ and $O(1)$ (and doing similar calculations for equation (C.6)) we get the differential equations (C.29) - (C.32), respectively. Notice that we have used the identity $\boldsymbol{\tau}^\alpha \times \partial_{\xi_\alpha} v = \overrightarrow{curl}_\Gamma(v)_N + curl_\Gamma(v)_T \boldsymbol{\nu}$.

Now, we derive the expressions for the *boundary conditions for the inner fields.* Since the transmission conditions (C.7)-(C.8) can be written as

$$
N^{\pm} \times \mathbf{E}^{\pm} = N^{\pm} \times \mathbf{E}^{\delta} \quad \text{on} \quad \Gamma_\pm, \qquad \text{(C.38)}
$$

$$
N^{\pm} \times \mathbf{H}^{\pm} = N^{\pm} \times \mathbf{H}^{\delta} \quad \text{on} \quad \Gamma_\pm, \qquad \text{(C.39)}
$$

where the vectors $N^{\pm}$ are the non-unit normal vectors on $\Gamma_\pm$ defined by (C.16), substituting the ansatz (C.22) and (C.18) in (C.38),

$$
\begin{aligned}
&\left(\boldsymbol{\nu} + \delta \Big\{ \pm f^{\pm}(l_1 + l_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^{\pm} \Big\} \right) \times (\mathbf{E}_0^{\pm} + \delta \{ \mathbf{E}_1^{\pm} \pm f^{\pm} \partial_\eta \mathbf{E}_0^{\pm} \})(\mathbf{x}_\Gamma, 0) \\
&= \left(\boldsymbol{\nu} + \delta \Big\{ \pm f^{\pm}(l_1 + l_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^{\pm} \Big\} \right) \times (\mathbf{E}_0 + \delta \mathbf{E}_1)(\mathbf{x}_\Gamma, \pm f^{\pm}) + O(\delta^2), \qquad \text{(C.40)}
\end{aligned}
$$

and hence, identifying terms of order $O(1)$ we get (C.33)-(C.34) and identifying terms of order $O(\delta)$ (and in the similar expression that we get when performing similar computations for the boundary condition (C.39)),we get:

$$\begin{aligned}
\boldsymbol{\nu} \times \mathbf{E}_1^{\pm}(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{E}_1(\mathbf{x}_\Gamma, \pm f^{\pm}) \mp f^{\pm} \boldsymbol{\nu} \times \partial_\eta \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) \\
&\pm \nabla_\Gamma f^{\pm} \times (\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{E}_0(\mathbf{x}_\Gamma, \pm f^{\pm})), && \text{(C.41)} \\
\boldsymbol{\nu} \times \mathbf{H}_1^{\pm}(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{H}_1(\mathbf{x}_\Gamma, \pm f^{\pm}) \mp f^{\pm} \boldsymbol{\nu} \times \partial_\eta \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) \\
&\pm \nabla_\Gamma f^{\pm} \times (\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{H}_0(\mathbf{x}_\Gamma, \pm f^{\pm})). && \text{(C.42)}
\end{aligned}$$

From the boundary conditions (C.33)-(C.34), the tangential components

$$((\mathbf{E}_0)_T, (\mathbf{H}_0)_T)(\mathbf{x}_\Gamma, \pm f^{\pm})$$

coincide with $((\mathbf{E}_0^{\pm})_T, (\mathbf{H}_0^{\pm})_T)(\mathbf{x}_\Gamma, 0)$, so that the last term in expressions (C.31)-(C.32) become:

$$\begin{aligned}
& \nabla_\Gamma f^{\pm} \times (\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{E}_0(\mathbf{x}_\Gamma, \pm f^{\pm})) \\
= \; & (\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{E}_0(\mathbf{x}_\Gamma, \pm f^{\pm}))_N \, \nabla_\Gamma f^{\pm} \times \boldsymbol{\nu} \\
= \; & (\mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{E}_0(\mathbf{x}_\Gamma, \pm f^{\pm}))_N \, \overrightarrow{curl}_\Gamma(f^{\pm}) && \text{(C.43)}
\end{aligned}$$

and

$$\begin{aligned}
& \nabla_\Gamma f^{\pm} \times (\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{H}_0(\mathbf{x}_\Gamma, \pm f^{\pm})) \\
= \; & (\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{H}_0(\mathbf{x}_\Gamma, \pm f^{\pm}))_N \, \nabla_\Gamma f^{\pm} \times \boldsymbol{\nu} \\
= \; & (\mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, 0) - \mathbf{H}_0(\mathbf{x}_\Gamma, \pm f^{\pm}))_N \, \overrightarrow{curl}_\Gamma(f^{\pm}). && \text{(C.44)}
\end{aligned}$$

Moreover, from equations (C.29)-(C.30), we see that the tangential components of $\mathbf{H}_0, \mathbf{E}_0$ do not depend on ζ, that is,

$$(\mathbf{E}_0)_T = \langle \mathbf{E}_0 \rangle_T \quad \text{and} \quad (\mathbf{H}_0)_T = \langle \mathbf{H}_0 \rangle_T \, , \qquad \text{(C.45)}$$

and therefore applying $\Pi_{||}$ to both sides in equations (C.31)- (C.32), and then using (C.45),

$$\begin{aligned} ik\epsilon_\delta(\mathbf{E}_0)_N &= -curl_\Gamma \langle \mathbf{H}_0 \rangle_T, && \text{(C.46)} \\ ik\mu_\delta(\mathbf{H}_0)_N &= curl_\Gamma \langle \mathbf{E}_0 \rangle_T, && \text{(C.47)} \end{aligned}$$

and using (C.29)-(C.30),

$$\begin{aligned} ik\epsilon_\delta \langle (\mathbf{E}_0)_N \rangle &= -curl_\Gamma \langle (\mathbf{H}_0)_T \rangle = -curl_\Gamma \left\langle (\mathbf{H}_0^\pm)_T \right\rangle = ik\epsilon_\pm (\mathbf{E}_0^\pm)_N, \text{ and} && \text{(C.48)} \\ ik\mu_\delta \langle (\mathbf{H}_0)_N \rangle &= curl_\Gamma \langle (\mathbf{E}_0)_T \rangle = curl_\Gamma \left\langle (\mathbf{E}_0^\pm)_T \right\rangle = ik\mu_\pm (\mathbf{H}_0^\pm)_N, && \text{(C.49)} \end{aligned}$$

thus

$$(\mathbf{E}_0^\pm)_N = \frac{\epsilon_\delta}{\epsilon_\pm} \langle (\mathbf{E}_0)_N \rangle \quad \text{and} \quad (\mathbf{H}_0^\pm)_N = \frac{\mu_\delta}{\mu_\pm} \langle (\mathbf{H}_0)_N \rangle, \tag{C.50}$$

so that (C.43)-(C.44) become

$$\begin{aligned} \nabla_\Gamma f^\pm \times (\mathbf{E}_0^\pm(\mathbf{x}_\Gamma, 0) - \mathbf{E}_0(\mathbf{x}_\Gamma, \pm f^\pm)) &= \langle (\mathbf{E}_0)_N \rangle \left(\frac{\epsilon_\delta}{\epsilon_\pm} - 1 \right) \overrightarrow{curl}_\Gamma(f^\pm), && \text{(C.51)} \\ \nabla_\Gamma f^\pm \times (\mathbf{H}_0^\pm(\mathbf{x}_\Gamma, 0) - \mathbf{H}_0(\mathbf{x}_\Gamma, \pm f^\pm)) &= \langle (\mathbf{H}_0)_N \rangle \left(\frac{\mu_\delta}{\mu_\pm} - 1 \right) \overrightarrow{curl}_\Gamma(f^\pm). && \text{(C.52)} \end{aligned}$$

Therefore, substituting these two last expressions into (C.41) and (C.42), one gets

$$\begin{aligned} \boldsymbol{\nu} \times \mathbf{E}_1^\pm(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{E}_1(\mathbf{x}_\Gamma, \pm f^\pm) \mp f^\pm \boldsymbol{\nu} \times \partial_\eta \mathbf{E}_0^\pm(\mathbf{x}_\Gamma, 0) \\ &\pm \langle (\mathbf{E}_0)_N \rangle \left(\frac{\epsilon_\delta}{\epsilon_\pm} - 1 \right) \overrightarrow{curl}_\Gamma(f^\pm), && \text{(C.53)} \\ \boldsymbol{\nu} \times \mathbf{H}_1^\pm(\mathbf{x}_\Gamma, 0) &= \boldsymbol{\nu} \times \mathbf{H}_1(\mathbf{x}_\Gamma, \pm f^\pm) \mp f^\pm \boldsymbol{\nu} \times \partial_\eta \mathbf{H}_0^\pm(\mathbf{x}_\Gamma, 0) \\ &\pm \langle (\mathbf{H}_0)_N \rangle \left(\frac{\mu_\delta}{\mu_\pm} - 1 \right) \overrightarrow{curl}_\Gamma(f^\pm), && \text{(C.54)} \end{aligned}$$

finishing the proof. □

Proposition C.2.1. *Given constant material properties $\epsilon_\pm$, $\mu_\pm$, μ_δ, and ϵ_δ in a neighborhood $\mathcal{N}_0 \subset \mathcal{N}$, the second order Approximate Transmission Conditions (ATCs) using the ansatz (C.22) and (C.18) are:*

$$\begin{aligned} [\boldsymbol{\nu} \times \mathbf{E}] &= \delta \tilde{\alpha}_1^* \langle \mathbf{H}_T \rangle + \delta \overrightarrow{curl}_\Gamma \left(\left\langle \tilde{\beta}_1^* f \right\rangle curl_\Gamma \langle \mathbf{H}_T \rangle \right) \quad on \quad \Gamma_0 && \text{(C.55)} \\ [\boldsymbol{\nu} \times \mathbf{H}] &= \delta \tilde{\alpha}_2^* \langle \mathbf{E}_T \rangle + \delta \overrightarrow{curl}_\Gamma \left(\left\langle \tilde{\beta}_2^* f \right\rangle curl_\Gamma \langle \mathbf{E}_T \rangle \right) \quad on \quad \Gamma_0, && \text{(C.56)} \end{aligned}$$

where $\tilde{\alpha}_1^* = 2ik\left\langle f(\mu_\delta - \mu)\right\rangle$*,* $\tilde{\alpha}_2^* = 2ik\left\langle f(\epsilon - \epsilon_\delta)\right\rangle$*,* $\tilde{\beta}_1^{\pm,*} = \frac{2}{ik}\left(\frac{1}{\epsilon_\delta} - \frac{1}{\epsilon_\pm}\right)$*, and* $\tilde{\beta}_2^{\pm,*} = \frac{2}{ik}\left(\frac{1}{\mu_\pm} - \frac{1}{\mu_\delta}\right)$.

Proof. We derive the proof into the following three steps.

1. From (C.45),

$$[\boldsymbol{\nu} \times \ \mathbf{E}_0] = 0 \text{ and } [\boldsymbol{\nu} \times \ \mathbf{H}_0] = 0,$$

so from the boundary conditions (C.33)-(C.34),

$$[\boldsymbol{\nu} \times \ \mathbf{E}_0^{\pm}] = 0, \quad [\boldsymbol{\nu} \times \ \mathbf{E}_0^{\pm}] = 0. \tag{C.57}$$

2. On the other hand from (C.45)-(C.46) we know that any component of the zeroth-order fields $(\mathbf{E}_0, \mathbf{H}_0)$ depend on the normal variable ζ, using this fact in equations (C.31)-(C.32), and integrating with respect to ζ along the interval $(-f^-, f^+)$, the fundamental theorem of calculus implies that

$$2\left\langle f\right\rangle ik\mu_\delta\left\langle \mathbf{H}_0\right\rangle - 2\left\langle f\right\rangle \overrightarrow{curl}_\Gamma\left\langle \mathbf{E}_0\right\rangle_N - 2\left\langle f\right\rangle curl_\Gamma\left\langle \mathbf{E}_0\right\rangle_T\boldsymbol{\nu} = [\boldsymbol{\nu} \times \mathbf{E}_1], \tag{C.58}$$

$$-2\left\langle f\right\rangle ik\epsilon_\delta\left\langle \mathbf{E}_0\right\rangle - 2\left\langle f\right\rangle \overrightarrow{curl}_\Gamma\left\langle \mathbf{H}_0\right\rangle_N - 2\left\langle f\right\rangle curl_\Gamma\left\langle \mathbf{H}_0\right\rangle_T\boldsymbol{\nu} = [\boldsymbol{\nu} \times \mathbf{H}_1]. \tag{C.59}$$

3. From the boundary conditions (C.35)-(C.36) and using the fact that in $\mathcal{N}_0 \subset \mathcal{N}$ all the material properties are piece-wise constant, we get

$$\begin{aligned}\left[\boldsymbol{\nu} \times \mathbf{E}_1^{\pm}\right] &= [\boldsymbol{\nu} \times \mathbf{E}_1] - 2\boldsymbol{\nu} \times \left\langle f^{\pm}\partial_\eta \mathbf{E}_0^{\pm}\right\rangle \\ &+ 2\left\langle \mathbf{E}_0\right\rangle_N \overrightarrow{curl}_\Gamma\left\langle f^{\pm}\left(\frac{\epsilon_\delta}{\epsilon_\pm} - 1\right)\right\rangle, \qquad \text{(C.60)}\\ \left[\boldsymbol{\nu} \times \mathbf{H}_1^{\pm}\right] &= [\boldsymbol{\nu} \times \mathbf{H}_1] - 2\boldsymbol{\nu} \times \left\langle f^{\pm}\partial_\eta \mathbf{H}_0^{\pm}\right\rangle \\ &+ 2\left\langle \mathbf{H}_0\right\rangle_N \overrightarrow{curl}_\Gamma\left\langle f^{\pm}\left(\frac{\mu_\delta}{\mu_\pm} - 1\right)\right\rangle. \qquad \text{(C.61)}\end{aligned}$$

To get an expression for the terms $\boldsymbol{\nu} \times \left\langle f^{\pm}\partial_\eta \mathbf{E}_0^{\pm}\right\rangle$ and $\boldsymbol{\nu} \times \left\langle f^{\pm}\partial_\eta \mathbf{H}_0^{\pm}\right\rangle$, we multiply by $\pm f^{\pm}$ the identities in Lemma C.2.1, applying $\Pi_{||}$ to both sides of the above identities, and adding them together we get

$$2ik\left\langle f^{\pm}\mu_\pm\right\rangle\left\langle (\mathbf{H}_0)_T\right\rangle - 2\left\langle f^{\pm}\frac{\epsilon_\delta}{\epsilon_\pm}\right\rangle \overrightarrow{curl}_\Gamma\left\langle (\mathbf{E}_0)_N\right\rangle = 2\boldsymbol{\nu} \times \left\langle f^{\pm}\partial_\eta \mathbf{E}_0^{\pm}\right\rangle, \tag{C.62}$$

$$-2ik\left\langle f^{\pm}\epsilon_\pm\right\rangle\left\langle (\mathbf{E}_0)_T\right\rangle - 2\left\langle f^{\pm}\frac{\mu_\delta}{\mu_\pm}\right\rangle \overrightarrow{curl}_\Gamma\left\langle (\mathbf{H}_0)_N\right\rangle = 2\boldsymbol{\nu} \times \left\langle f^{\pm}\partial_\eta \mathbf{H}_0^{\pm}\right\rangle, \tag{C.63}$$

and substituting these identities and (C.58)-(C.59) in (C.60)-(C.61), we get

$$\begin{aligned}
\left[\boldsymbol{\nu}\times\mathbf{E}_1^{\pm}\right] &= 2ik\left\langle f(\mu_\delta-\mu)\right\rangle\left\langle(\mathbf{H}_0^{\pm})_T\right\rangle \\
&+ 2\overrightarrow{curl}_\Gamma\left(\left\langle f\left(\frac{\epsilon_\delta}{\epsilon_\pm}-1\right)\right\rangle\left\langle(\mathbf{E}_0)_N\right\rangle\right), \qquad (C.64)\\
\left[\boldsymbol{\nu}\times\mathbf{H}_1^{\pm}\right] &= 2ik\left\langle f(\epsilon-\epsilon_\delta)\right\rangle\left\langle(\mathbf{E}_0^{\pm})_T\right\rangle \\
&+ 2\overrightarrow{curl}_\Gamma\left(\left\langle f\left(\frac{\mu_\delta}{\mu_\pm}-1\right)\right\rangle\left\langle(\mathbf{H}_0)_N\right\rangle\right), \qquad (C.65)
\end{aligned}$$

on Γ_0. Using (C.48)-(C.49) finally get

$$\begin{aligned}
\left[\boldsymbol{\nu}\times\mathbf{E}_1^{\pm}\right] &= 2ik\left\langle f(\mu_\delta-\mu)\right\rangle\left\langle(\mathbf{H}_0^{\pm})_T\right\rangle \\
&+ \frac{2}{ik}\overrightarrow{curl}_\Gamma\left(\left\langle f\left(\frac{1}{\epsilon_\delta}-\frac{1}{\epsilon_\pm}\right)\right\rangle curl_\Gamma\left\langle(\mathbf{H}_0^{\pm})_T\right\rangle\right) \quad \text{on } \Gamma_0 \qquad (C.66)\\
\left[\boldsymbol{\nu}\times\mathbf{H}_1^{\pm}\right] &= 2ik\left\langle f(\epsilon-\epsilon_\delta)\right\rangle\left\langle(\mathbf{E}_0^{\pm})_T\right\rangle \\
&+ \frac{2}{ik}\overrightarrow{curl}_\Gamma\left(\left\langle f\left(\frac{1}{\mu_\pm}-\frac{1}{\mu_\delta}\right)\right\rangle curl_\Gamma\left\langle(\mathbf{E}_0^{\pm})_T\right\rangle\right) \quad \text{on } \Gamma_0. \qquad (C.67)
\end{aligned}$$

The proof is completed by observing that from the asymptotic expanssion (C.18),

$$\left[\boldsymbol{\nu}\times\mathbf{E}^{\pm}\right](\mathbf{x}_\Gamma,0) = [\boldsymbol{\nu}\times\mathbf{E}_0^{\pm}](\mathbf{x}_\Gamma,0)+\delta[\boldsymbol{\nu}\times\mathbf{E}_1^{\pm}](\mathbf{x}_\Gamma,0)+O(\delta^2), \qquad (C.68)$$

$$\left[\boldsymbol{\nu}\times\mathbf{H}^{\pm}\right](\mathbf{x}_\Gamma,0) = [\boldsymbol{\nu}\times\mathbf{H}_0^{\pm}](\mathbf{x}_\Gamma,0)+\delta[\boldsymbol{\nu}\times\mathbf{H}_1^{\pm}](\mathbf{x}_\Gamma,0)+O(\delta^2), \qquad (C.69)$$

and by substituting (C.57) and (C.66)-(C.67). □

The *crack-type* model derived from the previous proposition, is related to the new configuration where the small opening Ω_δ is no longer present, and instead, we include the ATCs just derived (see Fig. C.5). More precisely, the associated *crack-type* model is defined by:

$$\nabla\times\mathbf{E}-ik\mathbf{H}=0 \quad \text{in} \quad \mathbb{R}^3\setminus\overline{\Omega_{ext}}, \qquad (C.70)$$

$$\nabla\times\mathbf{H}+ik\mathbf{E}=0 \quad \text{in} \quad \mathbb{R}^3\setminus\overline{\Omega_{ext}}, \qquad (C.71)$$

$$\nabla\times\mathbf{E}-ik\mu_\pm\mathbf{H}=0 \quad \text{in} \quad \Omega_\pm^b, \qquad (C.72)$$

$$\nabla\times\mathbf{H}+ik\epsilon_\pm\mathbf{E}=0 \quad \text{in} \quad \Omega_\pm^b, \qquad (C.73)$$

$$[\boldsymbol{\nu}\times\mathbf{E}] = \delta\tilde{\alpha}_1^*\left\langle\mathbf{H}_T\right\rangle+\delta\overrightarrow{curl}_\Gamma\left(\left\langle\tilde{\beta}_1^* f\right\rangle curl_\Gamma\left\langle\mathbf{H}_T\right\rangle\right) \quad \text{on} \quad \Gamma_0 \qquad (C.74)$$

$$[\boldsymbol{\nu}\times\mathbf{H}] = \delta\tilde{\alpha}_2^*\left\langle\mathbf{E}_T\right\rangle+\delta\overrightarrow{curl}_\Gamma\left(\left\langle\tilde{\beta}_2^* f\right\rangle curl_\Gamma\left\langle\mathbf{E}_T\right\rangle\right) \quad \text{on} \quad \Gamma_0, \qquad (C.75)$$

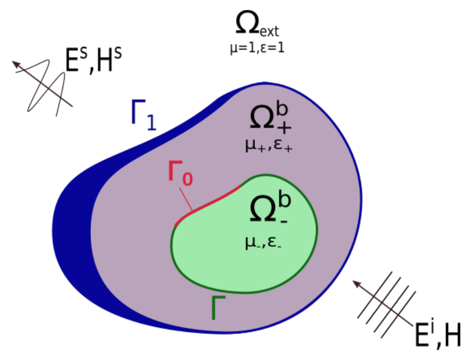


Figure C.5: Layered media with a crack Γ_0 at the interface of two layers Ω^b_- and Ω^b_+.

and in $\mathbb{R}^3 \setminus \overline{\Omega}$, $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ and $\mathbf{H} = \mathbf{H}^s + \mathbf{H}^i$, where $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy the Silver-Müller radiation condition (C.9).

This model is the analogue of a time domain model for electromagnetic scattering that has been analyzed by Chun et al. in [30], where the authors prove its unstablility.

This issue is reflected in our *crack-type* model in the fact that the signs of the coefficients appearing in the ATCs, are not compatible with the main operators.

Due to these complications, we derive in the following section a different model, that at least for a particular case leads to a stable model, as analyzed in Chapter 4.

C.3 Model II: Derivation of the Approximate Transmission Conditions for a Chun's type model

As mentioned in the last lines of the previous section, the aim of this part of the Appendx C is to derive a different set of ATCs for the electromagnetic scattering problem, which are sometimes referred to as *Chun's-type* ATCs (see [30, 40]). This *Chun's-type* ATCs differ from the *crack-type* ATCs in the fact that the jumps and average values of the fields are taken with respect to traces of the fields on the two different surfaces Γ_- and Γ_+. The complete set of ATCs is presented as Proposition C.3.1, and it is similar to the models analyzed in [30, 38, 40].

C.3.1 The formal asymptotic analysis

Let $\mathbf{u}$ with well-defined traces $\mathbf{u}|_{\Gamma_\pm}$, we define:

$$[\![\mathbf{u}]\!] := \mathbf{u}|_{\Gamma_+} - \mathbf{u}|_{\Gamma_-} \quad \text{and} \quad \langle\!\langle \mathbf{u} \rangle\!\rangle := \frac{\mathbf{u}|_{\Gamma_+} + \mathbf{u}|_{\Gamma_-}}{2}.$$

As in the previous section, we assume that if $0 < \delta$ is small enough, then in a neighborhood of Ω_δ, the expressions for the outer-fields (C.17) are true, and that after again considering $\zeta = \frac{\eta}{\delta}$, inside Ω_δ the fields have the expression (C.22).

Proposition C.3.1. *Assuming that $\mu_\delta\, \epsilon_\delta$ are constant, and assuming that the ansatz (C.22) and (C.17) are valid in a neighborhood $\mathcal{N}_0$ of Ω_δ, the second order ATCs are:*

$$\begin{aligned}
[\![\boldsymbol{\nu}^\pm \times \mathbf{E}^\pm]\!] &= 2\delta \langle f \rangle (c_1 + c_2) \boldsymbol{\nu} \times \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \\
&- 2\delta \nabla_\Gamma \langle f \rangle \times \langle\!\langle \mathbf{E}^\delta \rangle\!\rangle_T \\
&+ 2\delta i k \langle f \rangle \mu_\delta \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\
&+ \frac{2\delta}{ik\epsilon_\delta} \overrightarrow{curl}_\Gamma \Big(\langle f \rangle\, curl_\Gamma \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \Big) \qquad \text{(C.76)}\\
[\![\boldsymbol{\nu}^\pm \times \mathbf{H}^\pm]\!] &= 2\delta \langle f \rangle (c_1 + c_2) \boldsymbol{\nu} \times \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\
&- 2\delta \nabla_\Gamma \langle f \rangle \times \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\
&- 2\delta i k \langle f \rangle \epsilon_\delta \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \\
&- \frac{2\delta}{ik\mu_\delta} \overrightarrow{curl}_\Gamma \Big(\langle f \rangle\, curl_\Gamma \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \Big). \qquad \text{(C.77)}
\end{aligned}$$

Proof. We divide the proof into several steps.

1.Observing that from the ansatz (C.17) and (C.22), and (C.16):

$$\begin{aligned}
[\![N^\pm \times \mathbf{E}^\pm]\!] &= [\![\boldsymbol{\nu} \times \mathbf{E}_0^\pm]\!] + 2\delta(c_1 + c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{E}_0^\pm \rangle\!\rangle \\
&- 2\delta \langle\!\langle \nabla_\Gamma f^\pm \times \mathbf{E}_0^\pm \rangle\!\rangle + \delta [\![\boldsymbol{\nu} \times \mathbf{E}_1^\pm]\!] + O(\delta^2) \qquad \text{(C.78)}\\
[\![N^\pm \times \mathbf{H}^\pm]\!] &= [\![\boldsymbol{\nu} \times \mathbf{H}_0^\pm]\!] + 2\delta(c_1 + c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{H}_0^\pm \rangle\!\rangle \\
&- 2\delta \langle\!\langle \nabla_\Gamma f^\pm \times \mathbf{H}_0^\pm \rangle\!\rangle + \delta [\![\boldsymbol{\nu} \times \mathbf{H}_1^\pm]\!] + O(\delta^2). \qquad \text{(C.79)}
\end{aligned}$$

2. Moreover, from the continuity conditions (C.7)-(C.8),

$$N^{\pm} \times \mathbf{E}^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) \;=\; N^{\pm} \times \mathbf{E}^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}), \tag{C.80}$$

$$N^{\pm} \times \mathbf{H}^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) \;=\; N^{\pm} \times \mathbf{H}^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}),, \tag{C.81}$$

and therefore, from the expresion (C.16) andthe ansatz (C.17) and (C.22), when collecting terms of order $O(1)$ and $O(\delta)$:

$$\boldsymbol{\nu} \times \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) = \boldsymbol{\nu} \times \mathbf{E}_0^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}), \tag{C.82}$$

$$\boldsymbol{\nu} \times \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) = \boldsymbol{\nu} \times \mathbf{H}_0^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}), \tag{C.83}$$

and

$$\begin{aligned}\boldsymbol{\nu} \times \mathbf{E}_1^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) + \left\{ \pm f^{\pm}(c_1+c_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^{\pm} \right\} \times \mathbf{E}_0^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) \;&=\\ \boldsymbol{\nu} \times \mathbf{E}_1^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}) + \left\{ \pm f^{\pm}(c_1+c_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^{\pm} \right\} \times \mathbf{E}_0^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}) & \qquad \text{(C.84)}\\ \boldsymbol{\nu} \times \mathbf{H}_1^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) + \left\{ \pm f^{\pm}(c_1+c_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^{\pm} \right\} \times \mathbf{H}_0^{\pm}(\mathbf{x}_\Gamma, \pm\delta f^{\pm}) \;&=\\ \boldsymbol{\nu} \times \mathbf{H}_1^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}) + \left\{ \pm f^{\pm}(c_1+c_2)\boldsymbol{\nu} \mp \nabla_\Gamma f^{\pm} \right\} \times \mathbf{H}_0^{\delta}(\mathbf{x}_\Gamma, \pm f^{\pm}) & \qquad ,\text{(C.85)}\end{aligned}$$

and therefore:

$$[\![\boldsymbol{\nu} \times \mathbf{E}_0^{\pm}]\!] = [\![\boldsymbol{\nu} \times \mathbf{E}_0^{\delta}]\!], \tag{C.86}$$

$$[\![\boldsymbol{\nu} \times \mathbf{H}_0^{\pm}]\!] = [\![\boldsymbol{\nu} \times \mathbf{H}_0^{\delta}]\!], \tag{C.87}$$

and

$$\begin{aligned}[\![\boldsymbol{\nu} \times \mathbf{E}_1^{\pm}]\!] \;&=\; [\![\boldsymbol{\nu} \times \mathbf{E}_1^{\delta}]\!]\\ &+\; 2(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^{\pm}(\mathbf{E}_0^{\delta} - \mathbf{E}_0^{\pm})_T \rangle\!\rangle\\ &-\; 2\langle\!\langle \nabla_\Gamma f^{\pm} \times (\mathbf{E}_0^{\delta} - \mathbf{E}_0^{\pm}) \rangle\!\rangle, \qquad \text{(C.88)}\\ [\![\boldsymbol{\nu} \times \mathbf{H}_1^{\pm}]\!] \;&=\; [\![\boldsymbol{\nu} \times \mathbf{H}_1^{\delta}]\!]\\ &+\; 2(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^{\pm}(\mathbf{H}_0^{\delta} - \mathbf{H}_0^{\pm})_T \rangle\!\rangle\\ &-\; 2\langle\!\langle \nabla_\Gamma f^{\pm} \times (\mathbf{H}_0^{\delta} - \mathbf{H}_0^{\pm}) \rangle\!\rangle, \qquad \text{(C.89)}\end{aligned}$$

3. Therefore, substituting (C.88)-(C.89) in (C.78)-(C.79):

$$
\begin{aligned}
[\![N^\pm \times \mathbf{E}^\pm]\!] &= [\![\boldsymbol{\nu} \times \mathbf{E}_0^\pm]\!] + 2\delta(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{E}_0^\delta \rangle\!\rangle \\
&- 2\delta\langle\!\langle \nabla_\Gamma f^\pm \times \mathbf{E}_0^\delta \rangle\!\rangle + \delta[\![\boldsymbol{\nu} \times \mathbf{E}_1^\delta]\!] + O(\delta^2), \qquad \text{(C.90)} \\
[\![N^\pm \times \mathbf{H}^\pm]\!] &= [\![\boldsymbol{\nu} \times \mathbf{H}_0^\pm]\!] + 2\delta(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{H}_0^\delta \rangle\!\rangle \\
&- 2\delta\langle\!\langle \nabla_\Gamma f^\pm \times \mathbf{H}_0^\delta \rangle\!\rangle + \delta[\![\boldsymbol{\nu} \times \mathbf{H}_1^\delta]\!] + O(\delta^2). \qquad \text{(C.91)}
\end{aligned}
$$

4. Now, observe that since the ansatz of the inner fields $(\mathbf{E}^\delta, \mathbf{H}^\delta)$ are the same as in Model I, the PDEs (C.29)-(C.32) are still valid. Therefore:

$$(\mathbf{E}_0^\delta)_T = \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_T \quad \text{and} \quad (\mathbf{H}_0^\delta)_T = \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_T, \tag{C.92}$$

and hence,

$$[\![\boldsymbol{\nu} \times \ \mathbf{E}_0^\delta]\!] = 0 \text{ and } [\![\boldsymbol{\nu} \times \ \mathbf{H}_0^\delta]\!] = 0. \tag{C.93}$$

So from (C.86)-(C.87):

$$[\![\boldsymbol{\nu} \times \ \mathbf{E}_0^\pm]\!] = 0 \text{ and } [\![\boldsymbol{\nu} \times \ \mathbf{H}_0^\pm]\!] = 0. \tag{C.94}$$

Moreover, applying $P_{||}$ on both sides of equations (C.31)-(C.32) and integrating with respect to ζ along the interval $(-f^-, f^+)$,

$$
\begin{aligned}
2ik\,\langle f\rangle\,\mu_\delta \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_T - 2\,\langle f\rangle\,\overrightarrow{curl}_\Gamma \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_N &= [\![\boldsymbol{\nu} \times \mathbf{E}_1^\delta]\!], \qquad \text{(C.95)} \\
-2ik\,\langle f\rangle\,\epsilon_\delta \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_T - 2\,\langle f\rangle\,\overrightarrow{curl}_\Gamma \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_N &= [\![\boldsymbol{\nu} \times \mathbf{H}_1^\delta]\!]. \qquad \text{(C.96)}
\end{aligned}
$$

5. Substituting (C.94) and (C.95)-(C.96) in (C.90)-(C.91), we get

$$
\begin{aligned}
[\![N^\pm \times \mathbf{E}^\pm]\!] &= 2\delta(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{E}_0^\delta \rangle\!\rangle \\
&- 2\delta\langle\!\langle \nabla_\Gamma f^\pm \times \mathbf{E}_0^\delta \rangle\!\rangle \\
&+ 2\delta ik\,\langle f\rangle\,\mu_\delta \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_T - 2\delta\,\langle f\rangle\,\overrightarrow{curl}_\Gamma \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_N + O(\delta^2), \qquad \text{(C.97)} \\
[\![N^\pm \times \mathbf{H}^\pm]\!] &= 2\delta(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{H}_0^\delta \rangle\!\rangle \\
&- 2\delta\langle\!\langle \nabla_\Gamma f^\pm \times \mathbf{H}_0^\delta \rangle\!\rangle \\
&- 2\delta ik\,\langle f\rangle\,\epsilon_\delta \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_T - 2\delta\,\langle f\rangle\,\overrightarrow{curl}_\Gamma \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_N + O(\delta^2). \qquad \text{(C.98)}
\end{aligned}
$$

6. Observe that:

$$\nabla_\Gamma f^\pm \times \boldsymbol{\nu} \;=\; \overrightarrow{curl}_\Gamma(f^\pm), \tag{C.99}$$

so then for any vector $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_N \boldsymbol{\nu}$,

$$\nabla_\Gamma f^\pm \times \mathbf{u} \;=\; \overrightarrow{curl}_\Gamma(f^\pm)\mathbf{u}_N + \nabla_\Gamma f^\pm \times \mathbf{u}_T. \tag{C.100}$$

Therefore (C.97)-(C.98) become:

$$\begin{aligned}
[\![N^\pm \times \mathbf{E}^\pm]\!] \;&=\; 2\delta(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{E}_0^\delta \rangle\!\rangle \\
&-\; 2\delta\langle\!\langle \nabla_\Gamma f^\pm \times (\mathbf{E}_0^\delta)_T \rangle\!\rangle \\
&+\; 2\delta ik \left\langle f \right\rangle \mu_\delta \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_T - 2\delta \overrightarrow{curl}_\Gamma\Big(\left\langle f \right\rangle \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_N \Big) + O(\delta^2), \quad (C.101)\\
[\![N^\pm \times \mathbf{H}^\pm]\!] \;&=\; 2\delta(c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle f^\pm \mathbf{H}_0^\delta \rangle\!\rangle \\
&-\; 2\delta\langle\!\langle \nabla_\Gamma f^\pm \times (\mathbf{H}_0^\delta)_T \rangle\!\rangle \\
&-\; 2\delta ik \left\langle f \right\rangle \epsilon_\delta \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_T - 2\delta \overrightarrow{curl}_\Gamma\Big(\left\langle f \right\rangle \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle_N \Big) + O(\delta^2). \quad (C.102)
\end{aligned}$$

7. Observe now that

$$\begin{aligned}
ik\epsilon_\delta(\mathbf{E}_0^\delta)_N \;&=\; -curl_\Gamma \langle\!\langle \mathbf{H}_0^\delta \rangle\!\rangle = -curl_\Gamma \langle\!\langle \mathbf{H}_0^\pm \rangle\!\rangle, \quad (C.103)\\
ik\mu_\delta(\mathbf{H}_0^\delta)_N \;&=\; curl_\Gamma \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle = curl_\Gamma \langle\!\langle \mathbf{E}_0^\pm \rangle\!\rangle, \quad (C.104)
\end{aligned}$$

therefore (C.101)-(C.102) become

$$\begin{aligned}
[\![N^\pm \times \mathbf{E}^\pm]\!] \;&=\; 2\delta \left\langle f \right\rangle (c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle \mathbf{E}_0^\pm \rangle\!\rangle_T \\
&-\; 2\delta \nabla_\Gamma \left\langle f \right\rangle \times \langle\!\langle \mathbf{E}_0^\delta \rangle\!\rangle_T \\
&+\; 2\delta ik \left\langle f \right\rangle \mu_\delta \langle\!\langle \mathbf{H}_0^\pm \rangle\!\rangle_T \\
&+\; \frac{2\delta}{ik\epsilon_\delta} \overrightarrow{curl}_\Gamma\Big(\left\langle f \right\rangle curl_\Gamma \langle\!\langle \mathbf{H}_0^\pm \rangle\!\rangle_T \Big) + O(\delta^2), \quad (C.105)\\
[\![N^\pm \times \mathbf{H}^\pm]\!] \;&=\; 2\delta \left\langle f \right\rangle (c_1+c_2)\boldsymbol{\nu} \times \langle\!\langle \mathbf{H}_0^\pm \rangle\!\rangle_T \\
&-\; 2\delta \nabla_\Gamma \left\langle f \right\rangle \times \langle\!\langle \mathbf{H}_0^\pm \rangle\!\rangle_T \\
&-\; 2\delta ik \left\langle f \right\rangle \epsilon_\delta \langle\!\langle \mathbf{E}_0^\pm \rangle\!\rangle_T \\
&-\; \frac{2\delta}{ik\mu_\delta} \overrightarrow{curl}_\Gamma\Big(\left\langle f \right\rangle curl_\Gamma \langle\!\langle \mathbf{E}_0^\pm \rangle\!\rangle_T \Big) + O(\delta^2). \quad (C.106)
\end{aligned}$$

8. Finally, since

$$N^{\pm} \cdot N^{\pm} \;=\; 1 \pm 2\delta f^{\pm}(c_1 + c_2) + O(\delta^2), \tag{C.107}$$

we have

$$|N^{\pm}| \;=\; 1 \pm \delta f^{\pm}(c_1 + c_2) + O(\delta^2). \tag{C.108}$$

Thus since $\boldsymbol{\nu}^{\pm} = N^{\pm}/|N^{\pm}|$, (C.105) and (C.106) become:

$$\begin{aligned}
[\![\boldsymbol{\nu}^{\pm} \times \mathbf{E}^{\pm}]\!] \;=\;& 2\delta \left\langle f \right\rangle (c_1 + c_2)\boldsymbol{\nu} \times \langle\!\langle \mathbf{E}_0^{\pm} \rangle\!\rangle_T \\
-\;& 2\delta \nabla_\Gamma \left\langle f \right\rangle \times \langle\!\langle \mathbf{E}_0^{\delta} \rangle\!\rangle_T \\
+\;& 2\delta ik \left\langle f \right\rangle \mu_\delta \langle\!\langle \mathbf{H}_0^{\pm} \rangle\!\rangle_T \\
+\;& \frac{2\delta}{ik\epsilon_\delta} \overrightarrow{curl}_\Gamma \Big(\left\langle f \right\rangle curl_\Gamma \langle\!\langle \mathbf{H}_0^{\pm} \rangle\!\rangle_T \Big) + O(\delta^2), && \text{(C.109)} \\
[\![\boldsymbol{\nu}^{\pm} \times \mathbf{H}^{\pm}]\!] \;=\;& 2\delta \left\langle f \right\rangle (c_1 + c_2)\boldsymbol{\nu} \times \langle\!\langle \mathbf{H}_0^{\pm} \rangle\!\rangle_T \\
-\;& 2\delta \nabla_\Gamma \left\langle f \right\rangle \times \langle\!\langle \mathbf{H}_0^{\pm} \rangle\!\rangle_T \\
-\;& 2\delta ik \left\langle f \right\rangle \epsilon_\delta \langle\!\langle \mathbf{E}_0^{\pm} \rangle\!\rangle_T \\
-\;& \frac{2\delta}{ik\mu_\delta} \overrightarrow{curl}_\Gamma \Big(\left\langle f \right\rangle curl_\Gamma \langle\!\langle \mathbf{E}_0^{\pm} \rangle\!\rangle_T \Big) + O(\delta^2), && \text{(C.110)}
\end{aligned}$$

and this implies the statement by noticing that

$$\begin{aligned}
\delta \langle\!\langle \mathbf{E}^{\pm} \rangle\!\rangle_T \;&=\; \delta \langle\!\langle \mathbf{E}_0^{\pm} \rangle\!\rangle_T + O(\delta^2), && \text{(C.111)} \\
\delta \langle\!\langle \mathbf{H}^{\pm} \rangle\!\rangle_T \;&=\; \delta \langle\!\langle \mathbf{H}_0^{\pm} \rangle\!\rangle_T + O(\delta^2). && \text{(C.112)}
\end{aligned}$$

□

The new ATCs derived in the previous proposition, lead to the so called *Chun's-type* model (see [47, 40]). More precisely, it is defined by:

$$\nabla \times \mathbf{E} - ik\mathbf{H} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega_{ext}}, \tag{C.113}$$

$$\nabla \times \mathbf{H} + ik\mathbf{E} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega_{ext}}, \tag{C.114}$$

$$\nabla \times \mathbf{E} - ik\mu_\pm \mathbf{H} = 0 \quad \text{in} \quad \Omega_\pm, \tag{C.115}$$

$$\nabla \times \mathbf{H} + ik\epsilon_\pm \mathbf{E} = 0 \quad \text{in} \quad \Omega_\pm, \tag{C.116}$$

$$\begin{aligned}[\![\boldsymbol{\nu}^\pm \times \mathbf{E}^\pm]\!] = {} & 2\delta \langle f \rangle (c_1 + c_2)\boldsymbol{\nu} \times \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \\ & -2\delta \nabla_\Gamma \langle f \rangle \times \langle\!\langle \mathbf{E}^\delta \rangle\!\rangle_T \\ & +2\delta ik \langle f \rangle \mu_\delta \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\ & +\frac{2\delta}{ik\epsilon_\delta}\overrightarrow{curl}_\Gamma \Big(\langle f \rangle \, curl_\Gamma \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \Big) \end{aligned} \tag{C.117}$$

$$\begin{aligned}[\![\boldsymbol{\nu}^\pm \times \mathbf{H}^\pm]\!] = {} & 2\delta \langle f \rangle (c_1 + c_2)\boldsymbol{\nu} \times \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\ & -2\delta \nabla_\Gamma \langle f \rangle \times \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\ & -2\delta ik \langle f \rangle \epsilon_\delta \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \\ & -\frac{2\delta}{ik\mu_\delta}\overrightarrow{curl}_\Gamma \Big(\langle f \rangle \, curl_\Gamma \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \Big). \end{aligned} \tag{C.118}$$

and in $\mathbb{R}^3 \setminus \overline{\Omega}$, $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ and $\mathbf{H} = \mathbf{H}^s + \mathbf{H}^i$, where $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy the Silver-Müller radiation condition (C.9).

Let's then turn our attention to a very particular case, which constitute the ATCs used in Chapter 4 for the scattering of electromagnetic waves for planar delaminations of constant thickness.

Corollary C.3.1. *Assuming the ansatz (C.22) and (C.17), then for planar delaminations of constant thickness the second order* Chun's-type *ATCs are:*

$$\begin{aligned}
[\![\boldsymbol{\nu}^{\pm} \times \mathbf{E}^{\pm}]\!] &= \delta\widetilde{\alpha}_1 \langle\!\langle \mathbf{H}^{\pm} \rangle\!\rangle_T \\
&+ \delta\widetilde{\beta}_1 \, \overrightarrow{curl}_{\Gamma}\Big(curl_{\Gamma} \langle\!\langle \mathbf{H}^{\pm} \rangle\!\rangle_T\Big) \qquad (C.119)\\
[\![\boldsymbol{\nu}^{\pm} \times \mathbf{H}^{\pm}]\!] &= \delta\widetilde{\alpha}_2 \langle\!\langle \mathbf{E}^{\pm} \rangle\!\rangle_T \\
&+ \delta\widetilde{\beta}_2 \, \overrightarrow{curl}_{\Gamma}\Big(curl_{\Gamma} \langle\!\langle \mathbf{E}^{\pm} \rangle\!\rangle_T\Big), \qquad (C.120)
\end{aligned}$$

where $\widetilde{\alpha}_1 = 2ik\mu_\delta$, $\widetilde{\alpha}_2 = -2ik\epsilon_\delta$, $\widetilde{\beta}_1 = \frac{2}{ik\epsilon_\delta}$ *and* $\widetilde{\beta}_2 = \frac{2}{ik\mu_\delta}$.

Proof. Under this setting where Γ_0 is planar (implying that $c_1 = c_2 = 0$) and the thickness of Γ_0 is constant (so both $f^{\pm}$ are now -perhaps different -constants), this is an immediate consequence of Proposition C.3.1. □

Notice that under the conditions of Corollary , the shape of the thin domain Ω_δ is cylindrical, as shown in Fig. C.6.

Imposing *natural* boundary conditions on the side of the cylinder that we denote by

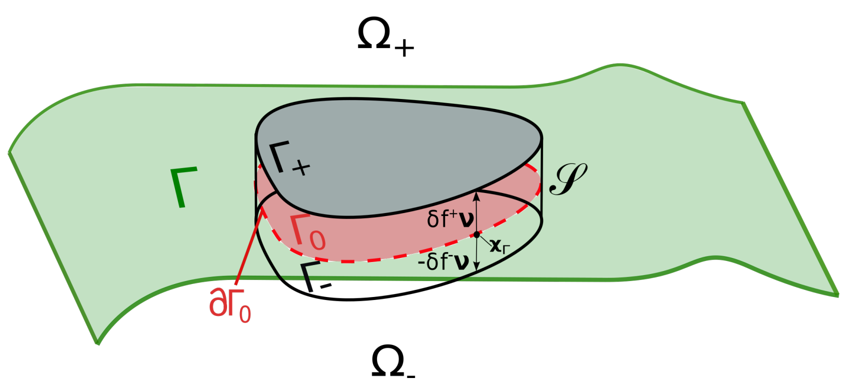


Figure C.6: Zoom on the planar delamination. Panel (b) Normal vectors on the boundary of the delamination.

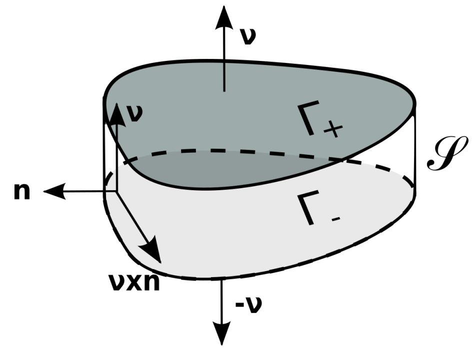


Figure C.7: Normal vectors on the boundary of the delamination.

$\mathscr{S}$, and where the normal unit vector is denoted by $\mathbf{n}$ (see Fig. C.7), the associated model with these ATCs consists of:

$$\nabla \times \mathbf{E} - ik\mathbf{H} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega_{ext}}, \tag{C.121}$$

$$\nabla \times \mathbf{H} + ik\mathbf{E} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega_{ext}}, \tag{C.122}$$

$$\nabla \times \mathbf{E} - ik\mu_\pm\mathbf{H} = 0 \quad \text{in} \quad \Omega_\pm, \tag{C.123}$$

$$\nabla \times \mathbf{H} + ik\epsilon_\pm\mathbf{E} = 0 \quad \text{in} \quad \Omega_\pm, \tag{C.124}$$

$$\begin{aligned}&[\![\boldsymbol{\nu}^\pm \times \mathbf{E}^\pm]\!] = \delta\widetilde{\alpha}_1 \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T \\ &+\delta\widetilde{\beta}_1\, \overrightarrow{curl}_\Gamma\Big(curl_\Gamma \langle\!\langle \mathbf{H}^\pm \rangle\!\rangle_T\Big)\end{aligned} \tag{C.125}$$

$$\begin{aligned}&[\![\boldsymbol{\nu}^\pm \times \mathbf{H}^\pm]\!] = \delta\widetilde{\alpha}_2 \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T \\ &+\delta\widetilde{\beta}_2\, \overrightarrow{curl}_\Gamma\Big(curl_\Gamma \langle\!\langle \mathbf{E}^\pm \rangle\!\rangle_T\Big),\end{aligned} \tag{C.126}$$

$$\mathbf{n} \times \mathbf{H} = 0 \text{ on } \mathscr{S}, \tag{C.127}$$

and in $\mathbb{R}^3 \setminus \overline{\Omega}$, $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ and $\mathbf{H} = \mathbf{H}^s + \mathbf{H}^i$, where $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy the Silver-Müller radiation condition (C.9).

This model is precisely the one analyzed in Chapter 4, for which a well posedness result Theorem 4.4.1, under the additional Assumptions 4.1.1 on the material properties μ and ϵ.

Appendix D

LIST OF SOBOLEV SPACES ON SURFACES

In this appendix we introduce the definitions of important Sobolev spaces on surfaces for the analysis in Chapter 4. Being consistent with the notation in that chapter, let $\Omega_- \subset \mathbb{R}^3$ be a bounded, simply connected domain with smooth boundary $\Gamma \subset \mathbb{R}^3$. If $\Gamma_0 \subset \Gamma$ is and open surface with Lipschitz countinuous relative boundary $\partial\Gamma_0$, then we define the following Sobolev spaces:

- First, we define the set of infinitely differentiable functions with compact support in Γ_0 by
$$\mathcal{D}(\Gamma_0) := \big\{\phi \in \mathcal{C}^\infty(\Gamma_0) \,|\, supp(\phi) \text{ is compact in } \Gamma_0\big\}, \tag{D.1}$$
endowed with the norm $\|\phi\|_\infty := \max\big\{|\phi(\mathbf{x}_\Gamma)| \,|\, \mathbf{x}_\Gamma \in \Gamma_0\big\}$.

- The dual space of $\mathcal{D}(\Gamma_0)$, is the space of distributions $\mathcal{D}'(\Gamma_0)$ on Γ_0 defined by
$$\begin{aligned}\mathcal{D}'(\Gamma_0) \;:=\; & \big\{u : \mathcal{D}(\Gamma_0) \to \mathbb{C} \,|\, u \text{ is linear and continuous} \\ & \qquad \text{in the sense of distributions } \big\},\end{aligned} \tag{D.2}$$
where $u \in \mathcal{D}'(\Gamma_0)$ is said to be continuous in the sense of distributions if and only if for every compact subset K of Γ_0 there exists a constant $C_K > 0$ and a non-negative integer N_K such that
$$|u(\phi)| \le C_K \max\big\{|\partial^{\boldsymbol{\alpha}}\phi(\mathbf{x}_\Gamma)| \,|\, \mathbf{x}_\Gamma \in K\big\},$$
for all test functions $\phi \in \mathcal{D}(\Gamma_0)$ with support contained in K and all multi-indices $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le N_K$.

- $$L^2(\Gamma_0) := \{u \in \mathcal{D}'(\Gamma_0) \,|\, \int_{\Gamma_0} |u|^2\, ds < \infty\}, \tag{D.3}$$
endowed with the inner product $(u, v)_{L^2(\Gamma_0)} := \int_{\Gamma_0} u\overline{v}\, ds$.

- additionally,
$$\mathbf{L}^2(\Gamma_0) := \{\mathbf{u} \in \mathcal{D}'(\Gamma_0)^3 \,|\, \int_{\Gamma_0} |\mathbf{u}|^2\, ds < \infty\}, \tag{D.4}$$
endowed with the inner product $(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Gamma_0)} := \int_{\Gamma_0} \mathbf{u} \cdot \overline{\mathbf{v}}\, ds$.

- and the tangential $\mathbf{L}^2$ vector fields are defined by:

$$\mathbf{L}^2_t(\Gamma_0) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma_0) \,|\, \boldsymbol{\nu} \cdot \mathbf{u} = 0\}, \tag{D.5}$$

endowed with the $\mathbf{L}^2(\Gamma_0)$ inner product.

- For $n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{H}^n(\Gamma_0) \quad := \quad & \{\mathbf{u} \in \mathcal{D}'(\Gamma_0)^3 \mid \partial^{\boldsymbol{\alpha}}\mathbf{u} \in L^2(\Gamma_0)^3 \\ & \text{for all multi-index } \boldsymbol{\alpha} \in \mathbb{N}^3 \text{such that } |\boldsymbol{\alpha}| \leq n\}, \end{aligned} \tag{D.6}$$

and endowed with the norm

$$||\mathbf{u}||^2_{\mathbf{H}^n(\Gamma_0)} := \sum_{|\boldsymbol{\alpha}|\leq n} ||\partial^{\boldsymbol{\alpha}}\mathbf{u}||^2_{L^2(\Gamma_0)^3}. \tag{D.7}$$

This definition can be extended for $n \in \mathbb{R}$, in the same way it is done for sub-domains in the Euclidean space $\mathbb{R}^n$, see [63].

- Let $s \in \mathbb{R}$, then we define

$$\widetilde{\mathbf{H}}^s(D_\Gamma, \Gamma_0) := \{\mathbf{u} \in \widetilde{H}^s(\Gamma_0)^3 \,|\, \boldsymbol{\nu} \cdot \mathbf{u} = 0 \text{ and } D_\Gamma \mathbf{u} \in \widetilde{H}^s(\Gamma_0)\}, \tag{D.8}$$

endowed with the $\mathbf{H}^s(D_\Gamma, \Gamma_0)$ norm, where $\widetilde{H}^s(\Gamma_0)$ are defined by

$$\begin{aligned} \widetilde{H}^s(\Gamma_0) \quad = \quad & \{u \in H^s(\Gamma_0) \,|\, \text{ the extension} \\ & \text{by zero of } u \text{ in } \Gamma, \ \widetilde{u}, \text{ is in } H^s(\Gamma)\}, \end{aligned} \tag{D.9}$$

endowed with the restricted $H^s(\Gamma_0)$ inner product.
It has been proven (see [63]) that for

$$s = \frac{2\ell + 1}{2},$$

where $\ell \in \mathbb{Z}$, the space $\widetilde{H}^s(\Gamma_0)$ is precisely the dual space of $H^{-s}(\Gamma_0)$, with respect to the duality pairing:

$$\langle v, u \rangle_{H^{-s}(\Gamma_0), \widetilde{H}^s(\Gamma_0)} \quad := \quad \langle v, \widetilde{u} \rangle_{H^{-s}(\Gamma), H^s(\Gamma)}, \tag{D.10}$$

where on the right-hand-side of (2.23) $\widetilde{u}$ is the extension by zero of u to Γ, and

$$\langle v, u \rangle_{\widetilde{H}^{-s}(\Gamma_0), H^s(\Gamma_0)} \quad := \quad \langle \widetilde{v}, u \rangle_{H^{-s}(\Gamma), H^s(\Gamma)}, \tag{D.11}$$

where $\widetilde{v} \in H^{-s}(\Gamma)$ is the extension by zero of v.

- If D_Γ denotes either the surface divergence div_Γ or the surface scalar $curl_\Gamma$ defined in Appendix C, and $s \in \mathbb{R}$, then

$$\mathbf{H}^s(D_\Gamma, \Gamma_0) := \{\mathbf{u} \in \mathbf{H}^s(\Gamma_0) \,|\, \boldsymbol{\nu} \cdot \mathbf{u} = 0, \text{ and } \ D_\Gamma \mathbf{u} \in H^s(\Gamma_0)\}, \tag{D.12}$$

with the graph norm $\|\mathbf{u}\|^2_{\mathbf{H}^s(D_\Gamma, \Gamma_0)} = \|\mathbf{u}\|^2_{\mathbf{H}^s(\Gamma_0)} + \|D_\Gamma \mathbf{u}\|^2_{H^s(\Gamma_0)}$.

Appendix E

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Sincerely,

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