THEORETICAL INVESTIGATIONS OF NEUTRINO AND DARK MATTER PROPERTIES

by

Heng-Yu Chen

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by

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ABSTRACT

There are several questions which couldn't be explained by the Standard Model of particle physics. For example, why the masses of particles of the third family are larger than those of the corresponding particles of the second family, which are larger than those of the first family? (the fermion mass hierarchy). Moreover, the different flavors mix with each other in a way described by mixing angles and complex phase angles. Why the inter-family mixing angles in the leptonic sector are bigger than the ones in the quark sector is still unknown. The problem of explaining the masses and mixing angles of the quarks and leptons is the flavor problem.

Another important unanswered question is that what is dark matter?. At this time, almost nothing is known about the dark matter particles, which constitute about 80% of the matter in the universe by mass. No one knows what kinds of particles dark matter is made of, what their masses are, and whether or how they mix or interact. This is the so called dark matter problem. These two questions are the main focus of this thesis. They touch the most fundamental questions in particle physics which involve postulating new physics, i.e. beyond the Standard Model physics.

1. The Flavor Problem

My adviser and I proposed a model in which the mixing angles in both quark and neutrino sectors are controlled by just one single matrix which arises from the mixture of regular Standard Model fermions with extra vector-like fermions in $5 + \overline{5}$ multiplets in SU(5). In the resulting model, all the presently unknown neutrino parameters are predicted, including Dirac neutrino CP phase. Why the inter-family mixing angles in the leptonic sector are larger than the ones in the quark sector is also explained.

The model predicts certain mixing angles within GUT fermion multiplets that are only observable in proton decay. The model also contains certain new scalar particles. If one of these scalars has mass near the weak scale, it will contribute an observable amount to such flavor-changing processes as $\mu \to e + \gamma$. The branching ratios for proton decay and flavor-changing lepton decays are calculated. The branching ratios from these processes could give several independent tests of the model.

Moreover,we have proposed a new version of the model in which the inter-family hierarchies among the fermion masses are controlled by another matrix. The combination of this idea with our previous model can give a complete and quite simple account of the entire flavor structure of the quarks and leptons, including mixing angles and mass ratios.

2. The Dark Matter Problem

One of the ideas that has attracted enormous interest in recent years is the idea of asymmetric dark matter. Another is the idea that ordinary matter and dark matter may have been generated together in the early universe by a single mechanism. The first paper on co-generation of dark matter, and the second on asymmetric dark matter was co-authored by my thesis adviser S.M. Barr in 1990, with E. Farhi and S. Chivukula. My adviser and I have proposed a co-generation mechanism that improves on the one proposed in that 1990 paper. This paper proposed that so-called sphaleron processes of the Standard Model could be responsible for co-generating dark matter. We show that sphalerons of a new non-abelian gauge interaction would more easily co-generate dark matter and also lead to definite predictions of the mass of the dark matter particles.

Chapter 1

A BRIEF INTRODUCTION TO THE PHYSICS OF THE STANDARD MODEL AND BEYOND

1.1 QED of A Fermion with Charge Q

There are many kinds of indices in this thesis, in fact seven kinds.

1. Lorentz vector indices, e.g. x^μ, ∂_μ, $\gamma^\mu$and A_μ
2. Lorentz spinor indices, e.g. ψ_λ, $(\gamma^\mu)^\lambda_{\lambda'}$ and $C^\lambda_{\lambda'}$
3. Color indices labelling components of triplets of SU(3)
4. Color indices labelling generators of SU(3)
5. electroweak indices labelling components of doublets of SU(2)
6. electroweak indices labelling generators of SU(2)
7. Family indices

If all indices were shown, the equations would look so messy they would be hard to understand. For example, for a left-handed quark, there are four types of indices on it. So I shall only show indices when necessary. They should be understood to be there even they are not shown. Finally, the summation convention are used throughout the thesis for all kinds of indices, for example, $\gamma^\mu A_\mu = \gamma^0 A_0 + \gamma^1 A_1 + \gamma^2 A_2 + \gamma^3 A_3$.

1.1.1 The Lagrangian Density of QED

In QED, the Lagrangian density is written as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{gauge} + \mathcal{L}_{f,kin} + \mathcal{L}_{f,mass} \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \big(\overline{\psi} i \gamma^\mu D_\mu \psi + h.c.\big) - m\overline{\psi}\psi \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \psi &\equiv \psi_\lambda = \text{ four component Dirac spinor} \\ \overline{\psi} &\equiv \psi^\dagger \gamma^0 \end{aligned}$$

and γ^μ are 4×4 gamma matrices having the following properties

$$\{\gamma^\mu,\, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^{0\dagger} = \gamma^0,\ \gamma^{i\dagger} = -\gamma^i$$

γ^0, γ^1, γ^3 are real and γ^2 is imaginary. Here we follow the Bjorken and Drell conventions. And

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu \\ \partial_\mu &\equiv \partial/\partial\, x^\mu \\ D_\mu &\equiv \partial_\mu - i\, e\, Q\, A_\mu \end{aligned}$$

The Lagrangian density is both Lorentz and gauge invariant, as we will be shown in the following sections.

1.1.2 Gauge Invariance

The gauge transformations of the fermion and gauge fields are

$$\begin{aligned} \psi(x^\lambda) &\rightarrow \psi'(x^\lambda) = e^{i\,Q\,\alpha(x^\lambda)}\psi(x^\lambda) \\ A_\mu(x^\lambda) &\rightarrow A'_\mu(x^\lambda) = A_\mu(x^\lambda) + \frac{1}{e}\partial_\mu\alpha(x^\lambda) \end{aligned}$$

Then

$$\begin{aligned}
D_\mu\psi \to D'_\mu\psi' &= \left(\partial_\mu - i\,e\,Q\,A'_\mu\right)\psi' \\
&= \left(\partial_\mu - i\,e\,Q\,[A_\mu(x^\lambda) + \frac{1}{e}\partial_\mu\alpha(x^\lambda)]\right)e^{i\,Q\,\alpha(x^\lambda)}\psi(x^\lambda) \\
&= e^{i\,Q\,\alpha}\left(\partial_\mu\psi + i\,Q\,(\partial_\mu\alpha)\psi - i\,e\,Q\,A_\mu\psi - i\,Q\,(\partial_\mu\alpha)\psi\right) \\
&= e^{i\,Q\,\alpha}\left(\partial_\mu - i\,e\,Q\,A_\mu\right)\psi \\
&= e^{i\,Q\,\alpha}D_\mu\psi
\end{aligned}$$

And

$$\overline{\psi}(x^\lambda) \to \overline{\psi}'(x^\lambda) = e^{-i\,Q\,\alpha}\overline{\psi}(x^\lambda)$$

therefore $\overline{\psi}i\gamma^\mu D_\mu\psi$ and $m\overline{\psi}\psi$ are both gauge invariant.

$$\begin{aligned}
F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
&= \partial_\mu\left(A_\nu + \frac{1}{e}\partial_\nu\alpha\right) - \partial_\nu\left(A_\mu + \frac{1}{e}\partial_\mu\alpha\right) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= F_{\mu\nu}
\end{aligned}$$

So $F_{\mu\nu}F^{\mu\nu}$ is also gauge invariant. We can conclude the Lagrangian density of QED is gauge invariant.

1.1.3 Lorentz Invariance

The Lorentz transformation of fermion field is

$$\psi \to \psi' = e^{i\theta_{\mu\nu}\sigma^{\mu\nu}}\psi$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\, \gamma^\nu]$ are the generators of Lorentz transformations and $\theta_{\mu\nu}$ are the transformation parameters, i.e. the magnitudes of the spatial rotations and boosts.

$$\begin{aligned}
\overline{\psi} = \psi^\dagger\gamma^0 &\to \psi^\dagger e^{-i\theta_{\mu\nu}(\sigma^{\mu\nu})^\dagger}\gamma^0 \\
&= \psi^\dagger\gamma^0 e^{-i\theta_{\mu\nu}\sigma^{\mu\nu}} \\
&= \overline{\psi}\,e^{-i\theta_{\mu\nu}\sigma^{\mu\nu}} \qquad (1.2) \\
\Rightarrow \overline{\psi}'\psi' &= \overline{\psi}\,e^{-i\theta_{\mu\nu}\sigma^{\mu\nu}}\,e^{+i\theta_{\mu\nu}\sigma^{\mu\nu}}\psi \\
&= \overline{\psi}\psi
\end{aligned}$$

so the mass term is Lorentz invariant. Notice that we went from the second line of eq. (1.2) to the third line by using the relation $\sigma^{\mu\nu\dagger}\gamma^0 = \gamma^0\sigma^{\mu\nu}$. This relation can be derived from the properties of γ^μ matrices and the definition of $\sigma^{\mu\nu}$. Similarly, it can be shown that the kinetic term is also Lorentz invariant, but the algebra is slightly longer because it also involves Lorentz transforming the gauge field A_μ.

1.2 More about Dirac Spinors

1.2.1 Charge Conjugation of Spinors

Given a Dirac spinor ψ that transforms as above, one can define a charge conjugate spinor ψ^c as follows:

$$\psi^c \equiv i\,\gamma^2\psi^*.$$

This can be shown to transform in the same way under Lorentz transformations as ψ itself:

$$\begin{aligned}\psi^c \to \psi'^c &= i\,\gamma^2\left(e^{i\theta_{\mu\nu}\sigma^{\mu\nu}}\psi\right)^* \\ &= i\,\gamma^2 e^{-i\theta_{\mu\nu}\sigma^{\mu\nu *}}\psi^* \\ &= e^{+i\theta_{\mu\nu}\sigma^{\mu\nu}}i\,\gamma^2\psi^* \\ &= e^{+i\theta_{\mu\nu}\sigma^{\mu\nu}}\psi^c,\end{aligned}$$

where we have used the fact that $-\sigma^{\mu\nu}\gamma^2 = \gamma^2\sigma^{\mu\nu *}$, which follows from the properties of the gamma matrices. It will be seen later that ψ^c has the opposite chirality to ψ. Under gauge transformations, if $\psi \to \psi' = e^{iQ\alpha}\psi$, then

$$\psi^c \to \psi'^c = e^{-i\,Q\,\alpha}\psi^c$$

so ψ^c and ψ also have opposite-sign charge. So if ψ is a left-handed electron, then ψ^c is a right-handed positron, etc. Therefore, one can write fermion mass terms using ψ^c

if we define $C \equiv i\,\gamma^2\gamma^0$,

$$
\begin{aligned}
m\,(\psi^c)^T\,C\psi &= m\left(i\,\gamma^2\psi^*\right)^T\left(i\,\gamma^2\gamma^0\right)\psi \\
&= -m\,\psi^{*T}\gamma^2\gamma^2\gamma^0\psi \\
&= +m\,\psi^{*T}\gamma^0\psi \\
&= m\overline{\psi}\psi
\end{aligned}
$$

As we will see, there are advantages to writing mass terms this way. For example, writing Majorana neutrino masses in this way will be less confusing, and so will writing quark and lepton masses in grand unified theories.

1.2.2 Chirality of Fermions

Define $\gamma^5 \equiv i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5$. We can easily derive the following properties of γ_5

$$
\left(\gamma^5\right)^2 = 1, \qquad \gamma^{5*} = \gamma^5 = \gamma^{5T}
$$

$$
\left\{\gamma^5,\,\gamma^\mu\right\} = 0,
$$

We also define the right-handed and left-handed projection matrices

$$
\frac{1+\gamma^5}{2} \equiv R, \qquad \frac{1-\gamma^5}{2} \equiv L,
$$

which have the properties

$$
L^2 = L, \quad R^2 = R,
$$

$$
LR = RL = 0, \quad L + R = I.
$$

We can then define the fields of the right-handed and left-handed particles as

$$
\begin{aligned}
\psi_R \equiv R\psi &= \left(\frac{1+\gamma^5}{2}\right)\psi \\
\psi_L \equiv L\psi &= \left(\frac{1-\gamma^5}{2}\right)\psi
\end{aligned}
\tag{1.3}
$$

$$
\Longrightarrow \quad \psi \;=\; \psi_R + \psi_L
$$

Note that $\gamma^5\psi_L = -\psi_L$ and $\gamma^5\psi_R = -\psi_R$.

1.2.3 Chirality and Fermion Kinetic Terms and Gauge Couplings to Fermions

We can rewrite the kinetic terms of the fermions using the definitions in eq. (1.3). First consider a fermion kinetic term with a projection matrix inserted into it:

$$
\begin{aligned}
\overline{\psi} i\gamma^\mu D_\mu \left(\frac{1\pm\gamma^5}{2}\right)\psi &= \overline{\psi} i\gamma^\mu D_\mu \left(\frac{1\pm\gamma^5}{2}\right)^2\psi \\
&= \psi^\dagger i\left(\gamma^0\gamma^\mu\right) D_\mu \left(\frac{1\pm\gamma^5}{2}\right)\left(\frac{1\pm\gamma^5}{2}\right)\psi \\
&= \psi^\dagger \left(\frac{1\pm\gamma^5}{2}\right) i\left(\gamma^0\gamma^\mu\right) D_\mu \left(\frac{1\pm\gamma^5}{2}\right)\psi \\
&= \left(\frac{1\pm\gamma^5}{2}\psi\right)^\dagger \gamma^0 i\,\gamma^\mu D_\mu \left(\frac{1\pm\gamma^5}{2}\right)\psi \\
&= \overline{\psi_R}\, i\,\gamma^\mu D_\mu \psi_R \quad \text{for the upper sign} \\
&\ \overline{\psi_L}\, i\,\gamma^\mu D_\mu \psi_L \quad \text{for the lower sign} \\
\Longrightarrow \overline{\psi} i\gamma^\mu D_\mu \psi &= \overline{\psi_R}\, i\,\gamma^\mu D_\mu \psi_R + \overline{\psi_L}\, i\,\gamma^\mu D_\mu \psi_L
\end{aligned}
$$

If one thinks of $\psi^\dagger$ as creating a fermion and ψ annihilating a fermion, these terms correspond to the following diagrams

Figure 1.1: Fermion and gauge boson coupling

Since D_μ contains an A_μ, so the fermion kinetic terms and gauge boson couplings to fermions do not change charility.

1.2.4 Chirality and Fermion Mass Terms (and Yukawa Couplings)

Now let us consider a fermion mass term with a projection matrix inserted into it:

$$
\begin{aligned}
m\overline{\psi}\left(\frac{1\pm\gamma^5}{2}\right)\psi &= m\overline{\psi}\left(\frac{1\pm\gamma^5}{2}\right)^2\psi \\
&= m\psi^\dagger\gamma^0\left(\frac{1\pm\gamma^5}{2}\right)^2\psi \\
&= m\psi^\dagger\left(\frac{1\mp\gamma^5}{2}\right)\gamma^0\left(\frac{1\pm\gamma^5}{2}\right)\psi \\
&= m\left[\left(\frac{1\mp\gamma^5}{2}\right)\psi\right]^\dagger\gamma^0\left(\frac{1\pm\gamma^5}{2}\right)\psi \\
\\
&= m\overline{\psi_L}\psi_R \quad \text{for the upper sign} \\
& m\overline{\psi_R}\psi_L \quad \text{for the lower sign} \\
\\
\Longrightarrow m\overline{\psi}\psi &= m\overline{\psi}_L\psi_R + m\overline{\psi}_R\psi_L
\end{aligned}
$$

Notice that these terms "flip" the chirality of the fermion: they destroy a left-handed fermion and create a right-handed one, or vice versa. Also for so-called "Yukawa couplings" of fermions to spin-0 fields (scalar fields), one can write

$$Y\overline{\psi}\psi\phi = Y\overline{\psi}_L\psi_R\phi + Y\overline{\psi}_R\psi_L\phi,$$

where the coefficient Y is called the Yukawa coupling constant. These fermion mass terms and Yukawa terms can be understood using the following Feynman diagrams

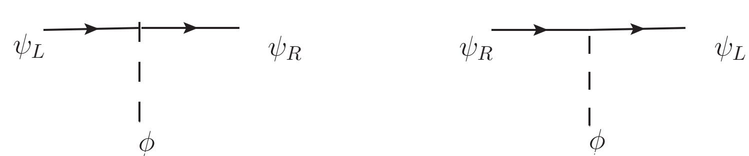

Figure 1.2: Fermion and scalar boson coupling

The reason Yukawa terms are called that is that they are similar in form to the nucleon-pion coupling term $\overline{N}N\pi$ proposed by Yukawa. Now let's consider other way of writing these mass terms with ψ^c

$$\begin{aligned}
(\psi^c)^T C \left(\frac{1 \pm \gamma^5}{2}\right) \psi &= (\psi^c)^T C \left(\frac{1 \pm \gamma^5}{2}\right)^2 \psi \\
&= (\psi^c)^T \left(\frac{1 \pm \gamma^5}{2}\right) C \left(\frac{1 \pm \gamma^5}{2}\right) \psi \\
&= \left[\left(\frac{1 \pm \gamma^5}{2}\right) \psi^c\right]^T C \left(\frac{1 \pm \gamma^5}{2}\right) \psi \\
&= (\psi_R^c)^T C \psi_R \quad \text{for the upper sign} \\
& \quad\ (\psi_L^c)^T C \psi_L \quad \text{for the lower sign.}
\end{aligned}$$

Therefore, one can draw the Feynman diagrams in another way

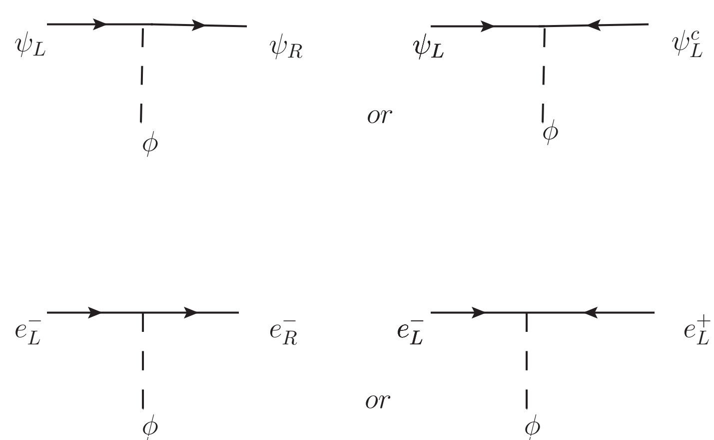

Figure 1.3: Yukawa coupling

There is an intuitive way to understand why the Yukawa coupling of a scalar field to a fermion does not conserve the chirality of the fermion. The Yukawa coupling gives a Feynman diagram that can describe the scalar decaying into a particle and an antiparticle, as we show in the last diagram in the Figure 1.3. Since the initial state in a spin-0 particle, the spins of the two particles in the final state must be pointing in opposite direction. But also (in the center of momentum frame) the momenta of the final state particles must be pointing in the opposite direction. Therefore, the spin and momentum must be aligned for both final state particles or anti-aligned. That is, they must be both left-handed or both right-handed. That corresponds to the Yukawa operator being of the form $\psi_L^c C \psi_L \phi$ or $\psi_R^c C \psi_R \phi$.

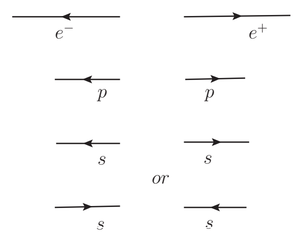

Figure 1.4: Handness of fermion

1.3 Complex Fermion Masses

Let's now consider complex fermion masses in QED

$$\begin{aligned}\mathcal{L}_{f,\,mass} &= -\left(m\overline{\psi_L}\psi_R + m^*\overline{\psi_R}\psi_L\right)\\ &= -\left(m\psi_L^\dagger\gamma^0\psi_R + m^*\psi_R^\dagger\gamma^0\psi_L\right)\end{aligned}$$

We have allowed the coefficient m (the fermion mass) to be complex, because there is no reason why it should not be. But the Lagrangian (like the kinetic and potential energy) should be real. One can see that $\mathcal{L}_{f,mass}$ is real if the coefficient of the second term is m^*, because then the second term can be obtained by taking the hermitian conjugate of the first. Let us write $m = |m|\, e^{i\alpha}$. Then we can eliminate the phase $e^{i\alpha}$ from the Lagrangian density by redefining fields. For example, if we define $\psi_R' \equiv e^{i\alpha}\psi_R$ and $\psi_L' \equiv \psi_L$, then the fermion mass term can be written

$$\mathcal{L}_{f,\,mass} = -|m|\left(\overline{\psi}_L'\psi_R' + \overline{\psi}_R'\psi_L'\right)$$

The fermion kinetic terms keep the same from

$$\begin{aligned}\mathcal{L}_{f,\,kin} &= \overline{\psi}_R i\,\gamma^\mu D_\mu \psi_R + \overline{\psi}_L i\,\gamma^\mu D_\mu \psi_L \\ &= \overline{\psi}'_R i\,\gamma^\mu D_\mu \psi'_R + \overline{\psi}'_L i\,\gamma^\mu D_\mu \psi'_L.\end{aligned}$$

So the phase α has no physical meaning. People say it can be "rotated away" or "absorbed" by redefinition of fermion fields. However, suppose we did not redefine the fields, and the phase α remained in the Lagrangian density. It would make no difference, because it would cancel in the calculation of any physical quantity. As we will see, some phases cannot be rotated away. Such phases are physical and lead to "CP violation." For example, suppose that a fermion had both a mass term and a Yukawa term coupling it to a real scalar field ϕ. Then

$$\mathcal{L}_{f,mass,Yuk} = -|m| \left(e^{i\alpha}\overline{\psi}_L\psi_R + e^{-i\alpha}\overline{\psi}_R\psi_L\right) \;\; - Y\left(e^{i\beta}\overline{\psi}_L\psi_R + e^{-i\beta}\overline{\psi}_R\psi_L\right)\phi. \tag{1.4}$$

One can define $\psi'_R = e^{i\alpha}\psi_R$, and make α disappear from the mass term but the Yukawa term would then be

$$Y\left(e^{i(\beta-\alpha)}\overline{\psi}_L\psi_R + e^{-i(\beta-\alpha)}\overline{\psi}_R\psi_L\right)\phi.$$

Or one could make the phase disappear from the Yukawa term, but it would then be in the mass term. In this case, the phase $(\beta-\alpha)$ is physical and leads to CP violation. One can see that the more interaction terms there are in $\mathcal{L}$, the more complex coefficients (more phases) there can be. But the more fields there are in the theory, the more phases can be absorbed by redefining these fields. So the net number of "physical phases" depends both on how many interaction terms there are and how many fields there are. This will be important in counting the number of physical phases there are in the Standard Model.

1.4 Mass Matrices

Now consider QED with N types of fermions, which we will call "flavors."

$$\psi_{Lm} = (\psi_{L1}, \psi_{L2}, \psi_{L3}, \cdots \psi_{LN}),$$

$$\psi_{Rm} = (\psi_{R1}, \psi_{R2}, \psi_{R3}, \cdots \psi_{RN}).$$

The kinetic and mass terms of fermions are

$$\begin{aligned}\mathcal{L}_{f,\,kin} + \mathcal{L}_{f,\,mass} &= Z^L_{mn}\left(\overline{\psi_{Lm}} i\,\gamma^\mu \overleftrightarrow{D}\,\psi_{Ln}\right) \\ &+ Z^R_{mn}\left(\overline{\psi_{Rm}} i\,\gamma^\mu \overleftrightarrow{D}\,\psi_{Rn}\right) \\ &+ M_{mn}\overline{\psi_{Lm}}\psi_{Rn} + M^\dagger_{mn}\overline{\psi_{Rm}}\psi_{Ln},\end{aligned}$$

where $\overleftrightarrow{D} = \overrightarrow{D} - \overleftarrow{D}$. One can easily check that for the kinetic terms to be real, Z^L and Z^R must be hermitian matrices. One can make $Z^L \to I$ and $Z^R \to I$ by a combination of unitary and scale transformations of the fields ψ_{Lm}, ψ_{Rn}. Let's look at the kinetic term for the left-handed fermions first. Suppressing the flavor indices, one has

$$\overline{\psi_L} Z^L i\,\gamma^\mu \overleftrightarrow{D}\,\psi_L.$$

Since Z^L is hermitian, it can be made real and diagonal by a unitary transformation

$$\begin{aligned}V_L Z^L V_L^\dagger &= \widetilde{Z}^L = \text{real diagonal} \\ \Longrightarrow Z^L &= V_L^\dagger \widetilde{Z}^L V_L \\ \Longrightarrow \overline{\psi_L} Z^L i\,\gamma^\mu \overleftrightarrow{D}\,\psi_L &= \left(\overline{\psi_L} V_L^\dagger \sqrt{\widetilde{Z}^L}\right) i\,\gamma^\mu D_\mu \left(\sqrt{\widetilde{Z}^L} V_L \psi_L\right).\end{aligned}$$

Therefore, if we define

$$\psi'_L = \sqrt{\widetilde{Z}^L} V_L \psi_L,$$

the term becomes

$$\overline{\psi'_L} i\,\gamma^\mu D_\mu \psi'_L.$$

This is called the "canonical form" of the kinetic term, i.e. the form with no matrix Z^L. Similarly, by redefinition of ψ_R, the kinetic term of ψ_R can be brought to canonical form. After that is done, the fermion mass term will in general *still* have a non-trivial matrix in it that is called the "mass matrix." Since one can always bring kinetic terms to canonical form, people usually start with them written in canonical form. Let us therefore do that, but assume that the mass matrix is non-trivial:

$$\overline{\psi_L}\; i\,\gamma^\mu \overleftrightarrow{D}\,\psi_L + \overline{\psi_R}\; i\,\gamma^\mu \overleftrightarrow{D}\,\psi_R + \overline{\psi_L}\; M\,\psi_R + \overline{\psi_R}\; M^\dagger \psi_L.$$

The fermion mass matrix M is not hermitian in general, but there is a theorem that complex matrices can be made diagonal and real by so-called "bi-unitary" transformations. So we may write

$$\mathcal{U}_L^\dagger M \mathcal{U}_R = \widetilde{M} = \text{real and diagonal}$$

$$\Longrightarrow M = \mathcal{U}_L \widetilde{M} \mathcal{U}_R^\dagger, \qquad M^\dagger = \mathcal{U}_R \widetilde{M}^\dagger \mathcal{U}_L^\dagger,$$

$$\Longrightarrow \mathcal{L}_{f,\,mass} = \overline{\psi_L} \mathcal{U}_L \widetilde{M} \mathcal{U}_R^\dagger \psi_R + \overline{\psi_R} \mathcal{U}_R \widetilde{M}^\dagger \mathcal{U}_L^\dagger \psi_L$$

Define

$$\begin{aligned}
\psi'_L &= \mathcal{U}_L^\dagger \psi_L, \\
\psi'_R &= \mathcal{U}_R^\dagger \psi_R \\
\Longrightarrow \mathcal{L}_{f,\,mass} &= \overline{\psi'_L} \widetilde{M} \psi'_R + \overline{\psi'_R} \widetilde{M} \psi'_L \\
&= \overline{\psi'} \widetilde{M} \psi'
\end{aligned}$$

This is called the "mass basis" because $\widetilde{M}$ is real and diagonal, so the ψ'_m are the mass eigenstates, i.e. fields of definite (real) mass. This is also called the "physical basis," because we identify the physical particles as those of definite mass. Notice that the unitary transformations $\mathcal{U}_L$ and $\mathcal{U}_R$ don't affect the kinetic terms, which remain in the canonical form:

$$\begin{aligned}
\overline{\psi'_L}\, i\gamma^\mu D_\mu \psi'_L &= \overline{\psi_L} \mathcal{U}_L i\gamma^\mu D_\mu \mathcal{U}_L^\dagger \psi_L \\
&= \overline{\psi_L}\, i\gamma^\mu D_\mu \psi_L,
\end{aligned}$$

and similarly for the kinetic term of the right-handed fermions.

1.5 Non-Abelian Gauge Theory: $SU(2)$

1.5.1 Review of Abelian Gauge Theory

Let us look at QED again. There is a gauge symmetry of $\mathcal{L}$. The symmetry transformation is

$$\begin{aligned}
\psi(x^\lambda) &\rightarrow \psi'(x^\lambda) = e^{i\,Q\,\alpha(x^\lambda)}\psi(x^\lambda), \\
A_\mu(x^\lambda) &\rightarrow A'_\mu(x^\lambda) = A_\mu(x^\lambda) + \frac{1}{e}\partial_\mu\alpha(x^\lambda), \\
D_\mu\psi &\rightarrow D'_\mu\psi' = \left(\partial_\mu - i\,e\,Q\,A'_\mu\right)\psi' = e^{i\,Q\,\alpha}D_\mu\psi, \\
\overline{\psi}(x^\lambda) &\rightarrow e^{-i\,Q\,\alpha}\overline{\psi}(x^\lambda).
\end{aligned}$$

This allows the kinetic term of gauge field A_μ to be written as

$$\begin{aligned}
\overline{\psi}i\gamma^\mu D_\mu\psi &\rightarrow \left(e^{-i\,Q\,\alpha}\overline{\psi}\right)i\gamma^\mu\left(e^{i\,Q\,\alpha}D_\mu\psi\right) \\
&= \overline{\psi}i\gamma^\mu D_\mu\psi
\end{aligned}$$

The transformations $\psi \rightarrow e^{i\,Q\,\alpha}\psi$ form a group whose elements are parameterized by a single angle $0 \leq \alpha \leq 2\pi$.

$$\mathcal{U}(\alpha) = e^{i\,Q\,\alpha}.$$

Note that this is a group of 1×1 unitary matrices. It is therefore called $U(1)$. It is abelian because $e^{i\,Q\,\alpha} \cdot e^{i\,Q\,\beta} = e^{i\,Q\,\beta} \cdot e^{i\,Q\,\alpha}$. Suppose we consider the commutator $[D_\mu,\, D_\nu]$

$$\begin{aligned}
[D_\mu,\, D_\nu]\,\phi &= \left(\partial_\mu - i\,e\,Q\,A_\mu\right)\left(\partial_\nu - i\,e\,Q\,A_\nu\right)\phi - (\mu \leftrightarrow \nu) \\
&= -i\,e\,Q\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\phi \\
&\equiv -i\,e\,Q\,F_{\mu\nu}\phi \\
&\Longrightarrow F_{\mu\nu} = \frac{i}{eQ}\,[D_\mu,\, D_\nu]
\end{aligned}$$

It is easy to check directly that $F_{\mu\nu}$ is gauge invariant. One can also see that (as operators)

$$
\begin{aligned}
D'_\mu \left(e^{iQ\alpha}\phi\right) = D'_\mu \phi' = e^{iQ\alpha} D_\mu \phi \;\Longrightarrow\; & D'_\mu e^{iQ\alpha} = e^{iQ\alpha} D_\mu \\
\Longrightarrow\; & e^{-iQ\alpha} D'_\mu e^{iQ\alpha} = D_\mu \\
\Longrightarrow [D_\mu,\, D_\nu] \;=\; & \left[e^{-iQ\alpha} D'_\mu e^{iQ\alpha},\, e^{-iQ\alpha} D'_\nu e^{iQ\alpha}\right] \\
=\; & e^{-iQ\alpha} \left[D'_\mu,\, D'_\nu\right] e^{iQ\alpha} \\
=\; & \left[D'_\mu,\, D'_\nu\right]
\end{aligned}
$$

In the last step we have used the fact that $[D_\mu,\, D_\nu] = -ieQF_{\mu\nu}$ has no free derivative operators in it but only derivatives of fields.

1.5.2 $SU(2)$ Gauge Theory

Now we will explain how the gauge transformations of QED can be generalized to "non-abelian" transformations. Consider a doublet of fermions:

$$
\psi^i = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}.
$$

One can consider transformations where this doublet is multiplied by a complex 2×2 matrix U: $\psi \longrightarrow \psi' = U\psi$. (It will simplify notation not to show the indices of the matrix U or of the fermion doublet ψ.) We will assume that both the left-handed and right-handed projections of ψ transform the same way, so $\psi_L \longrightarrow U\psi_L$ and $\psi_R \longrightarrow U\psi_R$. If one wants the mass term of the doublet to be invariant under this transformation, then one must have that

$$
m\overline{\psi}\psi \longrightarrow m\overline{(U\psi)}U\psi = m\overline{\psi}U^\dagger U\psi = m\overline{\psi}\psi.
$$

This implies that $U^\dagger U = I$, i.e. U that is unitary. Let us require not only that U be unitary, but that it be "unimodular", that is that $detU = 1$. The set of all 2×2 unimodular unitary matrices is closed under matrix multiplication and form a group called $SU(2)$. (U stands for unitary, 2 stands for 2×2, and S stands for "special," meaning determinant = 1.) Suppose that we demand that the Lagrangian density $\mathcal{L}$ is invariant under "local" $SU(2)$ transformations, i.e. transformations under which U is a function of space-time location: $U(x^\mu)$. Such a local transformation is called a "gauge transformation," and so such a theory would be said to have $SU(2)$ gauge invariance.

The most general unitary, unimodular 2×2 matrix can be written in the following form:

$$U(\alpha^a) = e^{i \sum_{a=1}^{3} T^a \alpha^a},$$

where α^a, $a = 1, 2, 3$, are three angles, and T^a, $a = 1, 2, 3$, are three 2×2 hermitian traceless matrices, which are called the generators of $SU(2)$ transformations. These generators can be written in terms of the well-known Pauli matrices:

$$T^a = \frac{1}{2}\sigma^a.$$

If the transformations are local, that means that the transformation angles depend on location, so we may write

$$U(x^\mu) = U(\alpha^a(x^\mu)) = e^{i \sum_{a=1}^{3} T^a \alpha^a(x^\mu)}.$$

Notice that these transformations are like those of QED, but with the charge Q replaced by the three generators T^a, and the single rotation angle α replaced by the three angles α^a.

$SU(2)$ is a non-abelian group, because its elements do not in general commute. That is, generally $U_1 U_2 \neq U_2 U_1$. That is a result of the fact that the generators of $SU(2)$ also do not commute with each other. In fact the generators satisfy commutation relations which are called the "algebra" of the group:

$$\left[T^a, T^b\right] = i\varepsilon^{abc} T^c$$

We have seen that the fermion mass terms are invariant under local $SU(2)$ transformations, but it is non-trivial to construct a kinetic term that is invariant. As with QED, making the kinetic terms gauge invariant requires replacing the ordinary partial derivatives with respect to x^μ by "covariant derivatives." The covariant derivative must have this property:

$$D_\mu \psi \to D'_\mu \psi' = U\,(D_\mu \psi).$$

In that case,

$$\begin{aligned}
\overline{\psi} i\gamma^\mu D_\mu \psi \longrightarrow \overline{\psi'} i\gamma^\mu D'_\mu \psi &= \overline{\psi} U^\dagger i\gamma^\mu U\,(D_\mu \psi) \\
&= \overline{\psi} i\gamma^\mu (D_\mu \psi).
\end{aligned}$$

To find such a covariant derivative, we will use the analogy of the covariant derivative of QED to write

$$D_\mu = \partial_\mu - ig \sum_{a=1}^{3} T^a A^a_\mu.$$

Notice that this is the QED covariant derivative with $Q \to T^a$, $e \to g$, and $A_\mu \to A^a_\mu$. A notation that makes some equations look simpler is $\mathbb{A}_\mu \equiv \sum_{a=1}^{3} T^a A^a_\mu$, so that the covariant derivative in $SU(2)$ can be written $D_\mu = \partial_\mu - ig\mathbb{A}_\mu$. We have therefore

$$\begin{aligned}
\mathbb{A}_\mu &\equiv \sum_{a=1}^{3} T^a A^a_\mu \\
&= \begin{pmatrix} \frac{A^3_\mu}{2} & \frac{A^1_\mu - iA^2_\mu}{2} \\ \frac{A^1_\mu + iA^2_\mu}{2} & -\frac{A^3_\mu}{2} \end{pmatrix} \equiv \begin{pmatrix} \frac{W^0_\mu}{2} & \frac{W^+_\mu}{\sqrt{2}} \\ \frac{W^-_\mu}{\sqrt{2}} & -\frac{W^0_\mu}{2} \end{pmatrix},
\end{aligned}$$

where we have renamed the "gauge fields" $\frac{A^1_\mu \mp iA^2_\mu}{\sqrt{2}} \equiv W^\pm_\mu$ and $A^3_\mu \equiv W^0_\mu$.

Requiring that $D_\mu \psi \longrightarrow D'_\mu \psi' = U(D_\mu \psi)$, as we have seen is needed to write gauge invariant kinetic terms, implies that

$$\begin{aligned}
D'_\mu \psi' = (\partial_\mu - ig\mathbb{A}'_\mu)(U\psi) &= U(\partial_\mu - ig\mathbb{A}_\mu)\psi \\
\Longrightarrow (\partial_\mu U)\psi + U(\partial_\mu \psi) - ig\mathbb{A}'_\mu U\psi &= U(\partial_\mu \psi) - U\,ig\mathbb{A}_\mu \psi \\
\Longrightarrow \mathbb{A}'_\mu U\psi &= U\,\mathbb{A}_\mu \psi - \frac{i}{g}(\partial_\mu U)\psi \\
\Longrightarrow \mathbb{A}'_\mu U &= U\,\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu U) \\
\Longrightarrow \mathbb{A}'_\mu &= U\,\mathbb{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \qquad (1.5)
\end{aligned}$$

Notice that eq. (5) also gives the transformation of QA_μ in QED (which is analogous to $A_\mu = \sum_{a=1}^{3} T^a A^a_\mu$ in the non-abelian gauge theory)

$$
\begin{aligned}
QA_\mu \to QA'_\mu &= e^{iQ\alpha} QA_\mu e^{-iQ\alpha} - \frac{i}{e}(\partial_\mu e^{iQ\alpha})e^{-iQ\alpha} \\
&= QA - -\frac{i}{e}(iQ\partial_\mu \alpha) \\
\Rightarrow A_\mu \to A'_\mu &= A_\mu + \frac{1}{e}\partial_\mu \alpha
\end{aligned}
$$

which is the same as our previous result.

We now can explain how to write kinetic terms for the gauge fields A^a_μ analogous to the kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for QED.

From the requirement that $D'_\mu \psi' = D'_\mu U\psi = UD_\mu \psi$ (see eq. (1.5)), one has the operator equation $D'_\mu U = UD_\mu$, or $D'_\mu = UD_\mu U^{-1}$. Let us define

$$\sum_{a=1}^{3} T^a F^a_{\mu\nu} \equiv F_{\mu\nu} \equiv \frac{i}{g}\left[D_\mu, D_\nu\right].$$

Then $F_{\mu\nu}$ transforms as

$$F_{\mu\nu} \longrightarrow \frac{i}{g}\left[D'_\mu, D'_\nu\right] = \frac{i}{g}\left[UD_\mu U^{-1}, UD_\nu U^{-1}\right] = \frac{i}{g}U\left[D_\mu, D_\nu\right]U^{-1} = UF_{\mu\nu}U^{-1}.$$

This means that the following expression is invariant under gauge transformations:

$$tr\left(F_{\mu\nu}F^{\mu\nu}\right) = F^a_{\mu\nu}F^{b\,\mu\nu} tr\left(T^a T^b\right) = F^a_{\mu\nu}F^{b\,\mu\nu}\frac{1}{2}\delta^{ab} = \frac{1}{2}\sum_{a=1}^{3} F^a_{\mu\nu}F^{a\,\mu\nu},$$

where we used the property of Pauli matrices $tr\sigma^a\sigma^b = 2\delta^{ab} \Longrightarrow trT^aT^b = \frac{1}{2}\delta^{ab}$, and for simplicity we did not show summation signs except in the last step. A canonically normalized kinetic term for the gauge fields of $SU(2)$ is therefore $-\frac{1}{4}\sum_{a=1}^{3} F^a_{\mu\nu}F^{a\,\mu\nu}$.

We can also see what $F^a_{\mu\nu}$ looks like in terms of A^a_μ:

$$\begin{aligned}
T^a F^a_{\mu\nu} = F_{\mu\nu} &= \tfrac{i}{g}\left[D_\mu, D_\nu\right] = \tfrac{i}{g}\left[(\partial_\mu - igT^b A^b_\mu), (\partial_\nu - igT^c A^c_\nu)\right] \\
&= \tfrac{i}{g}\left[-igT^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) - g^2\left[T^b, T^c\right] A^b_\mu A^c_\nu\right] \\
&= T^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu\right) \\
\Longrightarrow \qquad F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu
\end{aligned}$$

1.5.3 How Non-Abelian Gauge Interactions Can Turn One Field (Particle) into Another

Consider the kinetic term of a fermion doublet

$$\overline{\psi} i\gamma^\mu D_\mu \psi = \left(\overline{\psi^1}, \overline{\psi^2}\right)\ i\gamma^\mu \left(\partial_\mu - ig\begin{bmatrix} \frac{W^0_\mu}{2} & \frac{W^+_\mu}{\sqrt{2}} \\ \frac{W^-_\mu}{\sqrt{2}} & -\frac{W^0_\mu}{2}\end{bmatrix}\right)\begin{pmatrix}\psi^1 \\ \psi^2\end{pmatrix}. \tag{1.6}$$

This contains

$$\begin{aligned}
&\overline{\psi^1} i\gamma^\mu\left(\partial_\mu - ig\left(\frac{1}{2}\right)W^0_\mu\right)\psi^1 \ + \ \overline{\psi^2} i\gamma^\mu\left(\partial_\mu - ig\left(-\frac{1}{2}\right)W^0_\mu\right)\psi^2 \\
&+ \ \overline{\psi^1}\gamma^\mu \frac{g}{\sqrt{2}} W^+_\mu \psi^2 \ + \ \overline{\psi^2}\gamma^\mu \frac{g}{\sqrt{2}} W^-_\mu \psi^1.
\end{aligned} \tag{1.7}$$

from the terms on the first line, one sees that W^0_μ acts like the gauge field of an abelian gauge theory (like QED) with gauge coupling constant g, and ψ^1 and ψ^2 having opposite charges $\pm\frac{1}{2}$. One can see from the terms on the second line the very important fact that W^+_μ and W^-_μ can turn $\psi^1 \to \psi^2$ and $\psi^2 \to \psi^1$ which can be shown diagrammatically as in the Figure 1.5.

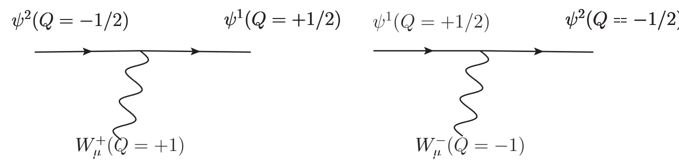

Figure 1.5: charge-current Weak interactions

The generator T^3 corresponds to the gauge field $A^3_\mu = W^0_\mu$. The eigenstates of $T^3(=\sigma^3/2)$ are just ψ^1 and ψ^2, with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. That is why these fields couple to W^0_μ with those charges in eq. (1.7). One can see from the second line of eq. (1.7) that W^+_μ and W^-_u must have T^3 charge of $+1$ and -1 respectively, and from the first line of eq. (1.7) that W^0_μ must have T^3 charge equal to 0. This is why these gauge fields are given those names. The $(W^+_\mu, W^0_\mu, W^-_\mu)$ form a *triplet* under $SU(2)$, and so the $SU(2)$ transformations and generators that act on them are 3×3 matrices. But we do not need to discuss that here.

1.6 The Strong Interactions (QCD)

The discussion in the last section can be generalized to larger groups. For example, consider fermions in triplets

$$\psi^a = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}$$

that transform by multiplication by 3×3 unitary matrices with $det = 1$

$$\psi^a \to \psi^{a'} = U^{a'}_a \psi^a.$$

this gives a $SU(3)$ gauge theory. Here U can be written as

$$U(\alpha^m) = e^{i\sum\limits_{m=1}^{8} \lambda^m \alpha^m}$$

where λ^m are 3×3, hermitian, traceless matrices, which are the generators of $SU(3)$ (in the triplet representation). There are eight such matrices and eight parameters α^m needed to parametrize the transformations. There are eight corresponding gauge fields A^m_μ. The strong interactions are described by such an $SU(3)$ gauge theory called Quantum Chromodynamics (QCD). The quark fields are triplets, e.g.

$$u^a = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} u^r \\ u^g \\ u^b \end{pmatrix}$$

The three components of a quark are usually called the three "colors," and the index is sometimes said to take the three values r (red), g (green), and b (blue) instead of 1,2,3. The eight gauge fields are called gluon fields. The $SU(3)$ of QCD is often called $SU(3)_c$ to remind us that the components are colors. We shall not discuss QCD much in the remainder of this thesis because our focus is more on the electroweak interactions. One can also consider larger groups than $SU(3)$. Later we shall discuss "grand unified theories" based on the the gauge group $SU(5)$. There are also grand unified theories based on other groups, such as $SO(10)$. In the last part of this thesis, grand unified theories with gauge groups $SU(6)$, $SU(7)$ and $E(6)$ are discussed.

1.7 The Electroweak Interactions

In the Standard Model of particle interactions (SM), the electroweak interactions are described by a theory very similar to the one we discussed in section 1.5 with two very significant differences: (1) In the SM, the left-handed fermions are in doublets of $SU(2)$, but the right-handed fermions are in singlets of $SU(2)$ (in other words, they do not transform at all, $\psi_R \to \psi_R$). (2) The gauge group of the electroweak interactions is actually $SU(2)\times U(1)$. Often the groups are called $SU(2)_L\times U(1)_Y$, where L reminds us that the $SU(2)$ acts only on left-handed fermions and Y reminds us that the generator of $U(1)_Y$ (called the weak hypercharge) is called Y. The generators of $SU(2)_L$ are called T^a, $a = 1, 2, 3$, as in section 1.5 (or sometimes T_L^a). The covariant derivative of a left-handed fermion doublet is given by

$$\left(\partial_\mu - ig \sum T^a\, A_\mu^a - \frac{ig'}{2} Y\, B_\mu\right) \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}.$$

For the right-handed fermions that are partners of these one has

$$\left(\partial_\mu - \frac{ig'}{2} Y\, B_\mu\right) \psi_R^1 + \left(\partial_\mu - \frac{ig'}{2} Y\, B_\mu\right) \psi_R^2.$$

In these expressions Y is thought of as an operator, so that it has the value of the weak hypercharge of the fermion it acts on. For historical reason, often people give the value of $Y/2$ for a field and call that the weak hypercharge.

Let us now consider the u and d (or "up" and "down") quarks. For each, both left-handed and right-handed components are triplets under $SU(3)_c$. (So in the terminology that will be explained shortly, quarks are "vector-like" under $SU(3)_c$ and the gluons couple to them with a γ^μ and no $\gamma^\mu\gamma^5$.) The left-handed parts of the u and d quarks form an $SU(2)_L$ doublet that has $Y/2 = +\frac{1}{6}$. This doublet is often called Q or Q_L:

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}.$$

One can see from this why these quarks were called "up" and "down." The right-handed parts of u and d are singlets under $SU(2)_L$, and have $Y/2$ equal to $+\frac{2}{3}$ and $-\frac{1}{3}$ respectively. We summarize the quantum numbers of every fermion in the Table 1.1.

	1st	2nd	3rd	T^3	$Y/2$
	$Q_{L1} = \begin{pmatrix} u_L \\ Ld_L \end{pmatrix}$	$Q_{L2} = \begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$Q_{L3} = \begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{6}$
quarks	u_R	c_R	t_R	0	$\frac{2}{3}$
	d_R	s_R	b_R	0	$-\frac{1}{3}$
leptons	$\ell_{L1} = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$	$\ell_{L2} = \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix}$	$\ell_{L3} = \begin{pmatrix} \nu_\tau \\ \tau_L \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$
	e_R	μ_R	τ_R	0	-1

Table 1.1: The quantum numbers of fermions under $SU(2)_L \times U(1)_Y$

What happens in the SM is that through the Higgs mechanism the group $SU(2)_L \times U(1)_Y$ is "spontaneously broken" to a subgroup $U(1)_Q$ whose generator is the electric charge Q. In the process $W^\pm_\mu \equiv \frac{A^1_\mu \mp i A^2_\mu}{\sqrt{2}}$ and a linear combination $Z^0_\mu \equiv \frac{-g' B_\mu + g A^3_\mu}{\sqrt{g^2+g'^2}}$ get large masses ($M_W \simeq 80\,\mathrm{GeV}, \quad M_Z \simeq 91\,\mathrm{GeV}$). The remaining linear combination $A_\mu = \frac{g B_\mu + g' A^3_\mu}{\sqrt{g^2+g'^2}}$ remains massless and is the gauge field of electromagnetism.

The generator of the unbroken $U(1)_Q$ is the electric charge and turns out to be given by

$$Q = T^3 + \frac{Y}{2},$$

so

$$\begin{aligned} Q\,(u_L) &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \\ Q\,(d_L) &= -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}, \\ Q\,(u_R) &= 0 + \frac{2}{3} = \frac{2}{3}, \\ Q\,(d_R) &= 0 - \frac{1}{3} = -\frac{1}{3}. \end{aligned}$$

Notice that the electric charges of the quarks come out to be the same for the left and right-handed components. (So, in the terminology that will be explained in a moment, the fermions are vector-like under $U(1)_Q$.) How the Higgs mechanism makes $W_\mu^\pm$, Z_μ^0 massive, how A_μ^3 and B_μ mix to give A_μ and Z_μ^0, and why $Q = T^3 + \frac{Y}{2}$ is explained in ref [1]. These things are not directly important for the topics dealt with in this thesis, which mostly are about the interactions of the $W_\mu^\pm$ to the fermions, through they are very important features of the Standard Model.

Notice that the $W_\mu^\pm$ are essentially the same fields discussed in the SU(2) theory in section 1.5, except they only couple to the left-handed doublets, not right-handed singlets. So eq. (1.7) becomes

$$\begin{aligned} & \overline{u_L}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ d_L \;+\; \overline{d_L}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^- u_L \\ = \; & \overline{u}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ \left(\frac{1-\gamma^5}{2}\right) d \;+\; \overline{d}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^- \left(\frac{1-\gamma^5}{2}\right) u \\ = \; & \overline{u}\frac{g}{2\sqrt{2}} W_\mu^+ \left(\gamma^\mu - \gamma^\mu\gamma^5\right) d \;+\; \overline{d}\frac{g}{2\sqrt{2}} W_\mu^+ \left(\gamma^\mu - \gamma^\mu\gamma^5\right) u. \end{aligned} \tag{1.8}$$

This gives the Feynman diagrams shown in the Figure 1.6.

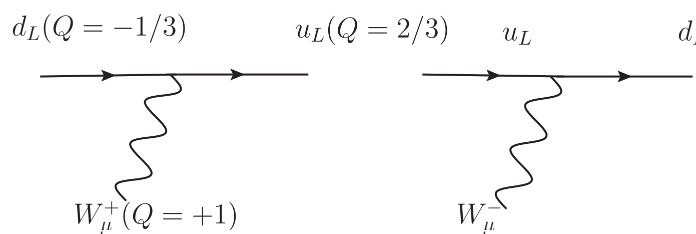

Figure 1.6: charged-current Weak interactions in the quark sector

Notice that the interactions of $W_\mu^\pm$ with the fermions contain $(\gamma^\mu - \gamma^\mu\gamma^5)$. Historically, this was called a "$V - A$" interactions, because γ^μ is a 4-vector (of 4×4 matrices) while $\gamma^\mu\gamma^5$ is an axial 4-vector because under parity $\gamma^5 \longrightarrow -\gamma^5$. (The reason for this is that under a parity transformation $x^0 \longrightarrow +x^0$, and $x^i \longrightarrow -x^i$, where

i is a spatial index. So $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$ flips sign.) The $V-A$ structure arises because only left-handed fermions transform under $SU(2)_L$. If only right-handed fermions did there would be $V+A$ interactions. Suppose left-handed and right-handed fermions transformed the same way as in section 5, then one would just get the sum of the two:

$$\overline{\psi}^1\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ \psi^2 + \overline{\psi}^2\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ \psi^1,$$

as in section 1.5, which is a purely vector or "V" interaction. This is why, for historical reasons, fermions are called "vector-like" if the left-handed and right-handed components transform the same way under the gauge group. So under $SU(3)_c$ and $U(1)_Q$ the fermions of the SM are vector-like, as pointed out previously. But, under $SU(2)_L$, left-handed and right-handed fermions of a given type transform differently, so such fermions are called "chiral," meaning "handed."

In the SM, fermions can be divided into two categories: the quarks, which we just discussed, and the leptons. For the first family, the leptons are

$$\ell_L \equiv \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix} \quad \text{with } \frac{Y}{2} = -\frac{1}{2}, \qquad e_R^- \quad \text{with } \frac{Y}{2} = -1$$

Notice that we use a script ℓ to denote the left-handed lepton doublet. In the SM there is no ν_{eR}, but if there were it would have $Y=0$ in order to be electrically neutral. As we will see, such a ν_R field, or equivalently a ν_L^c field, is present in many grand unified models. So the electric charges of the leptons are given by $Q = T^3 + Y/2$ are

$$\begin{aligned} Q\left(\nu_{eL}\right) &= \frac{1}{2} - \frac{1}{2} = 0, \\ Q\left(e_L^-\right) &= -\frac{1}{2} - \frac{1}{2} = -1, \\ Q\left(e_R^-\right) &= 0 - 1 = -1, \\ Q\left(\nu_{eR}\right) &= 0 - 0 = 0. \quad \text{(if it exists)} \end{aligned}$$

The lepton couplings give rise to the Feynman diagrams shown in the Figure 1.7.

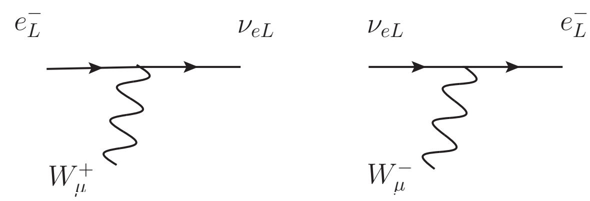

Figure 1.7: charged-current Weak interactions in the leptonic sector

The couplings of the quarks and leptons to the $W_\mu^\pm$ give rise to the phenomenon of beta decay, such as $n \to p + e^- + \overline{\nu}_e$, as one sees from the Figure 1.8.

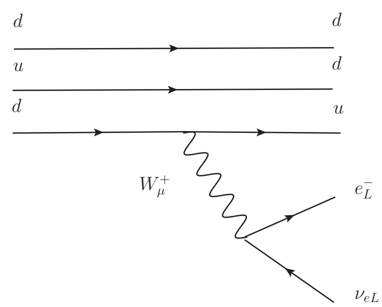

Figure 1.8: beta decay

1.8 Fermion Masses in The Standard Model

1.8.1 Electron Mass

In QED, one can write an explicit mass for the electron

$$m\,\overline{e_R^-}e_L^- + m\,\overline{e_L^-}e_R^- = m\left(e_L^+\right)^T Ce_L^- + m\left(e_R^+\right)^T Ce_R^-.$$

We have written it in two ways: (1) with $e_L^-$and e_R^-, and (2) with e_L^- and e_L^+ (the charge conjugate of e_R^-). Note that, as mentioned before, e_R^- and e_L^+ are really the same degrees of freedom. As we will see, the second way of writing the mass terms, with particle and antiparticle of the same chirality has some advantages.

In the Standard Model, we cannot write such a mass term for the electron as we do in QED because it would not be invariant under $SU(2)_L$ or $U(1)_Y$ gauge transformations. The field e_L^- has $T_L^3 = -\frac{1}{2}$, $\frac{Y}{2} = -\frac{1}{2}$, whereas e_R^- has $T_L^3 = 0$, $\frac{Y}{2} = -1$. Consequently, the term $\left(\overline{e_L^-} e_R^-\right)$ would have $T_L^3 = -\frac{1}{2}$, $\frac{Y}{2} = \frac{1}{2}$.

In the Standard Model, therefore, the mass of the electron must come from a Yukawa coupling of the electron to some scalar field (namely the Higgs field). And for that coupling to be invariant under $SU(2)_L \times U(1)_Y$, the Higgs field must transform in the right way under those groups. Therefore, let us introduce a complex $SU(2)_L$ doublet of scalar fields $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ which has $\frac{Y}{2} = +\frac{1}{2}$. Thus ϕ^+ has $T_L^3 = +\frac{1}{2}$ and $Q = T_L^3 + \frac{Y}{2} = \frac{1}{2} + \frac{1}{2} = 1$, and ϕ^0 has $T_L^3 = -\frac{1}{2}$ and $Q = T_L^3 + \frac{Y}{2} = -\frac{1}{2} + \frac{1}{2} = 0$. Then we can write the following $SU(2)_L \times U(1)_Y$ invariant (and Lorentz invariant) Yukawa term

$$\begin{aligned} & Y_e \overline{e_R^-} \phi^\dagger \ell_L \quad + \quad Y_e \overline{\ell_L} \phi e_R^- \\ = \quad & Y_e \left(e_L^+\right)^T C \phi^\dagger \ell_L \quad + \quad Y_e \left(\ell_R\right)^T C \phi e_R^-. \end{aligned}$$

Or, writing out the doublets ℓ_L and ϕ, one obtains

$$\begin{aligned} & Y_e \overline{e_R^-} \left(\phi^- \nu_L + \phi^{0*} e_L^-\right) \quad + \quad Y_e \left(\overline{\nu_L} \phi^+ + \overline{e_L^-} \phi^0\right) e_R^- \\ = \quad & Y_e \left(e_L^+\right)^T C \left(\phi^- \nu_L + \phi^{0*} e_L^-\right) \quad + \quad Y_e \left(\left(\nu_R\right)^T \phi^+ + \left(e_R^+\right)^T \phi^0\right) C e_R^-. \end{aligned}$$

We have used $(\phi^+)^* \equiv \phi^-$, $(\phi^-)^* \equiv \phi^+$. Suppose that in the ground state (or "vacuum state") the neutral component of ϕ has a non-zero expectation value (called a "vacuum expectation value" or "VEV"), and call this $v/\sqrt{2}$. Thus $\langle 0|\phi^0|0\rangle = v/\sqrt{2}$. By a gauge transformation, v can be made real. Then the above Yukawa terms give

$$\left(Y_e \frac{v}{\sqrt{2}}\right) \overline{e_R^-} e_L^- + \left(Y_e \frac{v}{\sqrt{2}}\right) \overline{e_L^-} e_R^- = \left(Y_e \frac{v}{\sqrt{2}}\right) \left(e_L^+\right)^T C e_L^- + \left(Y_e \frac{v}{\sqrt{2}}\right) \left(e_R^+\right)^T C e_R^-.$$

One sees that this is effectively a mass for the electron, and that its mass is given by $m_e = Y_e \langle 0|\phi^0|0\rangle = Y_e \frac{v}{\sqrt{2}}$.

1.8.2 Quark Mass

The d quark obtains a mass in a very similar way to the electron. Consider the Yukawa term

$$Y_d \overline{d_R} \phi^\dagger Q_L + Y_d \overline{Q_L} \phi d_R = Y_d \left(d_L^c\right)^T C \phi^\dagger Q_L + Y_d \left(Q_R^c\right)^T C \phi d_R.$$

Writing out the doublets Q_L and ϕ, one obtains

$$\begin{aligned} & Y_d \overline{d_R} \left(\phi^- u_L + \phi^{0*} d_L\right) &+& \quad Y_d \left(\overline{u_L} \phi^+ + \overline{d_L} \phi^0\right) d_R \\ =& Y_d \left(d_L^c\right)^T C \left(\phi^- u_L + \phi^{0*} d_L\right) &+& \quad Y_d \left(\left(u_R^c\right)^T \phi^+ + \left(d_R^c\right)^T \phi^0\right) C d_R. \end{aligned}$$

because of the VEV of ϕ^{*0}, the d quark obtains a mass $m_d = Y_d \frac{v}{\sqrt{2}}$.

To write a Yukawa term to give mass to u quark we need an $SU(2)$ doublet of scalar fields that has $\frac{Y}{2} = -\frac{1}{2}$, rather than $+\frac{1}{2}$. One can make such a doublet by "charge conjugating" ϕ. This done as follows

$$\phi^c \equiv i\sigma^2 \phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}.$$

This has opposite weak hypercharge Y to ϕ, because of the complex conjugation. But it transforms under $SU(2)_L$ in the same ways as ϕ:

$$\begin{aligned} \phi &\longrightarrow & \phi' = U\phi = e^{i\sum_{a=1}^{3} T^a \alpha^a} \phi = e^{i\sum_{a=1}^{3} \frac{\sigma^a}{2} \alpha^a} \phi \\ &\Longrightarrow & \phi^{c'} = i\sigma^2 \phi'^* = i\sigma^2 U^* \phi^* = i\sigma^2 e^{-i\sum_{a=1}^{3} \frac{\sigma^{a*}}{2} \alpha^a} \phi^* \\ &= & e^{i\sum_{a=1}^{3} \frac{\sigma^a}{2} \alpha^a} i\sigma^2 \phi^* \\ &= & U\phi^c, \end{aligned}$$

where we have used the relation $\sigma^2 \sigma^{a*} = -\sigma^a \sigma^2,\ a = 1, 2, 3$.

One can write an $SU(2)_L \times U(1)_Y$-invariant Yukawa coupling of ϕ^c to the u quarks:

$$Y_u \overline{u_R} \phi^{c\dagger} Q_L + Y_u \overline{Q_L} \phi^c u_R = Y_u \left(u_L^c\right)^T C \phi^{c\dagger} Q_L + Y_u \left(Q_R^c\right)^T C \phi^c u_R.$$

Or, writing out the doublets Q_L and ϕ^c:

$$\begin{aligned} & Y_u \overline{u_R} \left(\phi^0 u_L - \phi^+ d_L\right) \quad + \quad Y_u \left(\overline{u_L} \phi^{0*} - \overline{d_L} \phi^-\right) u_R \\ = \quad & Y_u \left(u_L^c\right)^T C \left(\phi^0 u_L - \phi^+ d_L\right) \quad + \quad Y_d \left(\left(u_R^c\right)^T \phi^{0*} - \left(d_R^c\right)^T \phi^-\right) C u_R. \end{aligned}$$

This gives $m_u = Y_u \frac{v}{\sqrt{2}}$.

One sees that the fermion masses come out proportional to the Yuakwa coupling constants denoted by Y_f, where f is the fermion type: $m_f = Y_f v/\sqrt{2}$. Or to put it the other way, the strength of the coupling between the Higgs boson ϕ and a type of fermion is proportional to the mass of that type of fermion.

1.8.3 Neutrino Mass

The kinds of fermion masses we have discussed so far, involve a term with both ψ_L and ψ_R (or equivalently ψ_L^c). Since ψ_L and ψ_R are not the same degrees of freedom, and they are each 2-component spinors, they make up together a 4-component "Dirac spinor." Such a mass term is therefore called a "Dirac mass term," and the resulting mass is called a "Dirac mass." As we will see, there is another type of mass term called a "Majorana mass term."

In the Standard Model, a neutrino cannot have a Dirac mass, because there is only a ν_L, but no ν_R (or equivalently ν_L^c). If there were such a field, as in many grand unified theories, one could obtain a Dirac mass term for a neutrino in the same way we did for the other types of fermion, through a Yukawa coupling to the Higgs field:

$$Y_\nu \overline{\nu_R} \phi^{c\dagger} \ell_L + Y_\nu \overline{\ell_L} \phi^c \nu_R = Y_\nu \left(\nu_L^c\right)^T C \phi^{c\dagger} \ell_L \quad + \quad h.c.$$

Or writing out the doublets

$$Y_\nu \overline{\nu_R} \left(\phi^0 \nu_L - \phi^+ e_L^-\right) + Y_\nu \left(\overline{\nu_L} \phi^{0*} - \overline{e_L^-} \phi^-\right) \nu_R = Y_\nu \left(\nu_L^c\right)^T C \left(\phi^0 \nu_L - \phi^+ e_L^-\right) + h.c.$$

Because $\langle 0|\phi^0|0\rangle = v/\sqrt{2}$, the neutrino gets a Dirac mass from these terms, which we will call $m_{D\nu}$, where the subscript D stands for "Dirac":

$$m_{D\nu}\, \overline{\nu_R}\, \nu_L + h.c. \quad = \quad m_{D\nu}\, (\nu_L^c)^T C \nu_L + h.c.,$$

where $m_{D\nu} = Y_\nu \frac{v}{\sqrt{2}}$.

For a long time, it looked like neutrinos were massless. Analysis of the kinematics of tritium beta decay gave an upper limit to the mass of the electron neutrino that was many orders of magnitude smaller than the electron mass. There were also strong limits on the muon neutrino's mass. The simplest way to explain the apparent masslessness of the neutrinos was to say that right-handed neutrinos (or equivalently left-handed anti-neutrinos) do not exist in nature. That was consistent with the fact that experiments showed that neutrinos were left-handed and anti-neutrinos were right-handed. That fact prevents a Dirac mass for neutrinos and seems to force neutrinos to be exactly massless.

However, another kind of mass term is possible for some fermions, called a "Majorana mass term." A Majorana mass term involves a coupling of a fermion ψ_L to itself. In other words, one has $m(\psi_L)^T C\psi_L$ rather than $m(\psi_L^c)^T C\psi_L$. Such a term can be written even if only ψ_L exists but no ψ_L^c (or equivalently ψ_R) exists. That means that a fermion with only a "Majorana mass" has only 2 components, not 4. It is not, therefore a full Dirac spinor.

Looking at the Majorana mass term $m(\psi_L)^T C\psi_L$, one sees that it is forbidden by electromagnetic gauge invariance, unless the electric charge of ψ_L is zero. Because neutrinos have zero electric charge, it seems like it may be possible for them to have Majorana masses. In fact, in most theories of neutrino mass, they do.

One cannot directly write down an explicit Majorana mass term $m\nu_L^T C\nu_L$ for the neutrino, because it would not be invariant under $U(1)_Y$. The left-handed neutrino has $Y/2 = -1/2$, so such a term would have $Y/2 = -1$, and not be invariant.

Similarly, one cannot just write down a Yukawa term to the neutral Higgs field ϕ^0, such as $Y\nu_L^T C\nu_L\phi^0$, because that also would not be invariant under $SU(2)_L \times U(1)_Y$. Each of the fields in that term have $Y/2 = \pm 1/2$, and all of them are in $SU(2)_L$ doublets, and one cannot multiply three doublets to obtain an invariant term.

There are two ways, however, that neutrinos could get Majorana masses. One way is that there might exist a new kind of Higgs field T that is a triplet under $SU(2)_L$

and has $Y/2 = +1$. That would be allowed to couple to neutrinos by a term of the form $Y\nu_L^T C\nu_L T$. In many grand unified theories, such triplet Higgs fields do exist, and they can give very small neutrino masses by means of a mechanism called the Type II see-saw mechanism.

Another way neutrinos can get a mass is by a term in $\mathcal{L}$ that is of higher order in the fields. The simplest term that can be written of this type that is $SU(2)_L \times U(1)_Y$ and Lorentz invariant is

$$\frac{1}{M_R}\left(\phi^{c\dagger}\ell_L\right)^T C\left(\phi^{c\dagger}\ell_L\right).$$

This is called the "Weinberg operator." By dimensional analysis one can show that the coefficient of such a term has dimension of inverse mass. The mass in that coefficient is often called M_R for reasons that we will explain later. Writing out the doublets ℓ_L and ϕ in this term, one obtains

$$\frac{1}{M_R}\left(\phi^0\nu_L - \phi^+ e_L^-\right)^T C\left(\phi^0\nu_L - \phi^+ e_L^-\right)$$

When ϕ^0 gets a VEV, it gives

$$\frac{\langle\phi^0\rangle^2}{M_R}\left(\nu_L^T C\nu_L\right) \implies m_\nu = \frac{v^2}{2M_R}$$

The trouble with terms that are so high order in fields is that they cause the theory to be "non-renomalizable." In other words quantum effects would give infinite corrections to it. The reason is that if one calculates quantum effects arising from such a term, the answers would have powers of the coefficient, which means powers of an inverse mass. That must be compensated by some mass (or energy) in the numerator. Generally, what appears in the numerator is the energy of virtual particles. But there is no upper limit in quantum field theory to how large the energy of a virtual particle can be. In quantum theory, one has to sum over all possibilities. That means including virtual particles of arbitrarily high energy. This gives infinite (and "non-renormalizable") answers.

However, such a term with coefficient that is an inverse power of mass can exist in an "effective theory" that is only supposed to describe physical processes up to some cutoff energy. In grand unified theories, for example, the Standard Model is an effective theory that is a good approximation only when the energies are less than the "unification scale," which is typically about 10^{15} GeV. Above that, one must use the full grand unified theory. So, in grand unified theories, one can get the Weinberg operator in the "effective theory" that works at energies below the unification scale. That does not cause infinities, because one should not allow virtual particles whose energy is above the unification scale. (Or, if one does, there are other processes that cancel their contributions.)

In fact, in most grand unified theories, the Weinberg operator does arise in the effective theory. This gives neutrinos mass. We will now explain one way this happens.

1.8.4 Type I See-Saw Mechanism for Neutrino Mass

Many grand unified theories (GUTs) predict that there should exist a right-handed neutrinos ν_R or equivalently left-handed anti-neutrinos ν_L^c that do not transform under $SU(2)_L \times U(1)_Y$, i.e. they are $SU(2)_L$ singlets with $Y = 0$.

Let us consider a case with one ν_L and one ν_L^c. Nothing prevents an explicit Majorana mass term for the ν_L^c, since it is neutral under all gauge symmetries.

$$M_R \nu_L^{cT} C \nu_L^c$$

The reason the mass is called M_R is that it gives the *right-handed* neutrino (left-handed anti-neutrinos) mass. This mass has no reason to be small. In fact, it should "naturally" be of order the highest energy scale that appears in the theory ($M_{Pl} \sim 10^{19}$ GeV or $M_{GUT} \sim 10^{15}$ GeV), because no symmetry forbids such a term or suppresses it.

The ν_L^c can also have a normal Yukawa coupling to ν_L:

$$Y_\nu \left(\nu_L^c\right)^T C\phi^{c\dagger}\ell_L + Y_\nu \left(\ell_L\right)^T \phi^c C\nu_L^c \Longrightarrow Y_\nu \left(\nu_L^c\right)^T C\phi^0\nu_L + Y_\nu \left(\nu_L\right)^T \phi^{0*}C\nu_L^c.$$

Since ϕ^0 has a VEV, this term gives a Dirac mass term $m_{D\nu}(\nu_L^c)^T C\nu_L + h.c.$, with $m_{D\nu} = Y_\nu v/\sqrt{2}$.

There are Yukawa and mass terms involving both ν_L and ν_L^c, which we may write in matrix form this way:

$$\begin{pmatrix} \nu_L & \nu_L^c \end{pmatrix}^T C \begin{pmatrix} 0 & Y_\nu \phi^0 \\ Y_\nu \phi^0 & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_L^c \end{pmatrix}. \tag{1.9}$$

Substituting for ϕ^0 its VEV gives a mass matrix for ν_L and ν_L^c:

$$\begin{pmatrix} \nu_L & \nu_L^c \end{pmatrix}^T C \begin{pmatrix} 0 & M_{D\nu} \\ M_{D\nu} & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_L^c \end{pmatrix},$$

where $M_{D\nu} = Y_\nu \frac{v}{\sqrt{2}} \cong \frac{1}{2} Y_\nu \times 174$ GeV, while $M_R \sim 10^{15}$ GeV. The determinant of this matrix is $-(m_{D\nu})^2$, while the largest eigenvalue is almost exactly equal to M_R (because $M_R \gg m_{D\nu}$). Therefore, if one diagonalizes this 2×2 matrix, one finds that the smaller eigenvalue must be very close to $-(m_{D\nu})^2/M_R$. Let us call the two eigenmasses

$$\begin{aligned} M_{heavy} &\cong M_R \\ M_{light} &\cong -\frac{M_D^2}{M_R}. \end{aligned}$$

One neutrino is superheavy (M_R), but the other is extremely light. The light neutrino is the one we see experimentally. It is almost exactly equal to ν_L, because only a tiny rotation angle is required to diagonalize this matrix. This way of the neutrino becoming massive is called the see-saw mechanism because the larger M_{heavy} is the smaller M_{light} is. If we assume that $Y_\nu \sim 1$, $M_R \sim 10^{15}$ GeV, one has $M_{light} \sim 10^{-11}$ GeV $\sim 10^{-2}$ eV. This is in the ballpark of the neutrino masses suggested by neutrino oscillation experiments. (Neutrino oscillation experiments only give *differences* of the squares of neutrino masses.)

If we return to eq. (9), we see that by diagonalizing, one gets an effective operators for ν_L that is of the form

$$\nu_L^T C \nu_L \left(\frac{(Y_\nu \phi^0)^2}{M_R} \right) = \left(\frac{Y_\nu^2}{M_R} \right) \nu_L^T C \nu_L (\phi^0)^2. \tag{1.10}$$

We recognize this as the Weinberg operator.

1.9 Quark Mass Matrices and CKM Mixing

We have shown how to write the mass terms for the up and down quarks in the SM. In reality, however, there are three families of quarks which are shown in the Table 1.2.

Family	Quarks	Leptons
I	$Q_{L1} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, u^c_L, d^c_L	$\ell_{L1} = \begin{pmatrix} \nu_{eL} \\ e^-_L \end{pmatrix}$, e^+_L
II	$Q_{L2} = \begin{pmatrix} c_L \\ s_L \end{pmatrix}$, c^c_L, s^c_L	$\ell_{L2} = \begin{pmatrix} \nu_{\mu L} \\ \mu^-_L \end{pmatrix}$, μ^+_L
III	$Q_{L3} = \begin{pmatrix} t_L \\ b_L \end{pmatrix}$, t^c_L, b^c_L	$\ell_{L3} = \begin{pmatrix} \nu_{\tau L} \\ \tau^-_L \end{pmatrix}$, e^+_L

Table 1.2: quarks and leptons in the standard model

The Yukawa interactions therefore have 3×3 complex Yukawa coupling matrices Y^u, Y^d

$$\mathcal{L}_{Yuk} = -\sum_{m,n=1}^{3}\left[Y^u_{mn} Q^T_{Lm}\phi^{c*} C\, u^c_{Ln} + Y^d_{mn} Q^T_{Lm}\phi^* C\, d^c_{Ln}\right] + h.c.$$

When ϕ^0 gets a VEV, $\langle\phi^0\rangle = \frac{v}{\sqrt{2}}$, there arise mass matrices M_u, M_d

$$-\sum_{n,m=1}^{3}[u^T_{Lm}\,(M_u)_{mn}\, C\, u^c_{Ln} + d^T_{Lm}\,(M_d)_{mn}\, d^c_{Ln} + h.c.].$$

For simplicity, let us now suppress family indices, and just write this as

$$u^T_L M_u C u^c_L + d^T_L M_d d^c_L + h.c.$$

One can make the matrices M_u, M_d real and diagonal by bi-unitary transformations, as explained in section 4.

$$\begin{aligned} V^\dagger_{uL} M_u V_{uR} = &\ \widetilde{M}_u &= \text{real and diagonal} \\ V^\dagger_{dL} M_d V_{dR} = &\ \widetilde{M}_d &= \text{real and diagonal} \end{aligned}$$

Define a new basis (the mass basis or physical basis) of left-handed and right-handed quark fields (denoted by tilde) by

$$\begin{aligned} u_L = V_{uL}^* \widetilde{u}_L &\leftrightarrow \tilde{u}_L = V_{uL}^T u_L \\ u_L^c = V_{uR} \widetilde{u}_L^c &\leftrightarrow \widetilde{u}_L^c = V_{uR}^\dagger u_L^c \\ d_L = V_{dL}^* \widetilde{d}_L &\leftrightarrow \widetilde{d}_L = V_{dL}^T d_L \\ d_L^c = V_{dR} \widetilde{d}_L^c &\leftrightarrow \widetilde{d}_L^c = V_{dR}^\dagger d_L^c \end{aligned}$$

Then the mass terms become

$$\begin{aligned} & \tilde{u}_L^T V_{uL}^\dagger M_u C\, V_{uR}\, \tilde{u}_L^c \;+\; \widetilde{d}_L^T V_{dL}^\dagger M_d C\, V_{dR}\, d_L^c \;+\; h.c. \\ = \;& \tilde{u}_L^T \tilde{M}_u \tilde{u}_L^c \;+\; \widetilde{d}_L^T \tilde{M}_d d_L^c \;+\; h.c. \end{aligned}$$

From eq. (1.7), the interactions of the $W_\mu^\pm$ bosons with the quarks — the so-called "charged-current Weak interactions" — can be written as

$$\begin{aligned} \overline{u_L}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ d_L + h.c &= \overline{\tilde{u}_L}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ (V_{uL}^\dagger V_{dL}) \widetilde{d}_L + h.c. \\ &= \overline{\tilde{u}_L}\gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ V_{CKM} \widetilde{d}_L + h.c. \end{aligned}$$

The $\tilde{u}_{Lm} = (\tilde{u}_L, \tilde{c}_L, \tilde{t}_L)$ are the mass eigenstates. that is, the physical u, c, and t quarks. And the $\tilde{d}_{Lm} = (\tilde{d}_L, \tilde{s}_L, \tilde{b}_L)$ are the mass eigenstates, that is, the physical d, s, and b quarks. Now let us drop the tildes to make the equations look cleaner. The charged-current Weak interactions in the physical basis then look like the following:

$$\mathcal{L}_{CC} = \overline{\begin{pmatrix} u_L & c_L & t_L \end{pmatrix}} \gamma^\mu \frac{g}{\sqrt{2}} W_\mu^+ \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + h.c.$$

The matrix appearing in this equation is called the CKM matrix, after Cabibbo, Kobayashi, and Maskawa. The CKM matrix describes how the $W^\pm$ bosons change the "flavor" of the quarks.

The reason that the CKM matrix appears in the interactions of the W bosons is that there is a "mismatch" between the physical up-type quarks and the physical down-type quarks. And this mismatch happens because the mass matrices M_u and M_d are independent of each other and not equal to each other, and therefore different unitary transformations of the up-type quarks and down-type quarks are needed to diagonalize them. The "weak eigenstates" match, but the mass eigenstates do not.

The CKM matrix is a 3×3 unitary matrix, and so has 9 parameters. Three of these are rotation angles, called the "CKM angles." The remaining 6 parameters are complex phase angles. But 5 of these complex phases can be "absorbed" or "rotated away" by redefining the six left-handed quark fields (u, c, t, d, s, b). (One cannot rotate 6 phases away, because rotating all the left-handed quarks by the same phase does nothing to V_{CKM}.) So there is just one physical phase angle in the CKM matrix, which is usually denoted δ or δ_{KM}, and is called the Kobayashi-Maskawa phase. This Kobayashi-Maskawa phase can account for all CP-violating effects ever seen in the laboratory. Summarizing, there are three rotation angles and one physical phase in V_{CKM}.

The CKM matrix has been parametrized in various ways. A popular way is the Wolfenstein parametrization, which expresses the elements of V_{CKM} in terms of powers of a small parameter λ and three other parameters:

$$\begin{aligned}
V_{CKM} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\
&= \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \\
&\sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix},
\end{aligned}$$

where $\lambda = 0.226 \pm 0.001$, $A = 0.814 \pm 0.02$, $\rho = 0.135 \pm 0.02$, and $\eta = 0.349 \pm 0.16$.

From this parametrization, we can observe an interesting pattern, $|V_{ub}| \sim |V_{us}||V_{cb}|$. The CKM matrix could also be parametrized as

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The component of CKM matrix, for example, V_{ud} acts as vertex coefficient for setting the probability that up quark will project into down quark via interacting with W gauge boson in the weak interaction or vice versa.

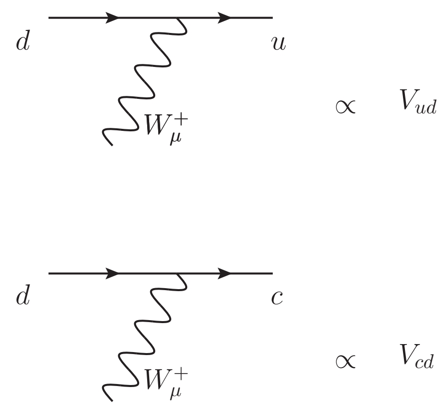

Figure 1.9: charged-current Weak interactions with three families

1.10 Lepton Mixing and The PMNS Matrix

As we see from the Table 1.2, there are also three families of leptons. The Yukawa terms can be written as

$$-\sum_{m,n=1}^{3} \left[Y^{\ell}_{mn} \ell^T_{mL} \phi^* C\, e^c_{nL} + \frac{y^{\nu}_{mn}}{M_R} \left(\ell^T_{mL} \phi^{c*} \right) C \, \left(\phi^{c\dagger} \ell_{mL} \right) \right] + h.c..$$

Note that for the neutrinos we have written the Weinberg operator. If we suppress the family indices this can be written

$$\ell^T_L Y^{\ell} \phi^* C e^c_L + (\ell^T_L \phi^{c*}) \frac{y^{\nu}}{M_R} C(\phi^{c\dagger} \ell_L) \quad + \quad h.c.$$

Writing out the doublets, we find that this contains

$$e_L^T Y^\ell C e_L^c \, \phi^{0*} + \nu_L^T \frac{y^\nu}{M_R} C \nu_L \, (\phi^0)^2 \quad + \quad h.c.$$

If the Weinberg operator comes from the Type I see-saw mechanism, as we explained in section 1.8, then its coefficient matrix, which we have written y^ν / M_R here, is equal to $Y_\nu^T Y_\nu / M_R$, as one can see by comparing to eq. (1.10).

Because $\langle \phi^0 \rangle = \frac{v}{\sqrt{2}}$, we will get mass terms of the following form for the leptons

$$e_L^T M_\ell C \, e_L + \nu_L^T M_\nu C \, \nu_L + h.c.$$

The matrices M_ℓ can be made real and diagonal by a bi-unitary transformation. The matrix M_ν is different, because it is complex and symmetric. It can also be made real and diagonal, but by multiplying by the same unitary matrix (transposed) on the right and left:

$$\begin{aligned} V_{\ell L}^\dagger M_\ell V_{\ell R} = & \quad \widetilde{M_\ell} \quad \text{Note that this is biunitary transformation} \\ V_{\nu L}^T M_\nu V_{\nu L} = & \quad \widetilde{M_\nu} \quad \text{Note this has the same unitary matrix on both sides} \end{aligned}$$

Once again we can redefine the fields

$$\begin{aligned} e_L = V_{\ell L}^* \widetilde{e}_L \quad &\leftrightarrow \quad \tilde{e}_L = V_{\ell L}^T e_L \\ e_L^c = V_{\ell R} \widetilde{e}_L^c \quad &\leftrightarrow \quad \widetilde{e}_L^c = V_{uR}^\dagger e_L^c \\ \nu_L = V_{\nu L} \widetilde{\nu}_L \quad &\leftrightarrow \quad \widetilde{\nu}_L = V_{\nu L}^\dagger \nu_L \end{aligned}$$

Therefore, the charged-current Weak interactions of the leptons can be written

$$\begin{aligned} \overline{e}_L \gamma^\mu \frac{g}{\sqrt{2}} W_\mu^- \nu_L + h.c \quad &= \quad \overline{\widetilde{e}}_L \gamma^\mu \frac{g}{\sqrt{2}} W_\mu^- (V_{\ell L}^T V_{\nu L}) \widetilde{\nu}_L + h.c. \\ &= \quad \overline{\widetilde{e}}_L \gamma^\mu \frac{g}{\sqrt{2}} W_\mu^- U_{PMNS} \widetilde{\nu}_L + h.c. \end{aligned}$$

The matrix $V_{\ell L}^T V_{\nu L} \equiv U_{PMNS}$ is sometimes also called U_{MNS}. The MNS stands for Maki, Nakagawa, and Sakata. The P stands for Pontecorvo. These were the people who first discussed mixing of leptons.

Now let us drop the tilde to simplify the expressions. Then we can write the charged-current Weak interactions of the leptons as

$$\mathcal{L}_{CC,lepton} = \overline{(e_L, \mu_L, \tau_L)}\, \gamma^\mu \frac{g}{\sqrt{2}} W_\mu^- \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} + h.c.$$

The PMNS matrix is a 3×3 unitary matrix, and therefore it has 3 rotation angles and 6 complex phases, like the CKM matrix. However, we cannot rotate away as many phases as we can for the CKM matrix. The reason is that the neutrino mass matrix M_ν is a Majorana matrix that couples the 3 left-handed neutrinos *to themselves*. Therefore, we cannot redefine the phases of the neutrino fields without making M_ν complex, which would mean that we were not in the physical basis of the neutrinos. So we can only redefine the phases of the 3 charged leptons. That allows us to rotate away only 3 of the 6 phases in the PMNS matrix. So the PMNS matrix has three "mixing angles," which are historically called "neutrino mixing angles," and *three* physical phases. One of these physical phases is analogous to the physical phase in the CKM matrix, and is called the "Dirac phase." The other two physical phases are called the "Majorana phases."

A standard way to parametrization the PMNS matrix is the following

$$U_{PMNS} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{atmospheric} \underbrace{\begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix}} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{solar} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.11 Patterns in The Quark and Lepton Masses and Mixing Angles

In the Standard Model, the masses of the quarks and leptons and the parameters of the CKM and PMNS matrix are free parameters. The only constraint is that the CKM and PMNS matrices are unitary. The Standard Model does not predict (or "post-dict") any of these parameters; they are known only by experiment. But, as we will now see, there are very interesting patterns in these parameters. Most particle theorists believe that there must be some explanation of these patterns; they are not just accidental. Because the Standard Model does not explain these patterns, they must point to some deeper theory.

Experimentally, the magnitudes of CKM matrix elements (they are complex, because of the KM phase) are given approximately by

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \cong \begin{pmatrix} 0.9742 & 0.226 & 0.0036 \\ 0.226 & 0.973 & 0.042 \\ 0.0087 & 0.041 & 0.9991 \end{pmatrix}.$$

One sees that the off-diagonal elements are much smaller than the diagonal elements. Moreover, the mixing between the first and third family is much smaller than the other inter-family mixings. In fact, roughly, $|V_{ub}| \sim |V_{us}V_{cb}|$, which is why the Wolfenstein parametrization looks the way it does.

Experimentally, the magnitudes of the PMNS matrix elements are given approximately by

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \cong \begin{pmatrix} 0.83 & 0.54 & 0.15 \\ -0.44 & 0.47 & 0.76 \\ 0.34 & -0.7 & 0.63 \end{pmatrix}$$

We see that the off-diagonal elements of the PMNS matrix (i.e. the neutrino mixing angles) are large, unlike the CKM mixing angles of the quarks. This was a big surprise, when the neutrino mixing angles started to be measured in 1998. Almost all theorists had expected that they would be small, like the CKM angles. It is interesting that for the leptons the mixing of the the first and third families is smaller than the others, as we found also for the quarks which could be observed in the Table 1.3.

	12 angle	23 angle	13 angle
quarks	$\lvert V_{us}\rvert = 0.226$	$\lvert V_{cb}\rvert = 0.042$	$\lvert V_{ub}\rvert = 0.0036$
leptons	$\lvert U_{e2}\rvert \approx \sin\theta_{sol} = 0.54$	$\lvert U_{\mu 3}\rvert \approx \sin\theta_{atm} = 0.76$	$\lvert U_{e3}\rvert = \sin\theta_{13} e^{i\delta_\ell} = 0.15$

Table 1.3: Inter-family mixing in the quark and leptonic sectors

From the Table 1.4 and Figure 1.10, it is easy to observe that there exists a "hierarchy" structure of the fermions masses. Note that here we assume that it is a "normal hierarchy" in the neutrino sector, i.e. that the third family is the heaviest and the the first family is the lightest. (Experiments so far only tell us two differences of the squared masses of neutrinos $\Delta m^2_{atmospheric} \equiv m_3^2 - m_2^2$ and $\Delta m^2_{solar} \equiv m_2^2 - m_1^2$). So another way to fit the data is with a so-called "inverted hierarchy" where the third family of neutrino is lightest.)

family I	family II	family III
$m_u \approx 2.4$ MeV	$m_c = 1.27$ GeV	$m_t = 171$ GeV
$m_d \approx 4.8$ MeV	$m_s = 104$ MeV	$m_b = 4.2$ GeV
$m_e = 0.51$ MeV	$m_\mu = 105$ MeV	$m_\tau = 1.78$ GeV
$m_{\nu_1} \sim 0$ meV	$m_{\nu_2} \sim 7$ meV	$m_{\nu_3} \sim 50$ meV

Table 1.4: masses of fermions in the Standard Model

There are several noticeable patterns, some of which we have already mentioned: (1) For each type of fermions, up quarks, down quarks and charged leptons, $m_3 \gg m_2 \gg m_1$. (We often refer to u, c, t as "up-type quarks" or just as "up quarks." Similarly, we often call d, s, b the "down-type quarks" or just as "down quarks.")

(2) Inter-family mass ratios for the up-type quarks (U, c, t) are much larger than for the down-type quarks (d, s, b) and charged leptons (e, μ, τ), which in turn are much larger than for the neutrinos (if the neutrino masses are hierarchical).
(3) The masses of the charged leptons are roughly similar to the masses of the down-type quarks.
(4) PMNS angles (also called neutrino mixing angles) are much larger than corresponding CKM angles.
(5) For CKM angles, $\theta_{13} \sim \theta_{12} \cdot \theta_{23}$.

Why these patterns exist is one of the greatest unsolved problems in particle physics. They are often called "the Flavor Problem." Many models have been proposed to explain these patterns.

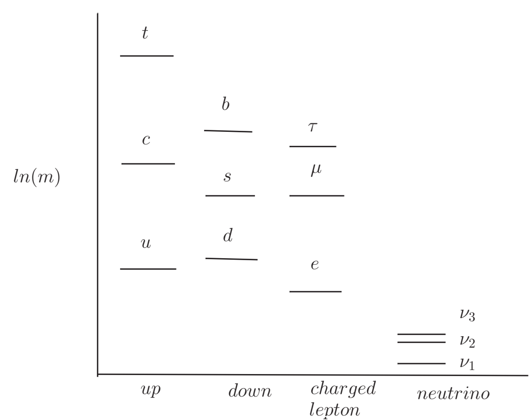

Figure 1.10: the hierarchy structure of fermions

1.12 Grand Unified Theories

1.12.1 Minimal $SU(5)$

In a theory with an $SU(5)$ gauge symmetry, a 5-plet of fields ϕ^α, $\alpha = 1 \cdots 5$, would transform under a gauge transformation as

$$\phi^\alpha \to \phi^{\alpha'} = U_\alpha^{\alpha'} \phi^\alpha$$

or, without indices,

$$\phi \longrightarrow \phi' = U\phi,$$

where U is a 5×5 unitary, unimodular matrix. The complex conjugate of such a field would be written as $(\phi^\alpha)^* \equiv \phi_\alpha$ and the complex conjugate of the matrix U would be written as $\left(U_\alpha^{\alpha'}\right)^* \equiv U_{\alpha'}^{\alpha}$. Therefore one has

$$\phi_\alpha \to \phi_{\alpha'} = U_{\alpha'}^{\alpha} \phi_\alpha,$$

or without indices

$$\phi^* \longrightarrow \phi'^* = U^* \phi^*.$$

The field ϕ^α is called a $\mathbf{5}$ multiplet (or "fundamental" multiplet) and ϕ_α is called a $\overline{\mathbf{5}}$ multiplet (some call it $\mathbf{5}^*$).

There are also higher-rank tensor multiplets. For example, there is an anti-symmetric rank-2 tensor $\phi^{\alpha\beta} = -\phi^{\beta\alpha}$, which is called a $\mathbf{10}$ multiplet. Its conjugate, $\phi_{\alpha\beta} = \left(\phi^{\alpha\beta}\right)^*$, is called a $\overline{10}$. (Notice that upper and lower indices are distinguished from each other, and that complex conjugation has the effect of turning one into the other. One speaks of a tensor with m upper indices and n lower indices as a rank-(m, n) tensor or sometimes as just a rank $m + n$ tensor.) The $\mathbf{10}$ multiplet transforms under the SU(5) symmetry as

$$\phi^{\alpha\beta} \to \phi^{\alpha'\beta'} = U_\alpha^{\alpha'} U_\beta^{\beta'} \phi^{\alpha\beta}$$

Other important multiplets in $SU(5)$, which will be mentioned in later sections of this thesis, are the $\mathbf{24}$, the $\mathbf{45}$ and the $\overline{\mathbf{45}}$.

The $\mathbf{24}$ multiplet, which is called the "adjoint" multiplet, is a traceless rank-(1,1) tensor: ϕ_β^α. The traceless condition is that $\sum_\alpha \phi_\alpha^\alpha = 0$. The $\mathbf{45}$ multiplet is a traceless rank-(2,1) tensor that is antisymmetric in its upper two indices: $\phi_\gamma^{\alpha\beta} = -\phi_\gamma^{\beta\alpha}$. The traceless condition is that $\sum_\alpha \phi_\alpha^{\alpha\beta} = 0$. The $\overline{\mathbf{45}}$ multiplet is just the conjugate of the $\mathbf{45}$ multiplet, and so is a traceless rank-(1,2) tensor that is antisymmetric in its lower two indices: $\psi_{\alpha\beta}^\gamma$.

Consider a 5-plet of $SU(5)$:

$$\phi^\alpha = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \\ \phi^5 \end{pmatrix}.$$

The $SU(5)$ symmetry group has an $SU(3)$ subgroup that acts upon the first three components of this 5-plet, and an $SU(2)$ subgroup that acts on the last two components. That is $\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}$ is a triplet under the $SU(3)$, but a singlet (i.e. it does not transform) under the $SU(2)$. And $\begin{pmatrix} \phi^4 \\ \phi^5 \end{pmatrix}$ is a singlet (does not transform) under $SU(3)$, but is a doublet under $SU(2)$.

The eight generators of the $SU(3)$ subgroup look like this

$$\begin{pmatrix} T^1_1 & T^1_2 & T^1_3 & 0 & 0 \\ T^2_1 & T^2_2 & T^2_3 & 0 & 0 \\ T^3_1 & T^3_2 & T^3_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where T^a_b, $a, b = 1, 2, 3$, are the elements of 3×3 traceless hermitian matrices.

The three generators of the $SU(2)$ subgroup look like this

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T^4_4 & T^4_5 \\ 0 & 0 & 0 & T^5_4 & T^5_5 \end{pmatrix}.$$

where T^i_j, $i, j = 4, 5$, are the elements of 2×2 traceless hermitian matrices (which can be written in terms of the Pauli matrices, as we showed before).

There is also a $U(1)$ subgroup of $SU(5)$, whose generator is the traceless hermitian matrix that commutes with all the generators of the $SU(3)$ and $SU(2)$ subgroups:

$$\begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Let us call this $U(1)$ generator $Y/2$. It is clear that the first three components of the 5-plet have $Y/2 = -\frac{1}{3}$, and the last two components have $\frac{Y}{2} = \frac{1}{2}$.

So the first three components of the 5-plet are an $SU(3)$ triplet, an $SU(2)$ singlet, and have $U(1)$ charge $-\frac{1}{3}$, so one usually denotes it $(3, 1, -\frac{1}{3})$. The last two components of the 5-plet are an $SU(3)$ singlet, $SU(2)$ doublet, and have $Y/2 = \frac{1}{2}$, so one usually denotes it $(1, 2, \frac{1}{2})$.

One sees that $SU(5)$ has a subgroup $SU(3) \times SU(2) \times U(1)$. These are just the gauge groups of the Standard Model. That suggests that the Standard Model can be "embedded" in an $SU(5)$ gauge theory, and in 1974 Georgi and Glashow discovered that it can be in a very simple way.

In an $SU(5)$ grand unified theory, there is a 5-plet of scalars whose $(1, 2, \frac{1}{2})$ part (i.e. the 4 and 5 components in our conventions) is the Higgs doublet of the Standard Model, which we have been calling ϕ. This 5-plet is sometimes denoted $\mathbf{5}_H$, where the subscript H means "Higgs". The conjugate of the Standard Model Higgs doublet, which we denoted ϕ^c (which has the same degrees of freedom as ϕ, because it is just $\phi^c \equiv i\sigma_2\phi^*$) is simply part of the complex conjugate of $\mathbf{5}_H$, which transforms, of course, as a $\overline{\mathbf{5}}$ of $SU(5)$.

1.12.2 First Family of Quarks and Leptons in $SU(5)$

Consider left-handed fermions in a $\overline{5}$ multiplet that will be called $\overline{5}_L$ and denote as $\psi_{L\alpha}$. It contains fermions in

$$\underbrace{\overline{5}_L}_{\psi_{L\alpha}} \rightarrow \underbrace{\left(\overline{3}, \quad 1, \quad \tfrac{1}{3} \right)}_{\psi_{La}} + \underbrace{\left(1, \quad 2, \quad -\tfrac{1}{2} \right)}_{\psi_{Lr}}$$

$$\overline{5}_L = \psi_{L\alpha} = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \\ \psi_{L3} \\ \psi_{L4} \\ \psi_{L5} \end{pmatrix} = \begin{pmatrix} d^c_{L1} \\ d^c_{L2} \\ d^c_{L3} \\ e^-_L \\ -\nu_{eL} \end{pmatrix}.$$

Notice the remarkable fact that the simple $\overline{5}$-plet has pieces that have just the right quantum numbers to be the left-handed anti-down quark d^c_L and the left-handed lepton doublet ℓ_L of the Standard Model! It is important to note that in this equation, the subscripts 1,2,3 on the left-handed anti-d quark d^c_L are $SU(3)$ *color* indices, *not* family indices as in some equations in earlier sections. Note also that we will be using lower-case Greek indices $\alpha, \beta = 1 \cdots 5$ for $SU(5)$ indices; Latin indices from the beginning of the alphabet $a, b = 1, 2, 3$ as color indices; and Latin indices from the middle of the alphabet $r, s = 4, 5$ as $SU(2)$ weak isospin indices.

Next consider left-handed fermions in a 10 multiplet that will be called 10_L and denoted $\psi^{\alpha\beta}_L$. It contains

$$\underbrace{10_L}_{\psi^{\alpha\beta}_L} \rightarrow \underbrace{\left(\overline{3}, \quad 1 \quad -\tfrac{2}{3} \right)}_{\psi^{ab}_L} + \underbrace{\left(3, \quad 2, \quad \tfrac{1}{6} \right)}_{\psi^{ar}_L} + \underbrace{\left(1, \quad 1, \quad 1 \right)}_{\varphi^{45}_L}$$

$$10_L = \psi^{\alpha\beta}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u^c_3 & -u^c_2 & -u_1 & -d_1 \\ -u^c_3 & 0 & u^c_1 & -u_2 & -d_2 \\ u^c_2 & -u^c_1 & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^+ \\ d_1 & d_2 & d_3 & e^+ & 0 \end{pmatrix}_L$$

Again, note the remarkable fact that the 10-plet has components with just the right quantum numbers to be the remaining quarks and leptons of a family. That is, the fermions of a Standard Model family ($Q_L = (u_L, d_L), u_L^c, d_L^c, \ell_L = (\nu_{eL}, e_L), e_L^c$) fit perfectly with two multiplets: $\mathbf{10} + \overline{\mathbf{5}}$.

This is a good time to mention an important historical point. When people knew only about the Standard Model, they were used to writing equations in terms of the left-handed particles and right-handed particles. This made it difficult for people to imagine how to unify the Standard Model particles in larger multiplets, because one cannot put left-handed and right-handed fermions in the same multiplet of a gauge group. (Gauge transformations do not change the handedness of a fermion.) But when one writes equations in terms of left-handed fermions ψ_L and left-handed anti-fermions ψ_L^c, the problem disappeared. Notice that the $\overline{5}$-plet contains the left-handed leptons and left-handed anti-d quark. And the 10-plet contains the left-handed quarks and the left-handed anti-u quark and left-handed positron. This shows how notation can make a big difference in how we think.

The weak hypercharges of the components of the 10-plet, which are written above, can be computed as follows. The $\frac{Y}{2}$ for elements of $\psi_L^{\alpha\beta}$ are just the sum of the $\frac{Y}{2}$ associated with each index, because $\psi^{\alpha'\beta'} = U_\alpha^{\alpha'} U_\beta^{\beta'} \psi^{\alpha\beta}$. For a $\frac{Y}{2}$ rotation

$$\psi^{\alpha'\beta'} = \left(e^{i\frac{Y}{2}\theta}\right)_\alpha^{\alpha'} \left(e^{i\frac{Y}{2}\theta}\right)_\beta^{\beta'} \psi^{\alpha\beta} = e^{i[\frac{Y(\alpha)+Y(\beta)}{2}]\theta}\psi^{\alpha\beta}$$

so $\frac{Y}{2}$ of $\psi^{ar} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, $\psi^{ab} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$ and $\psi^{45} = \frac{1}{2} + \frac{1}{2} = 1$.

Now the question arises how to write Yukawa coupling of the quarks and leptons to the Higgs field. It is amazing that this can be done with just two kinds of term that are often called the "10-10-5" term and the "10-5-bar-5-bar" term. The 10-5-bar-5-bar" term gives mass to the down-type quarks and charged leptons and has the form

$$\begin{aligned} Y\,\psi_L^{\alpha\beta} C\,\psi_{L\alpha}\phi_\beta = \left(10_L \overline{5}_L\right)\overline{5}_H \quad &\rightarrow \quad Y\,\psi_L^{5a} C\,\psi_{La}\phi_5 + Y\,\psi_L^{rs} C\,\psi_{Lr}\phi_s \\ &= \quad Y\,d_L C\,d_L^c \phi^{0*} + Y\,e_L^+ C\,e_L^- \phi^{0*} \end{aligned}$$

If one puts in three families, Y becomes a matrix, $M_d = Y/\sqrt{2}$ and $M_\ell^T = Y/\sqrt{2} \Rightarrow M_d = M_\ell^T$. (The transpose comes from the fact that we write mass matrices with the left-handed fermions on the left that left-handed anti-fermions on the right.) So "minimal SU(5)" predicts $m_e = m_d$, $m_\mu = m_s$, $m_\tau = m_b$. But these relations hold at the GUT scale $M_{GUT} \sim 10^{15}$GeV, and must be "run" using the renormalization group equation to find their values at low energies where they are measured.

The Yukawa terms needed to give mass to the up quarks has the form

$$\begin{aligned} Y\,\psi_L^{\alpha\beta} C\,\psi_L^{\gamma\delta}\phi^\zeta \varepsilon_{\alpha\beta\gamma\delta\zeta} = (10_L 10_L)5_H &\rightarrow Y\,\psi_L^{4a} C\,\psi_L^{bc}\phi^5 \varepsilon_{45abc} \\ &= Y\,u_L C\,u_L^c \phi^0 \end{aligned}$$

1.13 Anomalous Violation of B and L in The Standard Model

1.13.1 A Brief Introduction to Anomalies

The Standard Model Lagrangian is invariant under a global $U(1)$ symmetry that transforms the phases of all the quark fields in the same way: $q \longrightarrow q' = e^{i\alpha}q$. Noether's Theorem implies that there is a conserved "global charge," which is the quark number. People usually discuss "baryon number" (B) instead, where $B = \frac{1}{3}N_q$. So, as a classical theory, the Standard Model conserves B, and $\partial_\mu J_B^\mu = 0$, where J_B^μ is the baryon number current.

Similarly, the original Standard Model Lagrangian, without neutrino mass terms, is invariant under a $U(1)$ that rotates the phases of all lepton fields by the same amount. (Majorana neutrino masses, $m\nu_L^T C\nu_L$ are not invariant under this symmetry.) Therefore, ignoring the very small Majorana mass terms of the neutrinos, the Standard Model conserves "lepton number" L, so $\partial_\mu J_L^\mu = 0$.

However, it turns out that there are very subtle quantum effects called "anomalies" that cause B and L to be violated (i.e. not conserved) in the Standard Model. Anomalies in quantum field theories were discovered in the 1960s by Adler, Bell, and Jackiw. They discovered that in some quantum field theories, certain quantum mechanical amplitudes that should be zero because of the conservation of a global charge are actually not zero. The non-zero contributions to these amplitudes came from Feynman diagrams called triangle diagrams, in which virtual fermions go around a loop that has three vertices, with the current operator of the global current at one vertex and gauge bosons attached at the other two vertices. No one understood in the 1960s why these amplitudes were non-zero in spite of symmetries predicting that they should be zero. That is why they were called anomalies. In the 1970s, the explanation was found: These anomalies arise whenever the "measure of integration" over fermion fields in the Path Integral is *not* invariant, even though the Lagrangian and Action S are invariant.

The anomaly given by the triangle graphs is zero if left-handed and right-handed fermions going around the loop contribute equally, because they contribute with opposite sign — because the anomaly graph has a γ^5 at the current vertex. If the gauge bosons attached at the other two vertices are $W^{\pm}_{\mu}$ bosons, however, then only left-handed fermions can be in the loop, and there is a non-zero anomaly (unless the anomaly cancels for some other reason).

When the triangle graph is calculated for the Baryon number current one finds

$$\partial_\mu J^\mu_B = \frac{1}{3} n_{Q_L} \frac{g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma},$$

where $F^a_{\mu\nu}$ is the field strength tensor for the gauge fields of $SU(2)_L$, and n_{Q_L} stands for the number of left-handed quark doublets minus the number of left-handed anti-quark doublets (counting colors, e.g. counting (u_L, d_L) as three doublets).

Suppose that one calculates the amplitude for some process that goes from a state A at time T_i to a state B at time T_f. Let us integrate the above equation over all of space and over all times between T_i and T_f. By the divergence theorem in four dimensions, the left side of the equation gives a surface term. The "surface" here is the three-dimensional surface that consists of all of space at T_i and T_f. One obtains

$$\left(\int d^3x J^0_B \right)^{T_f}_{T_i} = B_f - B_i = \frac{1}{3} n_{Q_L} \int d^4x \frac{g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{1.11}$$

So baryon number can change if the integral on the right side of the equation is non-zero. Now we shall explain why that integral can be non-zero.

1.13.2 Vacuum "Winding Number"

Let us start by ignoring the Higgs field. For simplicity, assume that we can set it equal to zero everywhere and consider only the gauge fields of $SU(2)_L$. The "classical ground state" of the gauge fields of $SU(2)_L$ has $F^a_{\mu\nu} = 0$. One configuration of fields that has $F^a_{\mu\nu} = 0$ is simply the configuration $A^a_\mu = 0$. However, this configuration is not unique, because gauge transformations of A^a_μ do not change $F^a_{\mu\nu}$. As we saw in section 1.5.2, a gauge transformation changes $A_\mu \equiv \sum_a T^a A^a_\mu$ in the following way:

$$A_\mu \longrightarrow A'_\mu = U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1},$$

where $U(x^\lambda)$ is an $SU(2)_L$ transformation that depends on spacetime location. If we take $A_\mu = 0$, then

$$A'_\mu = -\frac{i}{g}(\partial_\mu U)U^{-1}, \tag{1.12}$$

is also a "classical ground state." In fact, it is equivalent by a gauge transformation to the one that has $A_\mu = 0$. That means that we can describe a classical ground state of the gauge fields by a function $U(x^\lambda)$ that maps points in spacetime (x^λ) onto elements of the group, which are represented by 2×2 unimodular unitary matrices U.

Let us consider a mapping from all the points in three-dimensional space at some time T to matrices U, such that at all points at spatial infinity $U = I$. That is like identifying the points at spatial infinity with each other. Topologically speaking, three-dimensional space with all points at spatial infinity identified with each other, is equivalent to the three-sphere S_3. (Analogously, an infinite line with points at infinity identified is topologically a circle S_1, and a plane with all points at infinity identified is topologically a sphere S_2.)

We will now show that the space of all 2×2 unimodular unitary matrices — the "group space of $SU(2)$ — is also topologically S_3. One can write the most general 2×2 unimodular unitary matrix in the following way:

$$U = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix}, \quad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1.$$

It is easy to check that this is unitary and has determinant = 1. Each matrix U is identified by four real numbers a_0, a_1, a_2, a_3, the sum of whose squares is 1. Those numbers could be considered the coordinates of a three-sphere S_3 in 4-dimensional space. So, the "group space" of $SU(2)$ clearly has the topology of S_3.

If we consider mappings from S_3 to S_3, they can be classified by a "winding number." This most easily thought about in the one-dimensional and two-dimensional analogies. Suppose "space" is a circle and "group space" is also a circle (topologically). One can map all the points of the first circle unto one point of the second circle. Or one can map the points on the first circle onto the points of the second circle that have the same "angle." So one is "winding" or "wrapping" the first circle once around the second circle. Or one can map the points of the first circle onto the points of the second circle that have N times the angle: then one is "winding" the first circle around the other N times. (Like winding a rubber band around your finger N times.) The same thing can be done with two-spheres, like winding a balloon around a basketball N times.

Suppose we imagine a process that begins at time T_i and ends at time T_f, where at both and T_i and T_f the gauge field is in a classical vacuum described by $U(x^\lambda)$ with $U \longrightarrow I$ at spatial infinity. But suppose that at T_i the mapping (of points in physical space to points in group space) has winding number N_i, but that at T_f it has winding number N_f. There is no way for the field to remain in a classical vacuum for all times between T_i and T_f. If it did, that would mean that the mapping went *continuously* from one winding number to another as t went from T_i to T_f. But this not possible, because the initial and final configurations are topologically different. That implies that somewhere in between T_i and T_f, the gauge field is not in a classical vacuum configuration. It must pass through configurations that have $F^a_{\mu\nu} \neq 0$. These configurations have positive energy compared to the classical vacuum. In other words, the process starts and ends in a classical vacuum, but passes over a "potential barrier" in between.

This can be thought about in the following way. One can imagine that there are an infinite number of classical vacuum states of the $SU(2)_L$ gauge fields, labelled by their "winding numbers." They are all equivalent under gauge transformations. This is like a mechanics problem of a rigid rotator with potential $V(\theta) = -V_0 \cos\theta$. There are an infinite number of minima ($\theta = 2\pi N$) that are all equivalent; but to go from one to the next one requires going over a potential barrier.

In the $SU(2)$ gauge theory, it turns out that in the process where the winding number changes by N units between T_i and T_f, the gauge field passes through configurations that have $F^a_{\mu\nu} \neq 0$ and also that the integral $\int d^4x \frac{g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$, which appears in eq. (1.11), is just N. In other words, that integral has a topological meaning: it is a winding number. What that means is that in processes that "go over" the barrier, baryon number is violated.

If temperature is high enough, thermal fluctuations can make the fields go over the barrier. If temperature is low, then the only way to get across the barrier is by quantum-mechanical tunnelling. At zero temperature, these tunneling processes are called "sphalerons." But people often call even the thermal processes that go over the barrier sphalerons. At low temperature, the Higgs field is non-zero (it has a VEV), so calculating a true sphaleron process (i.e. tunneling at low temperature) requires looking at configurations where both $F^a_{\mu\nu} = 0$ and $\phi \neq 0$. Those are the actual "sphaleron configurations." At very high temperature, the expectation value of the Higgs field is zero. So to discuss processes where thermal fluctuations cause the field to go over the barrier and violate baryon number, the Higgs field can be ignored.

The number of baryons that are created or destroyed in a "sphaleron" process (a true sphaleron or a thermal process) depends on the coefficient n_{Q_L}. That coefficient is the number of left-handed quark doublets in the model (minus the number of right-handed quark doublets, if there are any).

The discussion above can be extended to lepton number, or any global charge. For example, if the are fermions in $SU(2)_L$ doublets that carry a global charge X, then to calculate the amount by which an $SU(2)_L$ sphaleron violates X, one has to count every left-handed fermion doublet of $SU(2)_L$ (minus the right-handed doublets) weighted by the X charge of each.

Chapter 2

A SIMPLE GRAND UNIFIED RELATION BETWEEN NEUTRINO MIXING AND QUARK MIXING

2.1 Introduction

In this chapter, we will discuss a model in which all flavor mixing is caused by the mixing of the three usual chiral fermion families, which are in $10 + \overline{5}$ multiplets of $SU(5)$, with extra vector-like fermions in several pairs of $5' + \overline{5}'$ multiplets. As we shall see, this simple assumption, along with the assumption that there is an abelian family symmetry, can lead to the result that both V_{CKM} and U_{MNS} are generated by a single matrix. The entire 3×3 complex mass matrix of the neutrinos M_ν is then found to have a simple expression in terms of two complex parameters and an overall scale. Thus, all the presently unknown neutrino parameters are predicted.

As noted in chapter 1, experiment has shown that the angles of the MNS matrix, which describes the mixing among the left-handed leptons [2], are much larger than the corresponding angles of the CKM matrix of the quark sector [3, 4], which describe mixing among the left-handed quarks. This was unexpected. Since grand unification relates quarks and leptons by putting them in the same multiplets, theorists had expected that the leptonic mixing angles would be comparable in magnitude to the quark mixing angles. However, this expectation was somewhat simplistic. In SU(5), the left-handed quarks are related to the *right-handed* leptons within the 10-plets, so that the CKM angles should be related to the mixing angles of the right-handed leptons, which are not observable in the Standard Model. Similarly, the left-handed leptons are related to the *right-handed* quarks within the $\overline{5}$-plets, so that the MNS angles should be related to the mixing angles of the right-handed quarks, which are not observable in the Standard Model. If there is more mixing among the $\overline{5}$-plets of different families than among the 10-plets, the disparity between the MNS and CKM mixing angles would be explained. This idea can be implemented in models based on any grand unified group, since all such groups contain SU(5) as a subgroup. Several ways of implementing this basic idea have been proposed in the literature [5, 6, 7, 8, 9, 10, 11, 12, 13].

In section 2.2, we will propose a model in which there are three $10+\overline{5}$ families of fermions and N $5'+\overline{5}'$ vector-like fermions. (The possible existence of such additional vector-like fermions has been much discussed in the literature in a variety of contexts [14, 15, 16, 17, 18, 19, 20, 21, 22].) The central idea of the model proposed here is that all inter-family mixing is caused by the mixing between the $\overline{5}$ multiplets of the usual fermions and the $\overline{5}'$ multiplets of the extra vector-like fermions. And both quark mixing and lepton mixing are controlled in this model by a single matrix, which we will call A. In section 2.3, we will show that this matrix can be determined from the masses and mixing angles of the quarks alone, and this allows the entire 3×3 complex mass matrix M_ν of the known neutrinos (which contains 9 real physical observables) to be predicted in terms of just two complex parameters and an overall mass scale. We will also discuss the model's predictions of the quantities which are still unknown and the post-predictions of the quantities which have been measured, but still not precisely. In section 2.4, we will propose a new way to generalize the model which might completely explain the flavor problem. Also how the model could be embedded in a larger group SO(10) and E_6 without effecting the predictions of the model is discussed.

2.2 The Model of Flavor Mixing

There are three families of fermions denoted by $10_i + \overline{5}_i$, $i = 1, 2, 3$, and extra vector-like fermions denoted by $5'_A + \overline{5}'_A$, $A = 1, \ldots, N$. (N can be as small as 2.) It is assumed that in the absence of the vector-like pairs, the Yukawa couplings and mass matrices of the three families would be flavor diagonal, due to discrete symmetries, $K_1 \times K_2 \times K_3 \times K^{'}$, that distinguish the three families from each other. All flavor mixing is indirectly caused by mass terms that mix the $\overline{5}_i$ and $\overline{5}'_A$. The Yukawa term of the model are:

$$
\begin{aligned}
L_{Yuk} \;=\;& Y_i\left(10_i 10_i\right)\langle 5_H\rangle + y_i\left(10_i\overline{5}_i\right)\langle\overline{5}_H\rangle + \\
& \widetilde{Y}_i\left(10_i 10_i\right)\langle 45_H\rangle + \widetilde{y}_i\left(10_i\overline{5}_i\right)\langle\overline{45}_H\rangle + \\
& \frac{\lambda_i}{M_R}\left(\overline{5}_i\overline{5}_i\right)\langle 5_H\rangle\langle 5_H\rangle + \\
& Y'_{AB}\left(5'_A\overline{5}'_B\right)\langle 1_H\rangle + y_{Ai}\left(5'_A\overline{5}_i\right)\langle 1'_{Hi}\rangle
\end{aligned}
\qquad (2.1)
$$

where the subscript H denotes Higgs multiplets. The Yukawa terms in eq. (2.1) are the most general allowed by the abelian flavor symmetry $K_1 \times K_2 \times K_3 \times K'$, K_i is a Z_2 symmetry under which 10_i, $\overline{5}'_i$ and $1'_{Hi}$ are odd and all other fields even. The role of the $K_1 \times K_2 \times K_3$ symmetry is to prevent direct mixing among the usual chiral fermion families. K' is a Z_N symmetry ($N > 2$) under which $5'_A \to e^{2\pi i/N} 5'_A$, $\overline{5}'_A \to e^{2\pi i/N}\overline{5}'_A$, $1_H \to e^{-4\pi i/N} 1_H$ and $1'_{Hi} \to e^{-i2\pi/N} 1'_{Hi}$. One of the roles played by the K' symmetry is to forbid the Higgs fields that break $SU(5)$ at the superlarge scales from coupling to the fermions. (Typically, these include adjoint Higgs multiplets (i.e. 24-plets), which would be allowed by $SU(5)$ to couple to $5'_A\overline{5}'_B$, $5'_A\overline{5}_i$, etc. This would introduce several more parameters into the model of fermion masses and reduce its predictivity.)

The first four terms in eq. (2.1) are the standard Yukawa terms of SU(5) grand unification, and are the minimal terms needed to give mass to the known quarks and leptons. (As already noted in the original Georgi-Glashow paper on SU(5) unification, the presence of a 45-plet of Higgs fields avoids the unrealistic relations between down quark and charged lepton masses that would arise if only the VEV of a 5-plet of Higgs fields gave mass to the Standard Model fermions [23].) The fifth term is the standard dimension-5 Weinberg operator that gives the left-handed neutrinos Majorana masses in either the type-I or type-II seesaw mechanisms [24].

The last two terms in eq. (2.1) are the only ones peculiar to this model. The first of these simply gives masses to the vector-like fermions, and the second gives masses that mix these vector-like fermions with the three families. The Higgs fields in these two terms are gauge singlets, so that their VEVs would naturally be superlarge. All that matters for the purposes of the model is that the masses coming from these two terms be roughly of the same scale, which we shall call M_*. We assume that these masses are much heavier than the weak scale, and we will refer to them as "heavy." This scale should be large enough to explain why the extra vector-like fermions have not yet observed in the experiments. Any fermions whose masses are less than or at the weak scale (i.e. the Standard Model fermions) we will call "light." Note that the Yukawa matrices in the last two terms of eq. (2.1) are in general not diagonal. The reason for this is that the $\overline{5}'_A$ and $5'_A$ transform trivially under the family symmetry $K_1 \times K_2 \times K_3$, which therefore does not constrain the form of Y'_{AB} and y_{Ai}. This is how the extra vector-like $\overline{5}' + 5'$ multiplets cause family mixing in the model.

First, let us examine the mass matrix of the down quarks that emerges from eq. (2.1). There are left-handed anti-down quarks in both $\overline{5}_i$ and $\overline{5}'_A$, which will be denoted d^c_i and $D^{c'}_A$, respectively. There are left-handed down quarks in 10_i and $5'_A$, which will be denoted d_i and D'_A, respectively. There is a $(3+N) \times (3+N)$ mass matrix for the down quarks

$$L_{d\,mass} = \begin{pmatrix} d_i & D'_A \end{pmatrix} \begin{pmatrix} (m_d)_i \delta_{ij} & 0 \\ \Delta_{Aj} & M_{AB} \end{pmatrix} \begin{pmatrix} d^c_j \\ D^{c'}_B \end{pmatrix} \tag{2.2}$$

where $(m_d)_i = y_i \langle \overline{5}_H \rangle + \widetilde{y}_i \langle \overline{45}_H \rangle$, $M_{AB} = Y'_{AB} \langle 1_H \rangle$, and $\Delta_{Aj} = y'_{Aj} \langle 1'_{Hj} \rangle$. In this thesis, we follow the convention that Dirac mass matrices are multiplied from the left by the left-handed fermions and from the right by the right-handed fermions (or, equivalently, the left-handed anti-fermions).

The $(3+N)\times(3+N)$ matrix in eq. (2.2) can be block-diagonalized by multiplying it from the right by a unitary matrix $\mathcal{V}$ whose elements are of order one (since the elements of the matrices Δ and M are of the same order) and from the left by a unitary matrix whose angles are of order $m_d/M \ll 1$ and which therefore can be neglected. One can write the $(3+N) \times (3+N)$ unitary matrix $\mathcal{V}$ as

$$\mathcal{V} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.3}$$

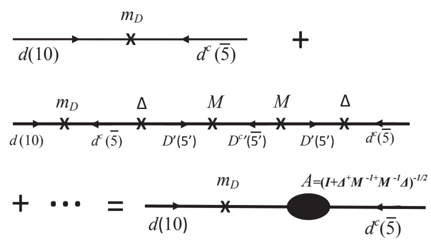

Figure 2.1: Diagrams showing how the mass terms Δthat mix the $\overline{5}_i$ with the $\overline{5}'_A$ lead to insertions of the matrix A on external $\overline{5}$ fermion lines.

where A is 3×3, D is $N \times N$, B is $3 \times N$, and C is $N \times 3$. This gives

$$\begin{pmatrix} m_d & 0 \\ \Delta & M \end{pmatrix} \mathcal{V} = \begin{pmatrix} m_d & 0 \\ \Delta & M \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cong \begin{pmatrix} m_d A & m_d B \\ 0 & \widetilde{M} \end{pmatrix}. \tag{2.4}$$

If we define $T \equiv M^{-1}\Delta$, it is easy to show that the blocks of $\mathcal{V}$ may be written $A = \left[I + T^\dagger T\right]^{-1/2}$, $B = \left[I + T^\dagger T\right]^{-1/2} T^\dagger$, $C = -T\left[I + T^\dagger T\right]^{-1/2}$, and $D = \left[I + TT^\dagger\right]^{-1/2}$.

The off-diagonal block $m_d B$ in eq. (2.4) can be removed by a rotation from the left that is of order $m_d/\widetilde{M}$, which is negligible, as already noted. The block-diagonalization separates out the light (i.e. electroweak scale or smaller) fermions, which are the fermions of the Standard Model, from the heavy fermions. The upper-left 3×3 block in eq. (2.4) therefore is the "effective" mass matrix of the three observed down quarks, which we will denote simply M_d:

$$M_d = m_d\, A \tag{2.5}$$

In other words, the net effect of the mixing of the three families with the heavy vector-like fermions is to multiply the diagonal mass matrix m_d from the right by a non-diagonal matrix A. This can be understood diagramatically from the Figure 2.1.

One can identify the effective mass matrices of the light charged leptons and neutrinos by a similar process of block-diagonalization. By doing so, one finds that factors of A appear in these effective matrices corresponding to fermions that are in the $\overline{5}$ multiplet of $SU(5)$. Since the mass matrices for the up quarks, down quarks, charged leptons and neutrinos come, respectively, from $(10\,10)\,5_H$, $(10\,\overline{5})\overline{5}_H$, $(\overline{5}\,10)\overline{5}_H$ and $(\overline{5}\,\overline{5})5_H 5_H$ terms, the effective mass matrices for these types of fermions end up having the form

$$\begin{aligned} M_u &= m_u \\ M_d &= m_d A \\ M_\ell &= A^T m_\ell \\ M_\nu &= A^T m_\nu A \end{aligned} \tag{2.6}$$

where the "underlying" mass matrices m_u, m_d, $m_\ell$and m_ν are all diagonal and given by $(m_u)_{ij} = \delta_{ij}\left(Y_i v_5 + \tilde{Y}_i v_{45}\right)$, $(m_d)_{ij} = \delta_{ij}\,(y_i v_{\overline{5}} + \tilde{y}_i v_{\overline{45}})$, $(m_\ell)_{ij} = \delta_{ij}\,(y_i v_{\overline{5}} - 3\tilde{y}_i v_{\overline{45}})$, and $(m_\nu)_{ij} = \delta_{ij}\lambda_i(v_5^2/M_R)$. These underlying mass matrices m_u, m_d, m_ℓ, and m_ν are diagonal because of the family symmetry $K_1 \times K_2 \times K_3$. On the other hand, the matrix A has no special form, because the extra vector-like multiplets transform trivially under the family symmetry. If we assume that the matrices we called M and Δ in eqs. (2.2) and (2.4) are of the same order (for example, the GUT scale), then the elements of $T \equiv M^{-1}\Delta$ are of order 1, and one expects that all the elements of A will also be of order 1.

Before moving to the next section, we would like to discuss the renormalization effects on the matrix A, which could be different for different types of fermions. At the unification scale, the same 3×3 matrices Δ and M appear in the $(N = 3) \times (N + 3)$ mass matrices of the charged leptons and the down quarks. But, due to gluon loops, the Δ and M of the down quarks should run more strongly between the unification scale and the scale M_* than the corresponding matrices of the leptons. The crucial point, however, is that A depends on the ratio $T \equiv M^{-1}\Delta$; and since gauge boson loops cause Δ and M to run in the same way, these effects cancel out in A. Moreover, the renormalization effects due to Yukawa couplings (which are small for the $\overline{\mathbf{5}}$ fermions) can be neglected. Thus, it really is the same matrix A (if one neglects very small effects) that appears in M_d, M_ℓ, and M_ν. This result makes the model simple and predictive.

2.3 The Predictions and Post-Predictions of The Model

First let us discuss the quark sector. The up quarks are already in the "physical basis" or "mass basis," since $M_u = m_u$ is diagonal. The down quark mass matrix is not diagonal, because it is has the form

$$M_d \;\; = \;\; m_d A = \mu_d \begin{pmatrix} \delta_d & 0 & 0 \\ 0 & \epsilon_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{2.7}$$

where μ_d is the 33 element of the diagonal matrix m_d.

The matrix A, like any matrix, can be written in the following form

$$A = \mathcal{D} A_{\Delta} \mathcal{U}, \tag{2.8}$$

where A_{Δ} has the form

$$A_{\Delta} = \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.9}$$

with a, b, and c real, and where $\mathcal{U}$ is unitary and $\mathcal{D}$ is diagonal. This is evident, because A can be made to have triangular form by multiplying from the right by a unitary matrix, and then the diagonal elements of that triangular matrix can be scaled to be 1 by multiplying from the left by a diagonal matrix. We can take a and b real, because the phases in the 12 and 23 elements of A_{Δ} can be absorbed into $\mathcal{U}$ and $\mathcal{D}$, while leaving its diagonal elements real.

By eqs. (2.7) and (2.8), the effective mass matrix of the observed down quarks is given by $M_d = m_d \mathcal{D} A_{\Delta} \mathcal{U}$, with $m_d = \mu_d \; diag(\delta_d, \epsilon_d, 1)$. The unitary matrix $\mathcal{U}$ can be absorbed into redefined right-handed down quarks. (It is important to note that since this is a transformation of right-handed quarks, it will have no effect on the CKM matrix.) The diagonal matrix $\mathcal{D}$ can be absorbed into a redefined diagonal mass matrix: $\overline{m}_d \equiv m_d \mathcal{D}$. The phases in $\overline{m}_d$ can then be absorbed into redefined left-handed down quarks. Finally, therefore, one may write

$$\begin{aligned}
\overline{M}_d &= \overline{m}_d \, A_\Delta \\
&= \overline{\mu}_d \begin{pmatrix} \overline{\delta}_d & 0 & 0 \\ 0 & \overline{\epsilon}_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \\
&= \overline{\mu}_d \begin{pmatrix} \overline{\delta}_d & \overline{\delta}_d b & \overline{\delta}_d ce^{i\theta} \\ 0 & \overline{\epsilon}_d & \overline{\epsilon}_d a \\ 0 & 0 & 1 \end{pmatrix}. \qquad (2.10)
\end{aligned}$$

To reach the physical basis of the down quarks, the matrix $\overline{M}_d$ must be diagonalized. As will be seen below, the rotations required to do this are small. Therefore, the eigenvalues $\overline{M}_d$ are to a very good approximation just equal to its diagonal elements, so that $\overline{\mu}_d \cong m_b$, $\overline{\epsilon}_d \overline{\mu}_d \cong m_s$, and $\overline{\delta}_d \overline{\mu}_d \cong m_d$.

We may therefore write

$$\overline{M}_d \cong \begin{pmatrix} m_d & m_d b & m_d ce^{i\theta} \\ 0 & m_s & m_s a \\ 0 & 0 & m_b \end{pmatrix}. \qquad (2.11)$$

The diagonalization of this matrix requires a "bi-unitary tranformation" of both the left-handed and right-handed down quarks. The unitary transformation of the right-handed quarks does not affect the CKM matrix (and anyway these rotation angles are quadratic in small ratios of quark masses and thus negligible). The unitary transformation of the left-handed down quarks is just the CKM mixing matrix, since the mass matrix $\overline{M}_u$ is already diagonal. (See the discussion of CKM mixing in the Introduction.) We then have

$$\overline{M}_d \cong V^*_{CKM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \qquad (2.12)$$

Comparing eq. (2.11) and eq. (2.12), it is easy to show that

$$\begin{aligned}
|V_{cb}| \approx \frac{a\, m_s}{m_b} \quad &\Longrightarrow \quad a \cong \frac{m_b}{m_s}|V_{cb}| \sim 2 \\
|V_{us}| \approx \frac{b\, m_d}{m_s} \quad &\Longrightarrow \quad b \cong \frac{m_s}{m_d}|V_{us}| \sim 4 \\
|V_{ub}| \approx \frac{ce^{i\theta}\, m_d}{m_b} \quad &\Longrightarrow \quad \theta c \cong \frac{m_b}{m_d}|V_{ub}| \sim 3, \;\; \theta \cong \delta_{CKM}
\end{aligned} \tag{2.13}$$

Turning to the lepton sectors, one sees that the mass matrix of the charged leptons can be writte

$$M_\ell \;\; = \;\; A^T m_\ell = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} \delta_\ell & 0 & 0 \\ 0 & \epsilon_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \mu_\ell \tag{2.14}$$

The hierarchy among the charged lepton masses tells us that $\overline{\delta}_\ell \ll \overline{\epsilon}_\ell \ll 1$. So, the diagonal matrix m_ℓ is hierarchical, just as m_d and m_u are. By comparing eqs. (2.7) and (2.14), we see how this model explains the disparity between the neutrino mixing angles and quark mixing angles. Because $M_d = m_d\, A$, whereas $M_\ell = A^T m_\ell$, the mass matrix of the down quarks has a hierarchy among the rows, whereas the charged lepton mass matrix has a hierarchy among the columns. Since rotations of the left-handed fermions (which are the ones relevant to the CKM and MNS mixing angles) are rotations among the rows, we see that small quark mixing angles and large lepton mixing angles arise. (This is a realization of the basic idea of "lopsided" models [5, 6, 7, 8, 9, 10, 11, 12, 13].)

We can make the matrix A in this equation have the same form as given in eq. (2.8) by doing the same combination of rotations to the left-handed leptons as we did to the right-handed down quarks, followed by analogous re-scalings and re-phasings. If we do the same rotations to the left-handed charged leptons and left-handed neutrinos, no MNS mixing is induced at this stage. We get

$$\begin{aligned}
\overline{M}_\ell &= A_\Delta^T \overline{m}_\ell \\
&= \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ ce^{i\theta} & a & 1 \end{pmatrix} \begin{pmatrix} \overline{\delta}_\ell & 0 & 0 \\ 0 & \overline{\epsilon}_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{\mu}_\ell \\
&= \begin{pmatrix} \overline{\delta}_\ell & 0 & 0 \\ \overline{\delta}_\ell b & \overline{\epsilon}_\ell & 0 \\ \overline{\delta}_\ell ce^{i\theta} & \overline{\epsilon}_\ell a & 1 \end{pmatrix} \overline{\mu}_\ell \cong \begin{pmatrix} m_e & 0 & 0 \\ m_\mu b & m_\mu & 0 \\ m_\tau ce^{i\theta} & m_\tau a & m_\tau \end{pmatrix}. \qquad (2.15)
\end{aligned}$$

This is not diagonal, but to a very good approximation it can be diagonalized by rotations done only to the righthanded charged leptons. Rotations of the left-handed charged leptons are also required, but they are rotations by angles that are proportional to squares of small lepton mass ratios, and can be neglected. Thus, to a very good approximation, the charged lepton mass matrix in eq. (2.15) is in the mass basis of the left-handed charged leptons. In this basis, the effective mass matrix of the three light neutrinos $\overline{M}_\nu$ has the form $\overline{M}_\nu = A_\Delta^T \overline{m}_\nu A_\Delta$ which is given by

$$\overline{M}_\nu \cong \begin{pmatrix} 1 & 0 & 0 \\ \frac{m_s}{m_d}|V_{us}| & 1 & 0 \\ \frac{m_b}{m_d}|V_{ub}|e^{i\delta} & \frac{m_b}{m_s}|V_{cb}| & 1 \end{pmatrix} \begin{pmatrix} qe^{i\beta} & 0 & 0 \\ 0 & pe^{i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{m_s}{m_d}|V_{us}| & \frac{m_b}{m_d}|V_{ub}|e^{i\delta} \\ 0 & 1 & \frac{m_b}{m_s}|V_{cb}| \\ 0 & 0 & 1 \end{pmatrix} \mu_\nu \qquad (2.16)$$

where $qe^{i\beta}$,$pe^{i\alpha}$ and the overall scale μ_ν are free parameters of the model, δ is the Kobayashi-Maskawa CP-violating phase, and the V_{ij} are the CKM matrix elements. Since we only have the freedom to re-phase the left-handed neutrinos, there are three physical phases in eq. (2.16), rather than one as in the other mass matrices. The extra two phases are the ones called α and β in eq. (2.16).

The mass matrix M_ν shown in eq. (2.16) contains all the information about the masses, mixings and CP-violating phases of the neutrinos. There are nine physical observables involved: the three neutrino masses, the three MNS angles, the Dirac CP phase, and the two majorana CP phases of the neutrinos. These are all determined by the five model parameters in eq. (2.16): μ_ν, $pe^{i\alpha}$and $qe^{i\beta}$.

Since five neutrino observables have already been measured (θ_{sol}, θ_{atm}, θ_{13}, Δm^2_{12} and Δm^2_{23}), we can use them to determine the five model parameters, and then predict the four as-yet-unmeasured neutrino observables. Since the equations are non-linear (they involve trigonometric functions), there is no guarantee that they can fit the five measured neutrino properties with five adjustable parameters. (To put it another way, the adjustable parameters α and β are bound within the range $[0,\, 2\pi)$.) Nevertheless, good fits are obtained. This is only true, however, if some of the measured neutrino properties have values that lie within a smaller range than that presently allowed by experiment. For example, although the current experimental range of the atmospheric neutrino mixing angle is $\theta_{atm} = 45 \pm 6.5°$[25], the model only obtains good fits for $\theta_{atm} \lesssim 43°$ with values near 40°preferred, as we shall see. The fits also prefer a value of θ_{sol} greater than or equal to 34°, i.e. greater than the present experimental central value. The quark properties are also constrained: the best fits are obtained with $m_s/m_d \lesssim 20$, and δ greater than or equal to its present experimental central value. Thus, in addition to predicting the four as-yetunmeasured neutrino observables, the model places non-trivial and testable constraints on the values of quantities that have been measured.

In the Table 2.1, we show a representative fit in which all the input quark parameters and the neutrino observables obtained as output are in their experimentally allowed ranges (and in most cases at their central values). The experimental values are taken from the 2012 Review of Particle Properties [25], except for δ_{lep}(the neutrino Dirac CP phase) where we use the result of a recent global analysis of neutrino data [26]. For m_b/m_s we have used the renormalization group results of [27] to obtain $m_s(m_b)$ from $m_s(2\text{GeV})$, which is given in [25].

Note that the model's prediction for $\delta_{lep}$is 1.15π radians, which accords remarkably well with the one-sigma range found in [26] of $1.1\pi^{+0.3\pi}_{-0.4\pi}$rad. The value of $(M_\nu)_{ee}$(to which the amplitude of neutrinoless double beta decay is proportional) is much smaller

Quantity	**Values in fit**	**Experiment**
μ_ν	0.1428 eV	—
$pe^{i\alpha}$	$0.1525e^{-2.734i}$	—
$qe^{i\beta}$	$0.01405e^{-0.352i}$	—
m_b/m_s	52.9	52.9 ± 2.6
m_s/m_d	19	17 to 22
$\lvert V_{us}\rvert$	0.2252	0.2252 ± 0.0009
$\lvert V_{cb}\rvert$	0.0409	0.0409 ± 0.0011
$\lvert V_{ub}\rvert$	0.00415	0.00415 ± 0.00049
δ	1.30 rad	$1.187^{+0.175}_{-0.192}$rad
θ_{sol}	$34.1°$	$33.89° \,^{+0.976°}_{-0.971°}$
θ_{atm}	$40°$	$45° \pm 6.5°$
θ_{13}	$9.12°$	$9.122° \,^{+0.609°}_{-0.647°}$
δm^2_{23}	$2.32 \times 10^{-3}\text{eV}^2$	$2.32^{+0.12}_{-0.08} \times 10^{-3}\text{eV}^2$
δm^2_{12}	$7.603 \times 10^{-5}\text{eV}^2$	$(7.5 \pm 0.2) \times 10^{-5}\text{eV}^2$
δ_{lep}	1.15πrad	$1.1\pi^{+0.3\pi}_{-0.4\pi}$rad
$(M_\nu)_{ee}$	0.0020 eV	

Table 2.1: A fit to the quark and neutrino data. μ_ν, $pe^{i\alpha}$and $qe^{i\beta}$are model parameters. $\delta_{lep}$is the neutrino Dirac CP phase, and $(M_\nu)_{ee}$the mass that comes into neutrinoless double beta decay.

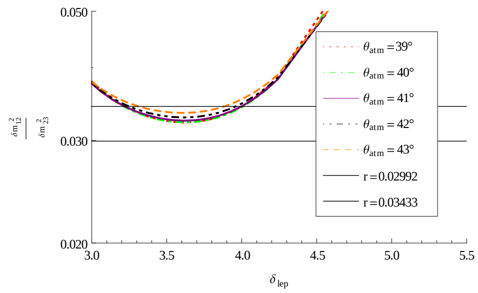

Figure 2.2: The result of fits with the values of quark parameters given in Table I, $\theta_{sol} = 34.2°$, and $\theta_{13} = 9.12°$. The curves are the relation of r ($\equiv \delta m^2_{12}/\delta m^2_{23}$) to the predicted δ_{lep} for different values of θ_{atm}. The horizontal lines are the one-sigma limits for r.

than the experimental limits, which tend to be in the range of a few tenths of an eV to several eVs for different experiments [25]. This prediction of the model is not very sensitive to variation of the model's input parameters.

Figures 2.2 - 2.4 show the degree of sensitivity of the δ_{lep} prediction to the values of θ_{atm},θ_{sol} and δ (the quark CP phase). In the Fgure 2.2, we have fixed the values of all the quark mass ratios and CKM parameters, and of θ_{sol} and θ_{13}, but have allowed θ_{atm} and the ratio $\Delta m^2_{12}/\Delta m^2_{23}$ (which we henceforth call r) to take different values. The curves are the relation of r to the predicted δ_{lep} for different values of θ_{atm}. The horizontal lines are the one-sigma limits for r. One sees that δ_{lep} is predicted to be roughly 1.15π radians and that values of $\theta_{atm} \lesssim 41°$are preferred.

In the Figure 2.3, we have done a similar thing, but this time fixing $\theta_{atm}$to be 40°and allowing θ_{sol} and r to vary. One can see a preference for values of θ_{sol} equal or above the present experimental central value. In the Figure 2.4, we have allowed the quark CP phase δ and r to vary. One sees that the best-fit value of δ_{lep} is rather insensitive to the assumed values of the measured quark and neutrino properties, but the width of the range of δ_{lep}values that give good fits is quite sensitive.

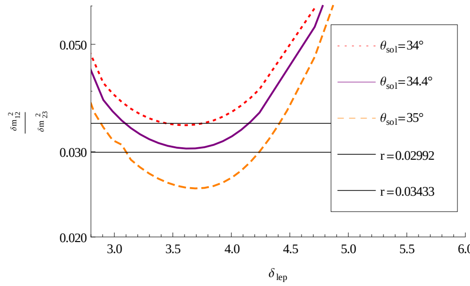

Figure 2.3: The result of fits with the values of quark parameters given in Table I, $\theta_{atm} = 40°$, and $\theta_{13} = 9.12°$. The curves are the relation of r ($\equiv \delta m_{12}^2/\delta m_{23}^2$) to the predicted δ_{lep} for different values of θ_{sol}.

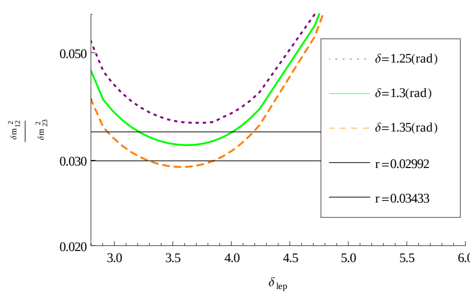

Figure 2.4: The result of fits with the values of quark parameters other than δ given in Table I, $\theta_{sol} = 34.2°$, and $\theta_{13} = 9.12°$, and $\theta_{atm} = 40°$. The curves are the relation of r ($\equiv \delta m_{12}^2/\delta m_{23}^2$) to the predicted δ_{lep} for different values of the quark phase δ.

2.4 Conclusions

The model proposed here gives an account of how the CKM and MNS flavor mixings arise, but does not explain the mass hierarchy among the families, since the hierarchies in the diagonal matrices m_u, m_d, m_ℓ and m_ν are simply assumed. There are, however, several simple ways in which the present model could be extended to give an explanation of the mass hierarchy. One way is to combine the structure in this model with the structure assumed in []. In that paper, the mass hierarchies were explained by the three usual families mixing with extra vector-like $\mathbf{10} + \overline{\mathbf{10}}$ fermion pairs in a way analogous to the mixing with $\mathbf{5} + \overline{\mathbf{5}}$ assumed here. Combining the structures of the two models would be appealing since it would mean that the vector-like fermions would comprise entire family-antifamily pairs. (It has been pointed out that this can lead in a simple way to unification of gauge couplings in non-SUSY models [22].) We will discuss a way to combine these two ideas which might be able to provide a complete explanation to the flavor problem in chapter 6.

It should be noted that the present model could be embedded in many grand unified schemes. For example, in an SO(10) model, the ordinary families could be in three 16 multiplets, while the vector-like fermions could be in three 10 multiplets. In E_6, one gets the extra vectorlike fermions "for free", since the $\mathbf{27}$ contains $\mathbf{16} + \mathbf{10} + \mathbf{1}$ of $SO(10)$. Different patterns of breaking of the grand unified group could be assumed without affecting the predictions for fermion masses and mixings. For example, in many unified models, an adjoint Higgs field does some of the breaking of the unification group. If that adjoint Higgs multiplet does not transform under the K' symmetry mentioned after eq. (2.1), it would not couple renormalizably to $(\overline{5}'_A \overline{5}'_B)$ or $(5'_A \overline{5}'_i)$ and hence not contribute to the matrices Δ and M in eq. (2.4) and the matrix A. Consequently, except for negligible higher-order corrections, the matrix A would not "know" that the unification group is broken, and the same A would appear in both the quark and lepton sectors, as is necessary for the model to be predictive.

In conclusion, if all flavor changing in both the quark and lepton sectors arises as a consequence of the mixing of the usual families with extra vector-like fermions that are in $5+\overline{5}$ of SU(5), a testable relationship arises between the quark and lepton mixing. This relationship allows the prediction of the four as-yet-unmeasured neutrino observables as well as testable constraints on several quantities that have been measured. Measurement of the Dirac CP phase of the neutrinos δ_{lep}, as well as more precise determinations of such quantities as $\theta_{atm}, \theta_{sol}$, $|V_{ub}|$, m_s/m_d and δ (the quark CP phase) would provide stringent tests of the model.

Chapter 3

PROTON DECAY AND THE ORIGIN OF QUARK AND LEPTON MIXING

3.1 Introduction

In the last chapter, we discussed a highly predictive model that relates the MNS mixing of the leptons [2] and the CKM mixing of the quarks [3, 4] by positing a single source for all flavor violation, namely mixing between the three "usual" fermion families with "extra" vector-like fermions in $\mathbf{5}' + \overline{\mathbf{5}}'$ multiplets of $SU(5)$. This was shown to lead to several testable predictions including neutrino masses and CP-violating phases.

Just as the CKM and MNS angles specify how fermion mass eigenstates are arranged within the multiplets of the electroweak $SU(2)_L$ group, there are mixing angles that specify how the fermions are arranged within the multiplets of the grand unification group. Since the model we discussed in the last chapter posits a single source for all flavor mixing, it predicts these grand unified mixing angles also. Proton decay branching ratios depend on these angles, and so discovery of proton decay and measurement of those angles would provide further non-trivial tests of the model. We extract the predictions for proton decay coming from the exchange of the grand unified gauge bosons, which would dominate in a non-supersymmetric version of the model.

What makes this model so predictive is the combination of SU(5) symmetry, which relates the quarks and leptons, and abelian flavor symmetries, which forbid flavor violation in the sector of the "usual" fermion multiplets, which consists of three families of $10 + \overline{5}$ multiplets. One of the interesting features of the model is that even though it unifies quarks and leptons, it explains in a simple and natural way why the CKM mixing angles of the former are much smaller than the MNS mixing angles of the latter, as we saw in the previous chapter. The point is that ultimate source of all flavor violation is the mixing among $\overline{\mathbf{5}}$ multiplets, which contain the left-handed leptons, but not the left-handed quarks. This is the basic idea that underlies so-called "lopsided models" [5, 6, 7, 8, 9, 10, 11, 12, 13], of which the present model is a particularly simple and predictive example.

This model is also very closely related to a model proposed in 1984 as a solution to the Strong CP Problem [16, 17, 18]. In that model, CP is assumed to be a symmetry of the Lagrangian that is spontaneously broken by the VEVS of the singlet scalars $\mathbf{1}'_{Hi}$ that produce the off-diagonal mass matrix Δ. In fact, the model proposed here has the same structure as the model originally proposed by A.E. Nelson in [16], except that here the 3×3 mass matrices of the ordinary three families are required to be flavor diagonal by family symmetries. Therefore, the mechanism for solving the Strong CP Problem proposed there can easily be implemented in the present model. If it is, then further predictions result, though they would require measuring proton decay branching ratios to less than 1 percent accuracy. That the mechanism can be tested at all, however, is quite striking and surprising, as the relevant physics happens at the GUT scale. Again, what makes that possible here is the powerful combination of $SU(5)$ symmetry and abelian flavor symmetry.

In section 3.2, we will go into more detail in our treatment of complex phases than we did in [28] as this is necessary to determine the relative phases of certain proton-decay amplitudes. In section 3.3, we derive the predictions for the $SU(5)$ mixing angles and proton decay branching ratios. In section 3.4, we see that by embedding the model in $SO(10)$, which is very simply done, even more interesting proton decay predictions result. The most interesting proton decay predictions are shown in eqs. (3.25) - (3.27).

3.2 The Complex Phases of The Model in Proton Decay

As we have already introduced in the chapter 2, the Yukawa terms of the model are

$$
\begin{aligned}
\mathcal{L}_{Yuk} &= Y_i(\mathbf{10}_i^U\mathbf{10}_i^U)\langle\mathbf{5}_H\rangle + y_i(\mathbf{10}_i^U\overline{\mathbf{5}}_i^U)\langle\overline{\mathbf{5}}_H\rangle \\
&+ \tilde{Y}_i(\mathbf{10}_i^U\mathbf{10}_i^U)\langle\mathbf{45}_H\rangle + \tilde{y}_i(\mathbf{10}_i^U\overline{\mathbf{5}}_i^U)\langle\overline{\mathbf{45}}_H\rangle \\
&+ (\lambda_i/M_R)(\overline{\mathbf{5}}_i^U\overline{\mathbf{5}}_i^U)\langle\mathbf{5}_H\rangle\langle\mathbf{5}_H\rangle \\
&+ Y'_{AB}(\mathbf{5}_A^E\overline{\mathbf{5}}_B^E)\langle\mathbf{1}_H\rangle + y'_{Ai}(\mathbf{5}_A^E\overline{\mathbf{5}}_i^U)\langle\mathbf{1}'_{Hi}\rangle,
\end{aligned}
\tag{3.1}
$$

Here we have used a somewhat different notation than in the last chapter. The "usual" chiral multiplets $10+\overline{5}$ are here given a superscript U and the "extra" vector-like fermions in $5+\overline{5}$ are given a superscript E. This will help keep track of the multiplets being discussed, as it will be necessary to carry out a series of changes of basis. The subscript H denotes Higgs multiplets. The last two terms give SU(5)-invariant masses to extra vector-like fermions in the $\mathbf{5^E}$ and $\overline{\mathbf{5}}^E$ multiplets which are denoted as $\mathbf{5}'$ and $\overline{\mathbf{5}}'$ in the chapter 2, so that these masses can be much larger than the weak scale and possibly even the GUT scale. It does not matter to the analysis in this chapter what the scale of these SU(5)-invariant masses are.

The Yukawa terms in eq. (3.1) that involve only the usual fermion multiplets give rise to mass matrices m_u, m_d, m_ℓ, and m_ν that are flavor diagonal

$$
u_{_{10_i^U}}\ (m_u)_{ij}\ u^c_{_{10_j^U}}, \qquad d_{_{10_i^U}}\ (m_d)_{ij}\ d^c_{_{\overline{5}_j^U}}, \qquad \ell_{_{\overline{5}_i^U}}\ (m_\ell)_{ij}\ \ell^c_{_{10_j^U}}, \qquad \nu_{_{\overline{5}_i^U}}\ (m_\nu)_{ij}\ \nu_{_{\overline{5}_j^U}} \tag{3.2}
$$

However, the fermions in $\overline{\mathbf{5}}_i^U$, $i=1,2,3$, are not simply the observed Standard Model fermions. The $\overline{\mathbf{5}}_i^U$ and the $\overline{\mathbf{5}}_A^E$ mix, with some linear combinations of them becoming "heavy", and the three combinations orthogonal to them remaining "light". We denote these respectively by $\overline{\mathbf{5}}^H$ and $\overline{\mathbf{5}}^L$. To identify these heavy and light multiplets one must look at the heavy fermion masses. From the last two terms in eq. (3.1), these are seen to have the for

$$
\mathbf{5}_A^E\left([y'_{Ai}\langle\mathbf{1}'_{Hi}\rangle]\,\overline{\mathbf{5}}_i^U + [Y'_{AB}\langle\mathbf{1}_H\rangle]\,\overline{\mathbf{5}}_B^E\right) \equiv \mathbf{5}_A^E\left(\Delta_{Ai}\overline{\mathbf{5}}_i^U + M_{AB}\overline{\mathbf{5}}_B^E\right), \tag{3.3}
$$

One sees that $\overline{\mathbf{5}}^H \propto (\Delta\overline{\mathbf{5}}^U + M\overline{\mathbf{5}}^E)$. One can easily write the light $\overline{\mathbf{5}}^L$ as the linear combinations orthogonal to these. The light $\overline{5}^L$are the standard model fermions we observed in the experiments. One can then invert to write the “usual" multiplets $\overline{\mathbf{5}}^U$ as linear combinations of $\overline{\mathbf{5}}^L$ and $\overline{\mathbf{5}}^H$. The result turns out to be (suppressing indices)

$$\overline{\mathbf{5}}^U = A\overline{\mathbf{5}}^L + B\overline{\mathbf{5}}^H, \tag{3.4}$$

where A and B are the same complex matrices given in the last chapter. (A is 3×3 and B is $3\times N$, where N is the number of “extra" $\overline{\mathbf{5}}$ multiplets.) For the $\mathbf{10}$ multiplets, there are no heavy mass terms or mixing with “extra” multiplets, so the “usual” multiplets are simply the same as the light multiplets: $\mathbf{10}^U = \mathbf{10}^L$. If we rewrite eq. (4.5) in terms of the light fermion multiplets using eq. (3.4), and suppress indices for clarity, we obtain

$$u^T_{{}_{10^L}}\, m_u\, u^c_{{}_{10^L}}, \qquad d^T_{{}_{10^L}}\, m_d\, A\, d^c_{{}_{\overline{5}^L}}, \qquad \ell^T_{{}_{\overline{5}^L}}\, A^T\, m_\ell\, \ell^c_{{}_{10^L}}, \qquad \nu^T_{{}_{\overline{5}^L}}\, A^T\, m_\nu\, A\, \nu_{{}_{\overline{5}^L}} \tag{3.5}$$

Note that the matrix A appears next to the fermions that are in $\overline{\mathbf{5}}$ multiplets, because A reflects the effects of the mixing of those multiplets. The terms in eq. (3.5) give the effective mass matrices of the three families of Standard Model fermions, which we will denote by *capital* M: M_u, M_d, M_ℓ, and M_ν.

So far the analysis is the same as presented in the last chapter. There we saw that

$$M_u = m_u, \qquad M_d = m_d A, \qquad M_\ell = A^T m_\ell, \qquad M_\nu = A^T m_\nu A, \tag{3.6}$$

where m_u, m_d, m_ℓ, and m_ν are the diagonal and hierarchical “underlying” mass matrices, and A is a matrix that can be brought to the form

$$A = \mathcal{D}\, A_\Delta\, \mathcal{U}, \tag{3.7}$$

where $\mathcal{D}$ is a complex diagonal matrix, $\mathcal{U}$ is unitary and

$$A_\Delta = \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

Here a and b have been made real by absorbing phases into $\mathcal{D}$ and $\mathcal{U}$.

It will be important to keep track of complex phases for the later analysis of proton decay, let us define $\mathcal{D} \equiv e^{i\Phi}D$, where D and Φ are real diagonal matrices, and similarly $m_u \equiv e^{i\Phi_u}m_u^R$, $m_d \equiv e^{i\Phi_d}m_d^R$, $m_\ell \equiv e^{i\Phi_\ell}m_\ell^R$, $m_\nu \equiv e^{i\Phi_\nu}m_\nu^R$, where Φ_u, m_u^R, etc. are real diagonal matrices.

It will be convenient to define a "flavor basis" of SU(5) multiplets (denoted by superscript F) by $\overline{\mathbf{5}}^F \equiv \mathcal{U}\overline{\mathbf{5}}^L$ and $\mathbf{10}^F \equiv e^{i(\Phi+\Phi_d)}\mathbf{10}^L$. (There are three families of these, but we are suppressing the family indices.) Therefore, if we use eq. (3.7) to rewrite the expressions in eq. (3.5), we can absorb the factors of $\mathcal{U}$ and some of the phases into redefined fermion multiplets, as follows.

$$
\begin{aligned}
u^T_{10^L}\ m_u\ u^c_{10^L} &= u^T_{10^L}\ (e^{i\Phi_u}m_u^R)\ u^c_{10^L} = u^T_{10^F}\ m_u^R\ e^{i(\Phi_u-2\Phi_d-2\Phi)}u^c_{10^F} \\
&= u^T_{10^F}\ \overline{m}_u\ e^{i\Theta_{u^c}}u^c_{10^F}, \qquad \text{where } \overline{m}_u \equiv m_u^R,\ \ \Theta_{u^c} \equiv \Phi_u - 2\Phi_d - 2\Phi, \\
d^T_{10^L}\ (m_d\ A)\ d^c_{\bar{5}^L} &= d^T_{10^L}\ (e^{i\Phi_d}m_d^R)\ (e^{i\Phi}DA_\Delta\mathcal{U})\ d^c_{\bar{5}^L} = d^T_{10^F}\ (m_d^R D)\ A_\Delta\ d^c_{\bar{5}^F} \\
&= d^T_{10^F}\ (\overline{m}_d\ A_\Delta)\ d^c_{\bar{5}^F}, \qquad \text{where } \overline{m}_d \equiv m_d^R D, \\
\ell^T_{\bar{5}^L}\ (A^T\ m_\ell)\ \ell^c_{10^L} &= \ell^T_{\bar{5}^L}\ (\mathcal{U}^T A_\Delta^T D e^{i\Phi})\ (e^{i\Phi_\ell}m_\ell^R)\ \ell^c_{10^L} = \ell^T_{\bar{5}^F}\ A_\Delta^T\ (D\overline{m}_\ell^R)\ e^{i(\Phi_\ell-\Phi_d)}\ \ell^c_{10^F} \\
&= \ell^T_{\bar{5}^F}\ (A_\Delta^T\overline{m}_\ell)\ e^{i\Theta_{\ell^c}}\ \ell^c_{10^F}, \qquad \text{where } \overline{m}_\ell \equiv Dm_\ell^R,\ \ \Theta_{\ell^c} \equiv \Phi_\ell - \Phi_d, \\
\nu^T_{\bar{5}^L}\ (A^T\ m_\nu\ A)\ \nu_{\bar{5}^L} &= \nu^T_{\bar{5}^L}\mathcal{U}^T\ A_\Delta^T\ (e^{2i(\Phi_\nu+\Phi)}D^2m_\nu^R)\ A_\Delta\ \mathcal{U}\ \nu_{\bar{5}^L} = \nu^T_{\bar{5}^F}\ A_\Delta^T\ e^{2i(\Phi_\nu+\Phi)}(D^2m_\nu^R)\ A_\Delta\ \nu_{\bar{5}^F} \\
&= \nu^T_{\bar{5}^F}\ (A_\Delta^T\ e^{i\Theta_\nu}\overline{m}_\nu\ A_\Delta)\ \nu_{\bar{5}^F}, \qquad \overline{m}_\nu \equiv D^2m_\nu^R,\ \ \Theta_\nu \equiv \Phi_\nu + \Phi.
\end{aligned}
\tag{3.9}
$$

This gives the mass matrices of the three Standard Model families in the $\mathbf{10}^F$, $\overline{\mathbf{5}}^F$ basis as

$$\overline{M}_u = \overline{m}_u e^{i\Theta_{u^c}}, \qquad \overline{M}_d = \overline{m}_d A_\Delta, \qquad \overline{M}_\ell = A_\Delta^T \overline{m}_\ell e^{i\Theta_{\ell c}}, \qquad \overline{M}_\nu = A_\Delta^T (e^{i\Theta_\nu} \overline{m}_\nu) A_\Delta. \tag{3.10}$$

Note that so far no transformations have been done that contribute to CKM or MNS mixing. To get to the $\mathbf{10}^F, \overline{\mathbf{5}}^F$ basis from the $\mathbf{10}^L, \overline{\mathbf{5}}^L$ basis, we have done a transformation by $\mathcal{U}$ to the $\overline{\mathbf{5}}$ multiplets, i.e. to (i) the left-handed anti-down quarks (right-handed quarks), which does not affect CKM mixing, and (ii) the left-handed charged leptons and neutrinos, which, because it was the *same* for the charged leptons and neutrinos, does not affect MNS mixing.

The parameters that come into the quark and lepton masses and the CKM and MNS mixing have been reduced to the matrix A_Δ; the four real diagonal matrices $\overline{m}_u$, $\overline{m}_d$, $\overline{m}_\ell$, and $\overline{m}_\nu$; and two relative phases in Θ_ν. The phases in Θ_{u^c} and Θ_{ℓ^c} only affect right-handed fermions, and for that reason do not affect the MNS and CKM angles. (This is why they were not discussed explicitly in the last chapter.) But they do matter, as will be seen, for proton decay.

Now, as usual, we have to diagonalize above mass matrices as by "bi-unitary" transformations, which means going from the $SU(5)$ "flavor basis" to the "mass basis" of the Standard Model fermions, i.e. the basis where the 3×3 effective mass matrices of the Standard Model fermions are real and diagonal. We will denote the fermions in this basis with a tilde. The matrix $\overline{M}_u$ is already diagonal. It can be made real by redefining the phases of the anti-up quarks as follows: $\tilde{u} \equiv u_{10^F}$ and $\tilde{u}^c \equiv e^{i\Theta_{u^c}} u_{10^F}$. (Here and in the following we are not showing family indices.) The matrix $\overline{M}_d = \overline{m}_d \; A_\Delta$ has the form

$$\overline{M}_d = \overline{m}_d \; A_\Delta = \overline{\mu}_d \begin{pmatrix} \overline{\delta}_d & 0 & 0 \\ 0 & \overline{\epsilon}_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = \overline{\mu}_d \begin{pmatrix} \overline{\delta}_d & \overline{\delta}_d b & \overline{\delta}_d c e^{-i\theta} \\ 0 & \overline{\epsilon}_d & \overline{\epsilon}_d a \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.11}$$

The matrix $\overline{M}_d$ must be diagonalized by a bi-unitary transformations. The unitary matrix that transforms it from the left is just the CKM matrix (since $\overline{M}_u$ is already diagonal), and the one that transforms it from the right is very close to being the identity matrix, as was explained in chapter 2. Thus we may write

$$\overline{M}_d = \overline{m}_d \; A_\Delta \cong V^*_{CKM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} . \tag{3.12}$$

The mass eigenstates for the down quarks are therefore given by $\tilde{d} = V^\dagger_{CKM} d_{10^F} \;\; \Rightarrow$ $d_{10^F} = V_{CKM} \tilde{d}$, whereas for the anti-down quarks, one has simply $\tilde{d}^c \cong d^c_{\overline{5}^F}$.

The diagonalization of $\overline{M}_\ell$ proceeds in an analogous way, except transposed. So the non-negligible rotation in this case is done from the right, i.e. to the left-handed anti-leptons ℓ^c, whereas a negligible transformation is needed of the left-handed leptons ℓ. We can thus write $A^T_\Delta \; \overline{m}_\ell = m_\ell \; V_{\ell^c}$. Therefore the mass eigenstates of the anti-leptons are $\tilde{\ell}^c = V_{\ell^c} e^{i\Theta_{\ell^c}} \ell^c_{10^F}$, $\tilde{\ell} \cong \ell_{\overline{5}^F}$. (See eq. (3.9).) Since negligible rotation of the left-handed charged leptons is needed to diagonalize $\overline{M}_\ell$, the MNS mixing comes almost entirely from the diagonalization of the neutrino mass matrix $\overline{M}_\nu$, which is given by

$$\begin{aligned} \overline{M}_\nu &= A^T_\Delta (e^{i\Theta_\nu} \overline{m}_\nu) A_\Delta \\ &\cong \begin{pmatrix} 1 & 0 & 0 \\ \frac{m_s}{m_d}|V_{us}| & 1 & 0 \\ \frac{m_b}{m_d}|V_{ub}|e^{i\delta} & \frac{m_b}{m_s}|V_{cb}| & 1 \end{pmatrix} \begin{pmatrix} qe^{i\beta} & 0 & 0 \\ 0 & pe^{i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{m_s}{m_d}|V_{us}| & \frac{m_b}{m_d}|V_{ub}|e^{i\delta} \\ 0 & 1 & \frac{m_b}{m_s}|V_{cb}| \\ 0 & 0 & 1 \end{pmatrix} \mu_\nu, \end{aligned} \tag{3.13}$$

where we have pulled out an overall factor μ_ν to scale the 33 element of $\overline{m}_\nu$ to 1. Since the transformation to diagonalize $\overline{M}_\nu$ is to a very good approximation just the MNS matrix, one has $\tilde{\nu} = U^\dagger_{MNS} \; \nu_{\overline{5}^F} \;\; \Rightarrow \;\; \nu_{\overline{5}^F} = U_{MNS} \tilde{\nu}$.

We can now write the content of the SU(5) multiplets that contain the Standard Model fermions, in the $\mathbf{10}^F$, $\overline{\mathbf{5}}^F$ basis. Using the facts that $u_{10^F} = \tilde{u}$, $\quad u^c_{10^F} =$

$\tilde{u}^c$, $d_{10^F} = V_{CKM}\,\tilde{d}$, $d^c_{\overline{5}^F} = \tilde{d}^c$, $\ell^c_{10^F} = V_{\ell^c}\,\tilde{\ell}^c$, $\ell_{\overline{5}^F} = \tilde{\ell}$, and $\nu_{\overline{5}^F} = U_{MNS}\tilde{\nu}$, one has

$$\mathbf{10}^F_I = \left(e^{-i\Theta_{\ell^c}} V_{\ell^c}\tilde{\ell}^c, \ \begin{pmatrix} \tilde{u} \\ V_{CKM}\tilde{d} \end{pmatrix}, \ e^{-i\Theta_{u^c}}\tilde{u}^c \right)_I, \qquad \overline{\mathbf{5}}^F_I = \begin{pmatrix} \tilde{d}^c \\ \begin{bmatrix} U_{MNS}\tilde{\nu} \\ \tilde{\ell} \end{bmatrix} \end{pmatrix}_I, \tag{3.14}$$

where I is the family index. So, for example, if $I = 1$, we see that the $SU(5)$ partners of the physical $\tilde{u}_L$ are (i) the physical $\tilde{u^c}_L$ times the phase $e^{-i(\Theta_{u^c})_{11}}$, (ii) the CKM linear combination of the physical down quarks, namely $V_{CKM}\tilde{d} = (V_{ud}\tilde{d}_L + V_{us}\tilde{s}_L + V_{ub}\tilde{b}_L)$, and (iii) the linear combination of the physical anti-leptons $((V_{\ell^c})_{11}\tilde{e^c} + (V_{\ell^c})_{12}\tilde{\mu^c} + (V_{\ell^c})_{13}\tilde{\tau^c})$ times the phase $e^{-i(\Theta_{\ell^c})_{11}}$.

3.3 Proton Decay Angles for $SU(5)$ Modes

In this section, we will consider proton decay caused by the exchange of the superheavy gauge bosons of $SU(5)$. These obviously only make transitions within the irreducible multiplets of $SU(5)$. From eq. (3.14), therefore, we see that the only mixing matrices that enter into such proton decay amplitudes are U_{MNS}, V_{CKM}, and V_{ℓ^c}. The CKM and MNS matrices can be measured at low-energy, and are fairly well known. The matrix V_{ℓ^c} cannot be measured at low energy, but is predicted by the model, since it comes from diagonalizing $A^T_\Delta \overline{m}_\ell$. A_Δ is known and determined by the masses and mxings angles of quarks. The diagonal matrix $\overline{m}_\ell$ can be determined from the masses of the charged leptons. One finds that to an excellent approximation

$$V_{\ell^c} = \begin{pmatrix} 1 & -\frac{m_e}{m_\mu}b & -\frac{m_e}{m_\tau}ce^{i\theta} \\ \frac{m_e}{m_\mu}b & 1 & -\frac{m_\mu}{m_\tau}a \\ \frac{m_e}{m_\tau}ce^{-i\theta} & \frac{m_\mu}{m_\tau}a & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -0.02 & -0.001e^{i\delta_{KM}} \\ 0.02 & 1 & -0.12 \\ 0.001e^{-i\delta_{KM}} & 0.12 & 1 \end{pmatrix}, \tag{3.15}$$

Since the τ lepton is too heavy to be involved in proton decay, only the 12 element of V_{ℓ^c} , which we shall call ζ, enters the proton decay amplitudes. ($\zeta \cong 0.02$.)

As can be seen from eq. (3.14), this model gives quite definite predictions for all the mixing angles that come into SU(5) fermion multiplets and thus into the amplitudes for proton decay via the exchange of the superheavy $SU(5)$ gauge bosons. In particular, one sees that the only mixings that come into those amplitudes are the CKM and MNS elements and the (small) 12 element of V_{ℓ^c}.

While this is a definite and (in principle) testable prediction, it is not very distinctive of this model, as all models will involve the CKM and MNS mixing. The one non-zero parameter that comes into proton decay that is distinctive of this model is ζ. Because ζ is small, however, it would be very hard to measure even if proton decay is seen. On the other hand, if ζ is measured, then, as we will show, it would allow a test of a well-known mechanism [] for solving the Strong CP Problem that is otherwise almost impossible to test.

If we embed this model in $SO(10)$, as is quite natural and simple to do, one finds predictions for proton decay branching ratios that are quite distinctive of this model. In fact, they would allow an independent measurement of the parameter b and a combination of a and c, which are also determined by low-energy physics. We shall consider the $SO(10)$ embedding of the model and the resulting proton decay predictions in the section 3.4. First, we shall find the proton-decay operators coming from the exchange of the $SU(5)$ gauge bosons.

Let us denote the superheavy gauge bosons of SU(5) by W_i^a, where $a = 1, 2, 3$ is an $SU(3)_c$ color index and $i = 4, 5$ is an $SU(2)_L$ weak index. The W_i^a could be the X or Y bosons and also $m_X = m_Y$ is assumed. The relevant couplings are $(\overline{\psi})_{ac} \not{W}_i^a \psi^{ic} + (\overline{\psi})_{aj} \not{W}_i^a \psi^{ij} + (\overline{\psi})^i \not{W}_i^a \psi_a$, in Standard Model notation. Dropping the color and weak indices and putting in the family indices I one has $\overline{u^c}_I \not{W} Q_I + \overline{Q}_I \not{W} \ell_I^c + \overline{L}_I \not{W} d_I^c$, where $Q_I = \begin{pmatrix} u \\ d \end{pmatrix}_I$ and $L_I = \begin{pmatrix} \nu \\ e \end{pmatrix}_I$. Here the family indices refer to the $\mathbf{10}^F$, $\overline{\mathbf{5}}^F$ basis, shown in eq. (3.14). These terms give the $d = 6$ operators

$$\frac{g_5^2}{2M_5^2}\ \left(\overline{Q}_I\gamma_\mu u_I^c\right)\left(\overline{L}_J\gamma^\mu d_J^c\right) - \frac{g_5^2}{2M_5^2}\ \left(\overline{Q}_I\gamma_\mu u_I^c\right)\left(\overline{Q}_J\gamma^\mu \ell_J^c\right)$$

$$= \frac{g_5^2}{2M_5^2}\ \left[\left(\overline{d}_I\gamma_\mu u_I^c\right)\left(\overline{\nu}_J\gamma^\mu d_J^c\right) + \left(\overline{u}_I\gamma_\mu u_I^c\right)\left(\overline{\ell}_J\gamma^\mu d_J^c\right)\right] \tag{3.16}$$

$$-\frac{g_5^2}{2M_5^2}\ \left[\left(\overline{u}_I\gamma_\mu u_I^c\right)\left(\overline{d}_J\gamma^\mu \ell_J^c\right) + \left(\overline{d}_I\gamma_\mu u_I^c\right)\left(\overline{u}_J\gamma^\mu \ell_J^c\right)\right].$$

Referring to eq. (3.14), and keeping only the fermions light enough to be decay products of a nucleon, the relevant operators are

$$\frac{g_5^2}{2M_5^2}\ \left(\overline{[\cos\theta_C\tilde{d} + \sin\theta_C\tilde{s}]}\gamma_\mu\tilde{u}^c\right)\left(\overline{\nu}_e\gamma^\mu\tilde{d}^c + \overline{\nu}_\mu\gamma^\mu\tilde{s}^c\right)$$

$$+\frac{g_5^2}{2M_5^2}\ \left(\overline{\tilde{u}}\gamma_\mu\tilde{u}^c\right)\left(\overline{\tilde{e}}\gamma^\mu\tilde{d}^c + \overline{\tilde{\mu}}\gamma^\mu\tilde{s}^c\right)$$

$$-\frac{g_5^2}{2M_5^2}\ \left(\overline{\tilde{u}}\gamma_\mu\tilde{u}^c\right)\left(\overline{[\cos\theta_C\tilde{d} + \sin\theta_C\tilde{s}]}\gamma^\mu[\tilde{e}^c + e^{-i\alpha_1}\zeta\tilde{\mu}^c]\right) \tag{3.17}$$

$$-\frac{g_5^2}{2M_5^2}\ \left(\overline{\tilde{u}}\gamma_\mu\tilde{u}^c\right)\left(\overline{[-\sin\theta_C\tilde{d} + \cos\theta_C\tilde{s}]}\gamma^\mu[-e^{-i\alpha_2}\zeta\tilde{e}^c + \tilde{\mu}^c]\right)$$

$$-\frac{g_5^2}{2M_5^2}\ \left(\overline{[\cos\theta_C\tilde{d} + \sin\theta_C\tilde{s}]}\gamma_\mu\tilde{u}^c\right)\left(\overline{\tilde{u}}\gamma^\mu[\tilde{e}^c + e^{-\alpha_1}\zeta\tilde{\mu}^c]\right),$$

where α_1 and α_2 are the 11 and 22 elements of the diagonal matrix of phases Θ_{ℓ^c}. Collecting like terms (and doing a Fierz transformation [] of the last term in eq. (3.17)) we obtain:

$$\frac{g_5^2}{2M_5^2}\ \left(\overline{[\cos\theta_C\tilde{d} + \sin\theta_C\tilde{s}]}\gamma_\mu\tilde{u}^c\right)\left(\overline{\nu}_e\gamma^\mu\tilde{d}^c + \overline{\nu}_\mu\gamma^\mu\tilde{s}^c\right)$$

$$+\frac{g_5^2}{2M_5^2}\ \left(\overline{\tilde{u}}\gamma_\mu\tilde{u}^c\right)\left(\overline{\tilde{e}}\gamma^\mu\tilde{d}^c + \overline{\tilde{\mu}}\gamma^\mu\tilde{s}^c\right)$$

$$-\frac{g_5^2}{2M_5^2}\ \left(\overline{\tilde{u}}\gamma_\mu\tilde{u}^c\right)\left[2c_\theta(\overline{\tilde{d}}\gamma^\mu\tilde{e}^c) + (2s_\theta - e^{-i\alpha_2}\zeta)(\overline{\tilde{s}}\gamma^\mu\tilde{e}^c) + (2e^{-i\alpha_1}\zeta - s_\theta)(\overline{\tilde{d}}\gamma^\mu\tilde{\mu}^c) - c_\theta(\overline{\tilde{s}}\gamma^\mu\tilde{\mu})\right], \tag{3.18}$$

where $s_\theta \equiv \sin\theta_C = V_{us}$, $c_\theta \equiv \cos\theta_C$, and where we have dropped terms of order ζs_θ (≈ 0.004) or smaller.

The phases α_1 and α_2 are unknown free parameters of the model. They enter, however, only in the small terms proportional to ζ, and thus their effect would not be significant unless the proton decay branching ratios were measurable to better than a percent accuracy. If the $O(\zeta)$ terms could be measured precisely enough, however, it would allow a test of a well-known mechanism for solving the Strong CP Problem, as we will now explain. The model we are describing here implements in a very simple way the mechanism for solving the Strong CP Problem proposed in [16, 17, 18]. Indeed, it is the model proposed in Nelson's paper, except that here we have imposed a flavor symmetry to make the Yukawa terms involving the "usual" fermion multiplets diagonal. To solve the Strong CP Problem (assuming no supersymmetry) all that is needed is to impose CP as an invariance of the Lagrangian that is spontaneously broken by the Higgs fields we denoted $\mathbf{1}'_{hI}$ in eq. (3.1). What would happen in that case is that the phases matrices Φ_u, Φ_d, Φ_ℓ, and Φ_ν would all vanish. (The phase matrix Φ would not vanish, however, as it comes from the matrix A that arises ultimately from $\langle \mathbf{1}'_{hI} \rangle$.) One can see from eq. (3.9), that Θ_{ℓ^c} is given by $\Phi_\ell - \Phi_d$ and therefore would vanish. Thus the mechanism for solving the Strong CP Problem [16, 17, 18] predicts that $\alpha_1 = \alpha_2 = 0$.

3.4 Proton Decay Angles in The $SO(10)$ Embedding of The Model

More interesting predictions for proton decay arise if the model is embedded in $SO(10)$. This embedding is very simple. The "usual" fermion multiplets are contained in spinors of $SO(10)$, while the "extra" vector-like fermion multiplets are contained in vectors of $SO(10)$:

$$\mathbf{10}^U + \overline{\mathbf{5}}^U \subset \mathbf{16}^U, \qquad \mathbf{5}^E + \overline{\mathbf{5}}^E \subset \mathbf{10}^E. \tag{3.19}$$

Then the Yukawa terms in $SO(10)$ are

$$\begin{aligned}
\mathcal{L}_{Yuk} &= Y_{_I}(\mathbf{16}_I^U\mathbf{16}_I^U)\langle\mathbf{10}_h\rangle \\
&+ \tilde{Y}_{_I}(\mathbf{16}_I^U\mathbf{16}_I^U)\langle\overline{\mathbf{126}}_h\rangle \\
&+ (\lambda_{_I}/M_R)(\mathbf{16}_I^U\mathbf{16}_I^U)\langle\overline{\mathbf{16}}_h\rangle\langle\overline{\mathbf{16}}_h\rangle \\
&+ Y'_{_{AB}}(\mathbf{10}_A^E\overline{\mathbf{10}}_B^E)\langle\mathbf{1}_h\rangle + y'_{_{AI}}(\mathbf{10}_A^E\mathbf{16}_I^U)\langle\mathbf{16}_{hI}\rangle.
\end{aligned} \tag{3.20}$$

The analysis presented in the previous section goes through without change. Now, however, there are additional superheavy gauge bosons in $SO(10)$ that mediate proton decay, namely those that make transitions between the $\mathbf{10}^U$ and $\overline{\mathbf{5}}^U$ within the spinors $\mathbf{16}^U$. These new gauge bosons are X' and Y' in $SO(10)$ which transform as $(3,2,\frac{1}{6})+$ *conj.* under the Standard Model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, and can be denoted by W^{ai} and W_{ai}, where a is a color index and i is a weak index. These new gauge bosons have coupling to the $\mathbf{10}^U$ and $\overline{\mathbf{5}}^U$ in the $\mathbf{16}^U$ of the form $\overline{\psi}_{\alpha\beta}\not{W}_{\gamma\delta}\psi_\eta\epsilon^{\alpha\beta\gamma\delta\eta}$ (where α, β, etc. are $SU(5)$ indices), which contains $(\overline{\psi}_{bc}\not{W}_{ai}\psi_j - \overline{\psi}_{bj}\not{W}_{ai}\psi_c)\epsilon_{ijabc}$. Translating this into the notation of the Standard Model, one has

$$(\overline{u^c})^a\not{W}_{ai}(L)_j\epsilon^{ij} + (\overline{Q})_{jb}\not{W}_{ai}(d^c)_c\epsilon^{abc}\epsilon^{ij}. \tag{3.21}$$

The fermions in these operators are in the "usual" multiplets $\mathbf{10}^U + \overline{\mathbf{5}}^U \subset \mathbf{16}^U$. Let us make this explicit, but suppress the $SU(5)$ indices for clarity: $\overline{u^c}_{10^U}\not{W}L_{\overline{5}^U} + \overline{Q}_{10^U}\not{W}d^c_{\overline{5}^U}$. Using eqs. (3.4) and (3.9), this can be rewritten in terms of the fields in the multiplets $\mathbf{10}^F$ and $\overline{\mathbf{5}}^F$ shown in eq. (3.14):

$$\begin{aligned}
& \overline{u^c}_{10^U}\not{W}L_{\overline{5}^U} + \overline{Q}_{10^U}\not{W}d^c_{\overline{5}^U} \\
\longrightarrow\ & \overline{u^c}_{10^L}\not{W}AL_{\overline{5}^L} + \overline{Q}_{10^L}\not{W}Ad^c_{\overline{5}^L} \\
=\ & \overline{u^c}_{10^L}\not{W}\mathcal{D}A_\Delta(\mathcal{U}L_{\overline{5}^L}) + \overline{Q}_{10^L}\not{W}\mathcal{D}A_\Delta(\mathcal{U}d^c_{\overline{5}^L}) \\
=\ & e^{i(\Phi_d+\Phi)}\left[\overline{u^c}_{10^F}\not{W}[\mathcal{D}A_\Delta]L_{\overline{5}^F} + \overline{Q}_{10^F}\not{W}[\mathcal{D}A_\Delta]d^c_{\overline{5}^F}\right]
\end{aligned} \tag{3.22}$$

The gauge bosons in eq. (3.22) form a weak doublet with electric charges $\frac{1}{3}$ and $-\frac{2}{3}$, which we will denote $W_{(1/3)}$ and $W_{(-2/3)}$. First, let us consider the couplings of $W_{(1/3)}$, which will give the proton decay modes with charged leptons. Writing out the terms in the last line of eq. (3.22), referring to eq. (3.14), and keeping only those quarks and leptons that are lighter than a nucleon, one has

$$\begin{aligned} & e^{i(\Phi_u-\Phi_d)_1}[\overline{\tilde{u}^c}\not{W}_{(1/3)}D_1\sum_{J=1,2}(A_\Delta)_{1J}\tilde{\ell}_J+\overline{\tilde{u}}\not{W}_{(1/3)}D_1\sum_{J=1,2}(A_\Delta)_{1J}\tilde{d}^c_J] \\ = \;& e^{i(\Phi_u-\Phi_d)_1}D_1[\overline{\tilde{u}^c}\not{W}_{(1/3)}(\tilde{e}+b\tilde{\mu})+\overline{\tilde{u}}\not{W}_{(1/3)}(\tilde{d}^c+b\tilde{s}^c)], \end{aligned} \tag{3.23}$$

where D_J stands for the JJ element of the diagonal matrix D. The couplings in eq. (3.23) give the $d=6$ nucleon decay operators

$$\begin{aligned} & \left(\tfrac{g_{10}^2}{2M_{10}^2}(D_1)^2\right)(\overline{[\tilde{e}+b\tilde{\mu}]}\gamma_\mu\tilde{u}^c)\;(\overline{\tilde{u}}\gamma^\mu[\tilde{d}^c+b\tilde{s}^c]) \\ = \;& \kappa\left(\tfrac{g_5^2}{2M_5^2}\right)(\overline{\tilde{u}}\gamma_\mu\tilde{u}^c)\;(\overline{[\tilde{e}+b\tilde{\mu}]}\gamma^\mu[\tilde{d}^c+b\tilde{s}^c]), \end{aligned} \tag{3.24}$$

where the second line was obtained by a Fierz transformation, and we have defined the real parameter $\kappa\equiv(D_1)^2\frac{g_{10}^2/M_{10}^2}{g_5^2/M_5^2}$.

Combining these $SO(10)$ operators with the $SU(5)$ operators given in eq. (3.18), we can compute the proton-decay rates for the two-body decay modes having charged leptons in the final state, namely $p\to\pi^0e^+$, $p\to\pi^0\mu^+$, $p\to K^0e^+$, and $p\to K^0\mu^+$. If we consider the three ratios of these four rates, most of the unknown quantities cancel out. Using the results of [29] we obtain

$$\frac{\Gamma(p\to\pi^0\mu^+)}{\Gamma(p\to\pi^0e^+)}=\frac{(b\kappa)^2+|s_\theta-2e^{-i\alpha_1}\zeta|^2}{(1+\kappa)^2+(2c_\theta)^2}, \tag{3.25}$$

$$\frac{\Gamma(p\to K^0e^+)}{\Gamma(p\to\pi^0e^+)}=R\frac{(b\kappa)^2+|2s_\theta-e^{-i\alpha_2}\zeta|^2}{(1+\kappa)^2+(2c_\theta)^2}, \tag{3.26}$$

$$\frac{\Gamma(p\to K^0\mu^+)}{\Gamma(p\to\pi^0e^+)}=R\frac{(1+b^2\kappa)^2+c_\theta^2}{(1+\kappa)^2+(2c_\theta)^2}, \tag{3.27}$$

with

$$R\equiv2\left(1-\frac{m_K^2}{m_p^2}\right)^2\left(\frac{1+\frac{m_p}{m_B}(D-F)}{1+D+F}\right)^2=0.105\pm0.005 \tag{3.28}$$

where D and F are chiral lagrangian parameters found in [30] to be $D + F = 1.267 \pm 0.003$ and $D - F = -0.341 \pm 0.016$ and $m_B = 1150$ MeV is an average baryon mass.

One sees that if the small effects of ζ are neglected, the three measurable ratios given in eqs. (3.25)-(3.27) depend in the $SU(5)$ model ($\kappa = 0$) on no unknown model parameters, giving three testable predictions. In the $SO(10)$ model they depend on only the one unknown model parameter κ, giving two testable predictions. The $SO(10)$ version is more interesting, however, in that the value of b can be extracted in two independent ways from these two predictions and compared to the value of b obtained from fitting the CKM mixing, which is $b = \frac{m_s}{m_d} V_{us}$. This is a highly non-trivial test of the model. And, indeed, this is one of the few models in the literature (maybe the only one) where proton decay allows a direct test of a model of quark and lepton masses and the mechanism of flavor mixing.

Let us now turn to the operators involving $W_{(-1/3)}$, which give the nucleon decay modes with neutrinos.

$$
\begin{aligned}
& e^{i(\Phi_u-\Phi_d)_1}[\overline{\tilde{u}^c}\not{W}_{(-2/3)}D_1\textstyle\sum_{J=1,2,3}(A_\Delta)_{1J}\nu_J] \\
& +\ e^{i(\Phi_d+2\Phi)_1}[\overline{[c_\theta\tilde{d}+s_\theta\tilde{s}]}\not{W}_{(-2/3)}D_1\textstyle\sum_{J=1,2}(A_\Delta)_{1J}\tilde{d}^c_J] \\
& +\ e^{i(\Phi_d+2\Phi)_2}[\overline{[-s_\theta\tilde{d}+c_\theta\tilde{s}]}\not{W}_{(-2/3)}D_2\textstyle\sum_{J=1,2}(A_\Delta)_{2J}\tilde{d}^c_J] \\
= \quad & e^{i(\Phi_u-\Phi_d)_1}[\overline{\tilde{u}^c}\not{W}_{(-2/3)}D_1(\nu_e+b\nu_\mu+ce^{-i\theta}\nu_\tau)] \\
& +\ e^{i(\Phi_d+2\Phi)_1}[\overline{[c_\theta\tilde{d}+s_\theta\tilde{s}]}\not{W}_{(-2/3)}D_1(\tilde{d}^c+b\tilde{s}^c)] \qquad (3.29)\\
& +\ e^{i(\Phi_d+2\Phi)_2}[\overline{[-s_\theta\tilde{d}+c_\theta\tilde{s}]}\not{W}_{(-2/3)}D_2\tilde{s}^c] \\
\longrightarrow \quad & e^{i(\Phi_u-\Phi_d)_1}D_1[\overline{\tilde{u}^c}\not{W}_{(-2/3)}(\nu_e+b\nu_\mu+ce^{-i\theta}\nu_\tau)] \\
& +\ e^{i(\Phi_d+2\Phi)_1}D_1[c_\theta\overline{\tilde{d}}\not{W}_{(-2/3)}\tilde{d}^c+(s_\theta+\lambda c_\theta)\overline{\tilde{s}}\not{W}_{(-2/3)}\tilde{d}^c \\
& +\ (c_\theta b-\lambda s_\theta)\overline{\tilde{d}}\not{W}_{(-2/3)}\tilde{s}^c+(s_\theta b)\overline{\tilde{s}}\not{W}_{(-2/3)}\tilde{s}^c],
\end{aligned}
$$

where λ is a complex number with magnitude D_2/D_1, and phase $(\Phi_d+2\Phi)_2-(\Phi_d+2\Phi)_1$. Both the magnitude and phase of λ are free parameters in this model, even if the mechanism of [16, 17, 18] is implemented.

The terms in eq. (3.29) allow us to write down the $d=6$ nucleon decay operators that come from the exchange of $W_{(-2/3)}$:

$$
e^{i(-\Phi_u+2\Phi_d+2\Phi)_1}\kappa\left(\frac{g_5^2}{2M_5^2}\right)\left[c_\theta\overline{\tilde{d}}\gamma_\mu\tilde{d}^c+(s_\theta+\lambda c_\theta)\overline{\tilde{s}}\gamma_\mu\tilde{d}^c+(c_\theta b-\lambda s_\theta)\overline{\tilde{d}}\gamma_\mu\tilde{s}^c+(s_\theta b)\overline{\tilde{s}}\gamma_\mu\tilde{s}^c\right][\overline{\nu}_*\gamma^\mu\tilde{u}^c]\,, \qquad (3.30)
$$

where $\nu_*\equiv(\nu_e+b\nu_\mu+ce^{-i\theta}\nu_\tau)$. This can be compared to the operator containing the neutrino fields coming from the exchange of $SU(5)$ gauge bosons, given in eq. (3.18), which after Fierzing is

$$\left(\frac{g_5^2}{2M_5^2}\right)\left[\left(c_\theta\bar{\tilde{d}}\gamma_\mu\tilde{d}^c+s_\theta\bar{\tilde{s}}\gamma_\mu\tilde{d}^c\right)\left(\overline{\nu}_e\gamma^\mu\tilde{u}^c\right)+\left(c_\theta\;\bar{\tilde{d}}\gamma_\mu\tilde{s}^c+s_\theta\;\bar{\tilde{s}}\gamma_\mu\tilde{s}^c\right)\left(\overline{\nu}_\mu\gamma^\mu\tilde{u}^c\right)\right]. \quad (3.31)$$

There are two measurable two-body decays of the proton that involve neutrinos, $p \to \pi^+\nu$ and $p \to K^+\nu$, since the flavor of the neutrino is not observable in practice. Thus, two more ratios of rates can be measured. In the $SU(5)$ version of the model, these do not depend on any additional model parameters, so two predictions result for the neutrino modes. In the $SO(10)$ version of the model, an additional complex model parameter enters, namely λ. For some ranges of $|\lambda|$, the phase of λ doesn't make much difference, so there would be one prediction for the neutrino modes. One can see form the definition of ν_* that this prediction would test the values of the parameters a and c that are given in eq. (2.13).

3.5 Conclusions

To sum up, in the $SU(5)$ version of the model, there are five measurable ratios of rates for the two-body proton decays, and so there are five predictions if the effects of the small parameter ζ can be neglected. These five predictions do not test the values of the quantities a, b, and c. On the other hand, if the effects of ζ could be measured with enough precision, the values of the phases α_1 and α_2 could be determined. If they are consistent with 0 or π, then it would support the mechanism for solving the Strong CP Problem proposed in []. In the $SO(10)$ version of the model, the same five measurable ratios of proton decay rates depend (if ζ is neglected) on two unknown model parameters, κ and λ. If the value of the latter is such that its phase does not matter, then there are three predictions. These do test the quantities a, b, and c, and thus would provide a highly significant discriminant between this model and others.

Chapter 4

A SINGLE SOURCE FOR ALL FLAVOR VIOLATION

4.1 Introduction

In this chapter, we are going to discuss another way to test the model proposed in the chapter 2. This model was shown to give several predictions for quark and lepton masses and mixing angles and for mixing angles within grand unified multiplets that are observable in proton decay. Here it is shown that the same master matrix A controls the flavor-changing processes mediated by a Standard-Model-singlet scalar that exists in the model, giving predictions for $\tau \to \mu\gamma$, $\tau \to e\gamma$, and $\mu \to e\gamma$. Therefore, certain parameters of the model may be measurable in three independent ways: by precise determination of neutrino and quark properties, by proton decay branching ratios, and through flavor-changing lepton decays.

In section 4.2, we will show how the model leads to predictions for flavor changing in the lepton sector. In section 4.3, we will analyze the flavor-changing effects that arise from the exchange of a Standard-Model-singlet scalar that exists in the model.

4.2 The Model with A Single Source for All Flavor Violation

We will start our discussion again from the Yukawa terms of the model

$$\begin{aligned}
\mathcal{L}_{Yuk} &= Y_i(\mathbf{10}_i\mathbf{10}_i)\langle\mathbf{5}_H\rangle + y_i(\mathbf{10}_i\overline{\mathbf{5}}_i)\langle\overline{\mathbf{5}}_H\rangle \\
&+ \tilde{Y}_i(\mathbf{10}_i\mathbf{10}_i)\langle\mathbf{45}_H\rangle + \tilde{y}_i(\mathbf{10}_i\overline{\mathbf{5}}_i)\langle\overline{\mathbf{45}}_H\rangle \\
&+ (\lambda_i/M_R)(\overline{\mathbf{5}}_i\overline{\mathbf{5}}_i)\langle\mathbf{5}_H\rangle\langle\mathbf{5}_H\rangle \\
&+ Y'_{AB}(\mathbf{5}_A\overline{\mathbf{5}}_B)\langle\mathbf{1}_H\rangle + y'_{Ai}(\mathbf{5}_A\overline{\mathbf{5}}_i)\langle\mathbf{1}'_{Hi}\rangle,
\end{aligned} \tag{4.1}$$

where the subscript H denotes Higgs multiplets. Repeated indices (i, A, or B) are summed over. Note that the vacuum expectation values of $\mathbf{1}'_{Hi}$ spontaneously break the abelian family symmetries; so that the last term in eq. (4.1), which mixes the $\overline{\mathbf{5}}_i$ and $\overline{\mathbf{5}}'_A$, does not respect the family symmetries and can give flavor violation. It is important for the predictivity of the model that the last two terms in eq. (4.1) involve only $SU(5)$-singlet Higgs fields, as otherwise the master matrix would be different for quarks and leptons. This can be ensured by another abelian symmetry that prevents the SU(5) adjoint Higgs field from coupling in these terms [28].

Let us recall that $[Y'_{AB}\langle \mathbf{1}_H \rangle] \equiv M_{AB}$ and $[y'_{Ai}\langle \mathbf{1}'_{Hi} \rangle] \equiv \Delta_{Ai}$ and first examine the down-type quarks. These have a $(3+N) \times (3+N)$ mass matrix of the form

$$\left(d_{(\mathbf{10})}, D_{(\mathbf{5}')}\right) \begin{pmatrix} m_d & 0 \\ \Delta & M \end{pmatrix} \begin{pmatrix} d^c_{(\overline{\mathbf{5}})} \\ D^c_{(\overline{\mathbf{5}}')} \end{pmatrix}, \tag{4.2}$$

The block diagonalization is carried out by a bi-unitary transformation of the $(3+N) \times (3+N)$ mass matrix:

$$\begin{pmatrix} m_d & 0 \\ \Delta & M \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} I & G \\ -G & I \end{pmatrix}}_{\cong U_L} \begin{pmatrix} m_d & 0 \\ \Delta & M \end{pmatrix} \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\cong U_R} = \begin{pmatrix} M_d & 0 \\ 0 & M' \end{pmatrix}. \tag{4.3}$$

Here the elements of G are small and U_L is approximately diagonal, because the elements of m_d are very small compared to those of M and Δ. In earlier chapters, we could neglect the matrix G, but for the calculations we will do in this chapter, it must be included. One can give exact expressions for the matrices A, B, C, D, and G, which will be useful in section 4.3. Defining $T \equiv M^{-1}\Delta$, one can write

$$\begin{aligned} A &\equiv [I + T^\dagger T]^{-1/2}, \\ B &\equiv [I + T^\dagger T]^{-1/2} T^\dagger = A T^\dagger = T^\dagger [I + T T^\dagger]^{-1/2} \equiv T^\dagger D \\ C &\equiv -T[I + T^\dagger T]^{-1/2} = -TA = -[I + T T^\dagger]^{-1/2} T \equiv -DT \\ D &\equiv [I + T T^\dagger]^{-1/2} \\ G &\equiv -M^{-1\dagger} D^{2\dagger} T m_d^\dagger. \end{aligned} \tag{4.4}$$

Since the elements of Δ and M are of the same order, the elements of T are of $O(1)$, and the matrices A, B, C, D have off-diagonal elements of $O(1)$. As we showed in earlier chapters, one can write the effective 3×3 mass matrices of the Standard Model up-type quarks, down-type quarks, charged leptons, and neutrinos a

$$M_u = m_u, \qquad M_d = m_d A, \qquad M_\ell = A^T m_\ell, \qquad M_\nu = A^T m_\nu A, \tag{4.5}$$

where A is of the form $A = \mathcal{D}\ A_\Delta\ \mathcal{U}$. And as discussed before, the matrix $\mathcal{U}$ can be absorbed into redefined right-handed down quarks and the left-handed lepton doublets. Similarly, the phases in $\mathcal{D}$ can be absorbed into redefined fields. The diagonal real matrix $|\mathcal{D}|$ can be absorbed into redefinitions of the original diagonal mass matrices as follows: $\overline{m}_d \equiv m_d|\mathcal{D}|$, $\overline{m}_\ell \equiv m_\ell|\mathcal{D}|$, $\overline{m}_\nu \equiv m_\nu|\mathcal{D}|^2$, and $\overline{m}_u \equiv m_u$. After these redefinitions, the mass matrices of the three light families take a new form and eq. (4.5) can be rewritten a

$$\overline{M}_u = \overline{m}_u, \qquad \overline{M}_d = \overline{m}_d A_\Delta, \qquad \overline{M}_\ell = A_\Delta^T \overline{m}_\ell, \qquad \overline{M}_\nu = A_\Delta^T \overline{m}_\nu A_\Delta. \tag{4.6}$$

4.3 Flavor Changing from Singlet Scalar Exchange

In this section we consider the effects of the scalar field $\mathbf{1}_H$ that couples to the vector-like fermions to produces the $N \times N$ mass matrix $M_{AB} = Y'_{AB}\langle \mathbf{1}_H \rangle$. We will henceforth call this singlet Higgs field $\Omega = \langle \Omega \rangle + \tilde{\Omega}$. The exchange of the $\tilde{\Omega}$ will mediate flavor-changing processes. For these effects to be observable in practice, in this chapter, we must assume that the scale M_*, which characterizes the mass and vacuum expectation value of Ω, is not too much larger than the weak scale. We will assume that it is of order 1 TeV to several TeV.

Let us look first at the Yukawa couplings of $\tilde{\Omega}$ to the down-type quarks. In the same notation of eq. (4.2), the Yukawa couplings of $\tilde{\Omega}$ to the down-type quarks is given by

$$\left(d_{(\mathbf{10})}, D_{(\mathbf{5'})}\right) \begin{pmatrix} 0 & 0 \\ 0 & M/\langle \Omega \rangle \end{pmatrix} \begin{pmatrix} d^c_{(\overline{\mathbf{5}})} \\ D^c_{(\overline{\mathbf{5}}')} \end{pmatrix} \tilde{\Omega}, \tag{4.7}$$

When one block-diagonalizes to separate the light and heavy fermion stats, this Yukawa matrix is transformed by the unitary matrices U_L and U_R as in eq. (4.3):

$$\begin{pmatrix} 0 & 0 \\ 0 & M/\langle\Omega\rangle \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} I & G^\dagger \\ -G & I \end{pmatrix}}_{\cong U_L^\dagger} \begin{pmatrix} 0 & 0 \\ 0 & M/\langle\Omega\rangle \end{pmatrix} \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\cong U_R} = \frac{1}{\langle\Omega\rangle} \begin{pmatrix} G^\dagger MC & G^\dagger MD \\ MC & MD \end{pmatrix}. \tag{4.8}$$

So the effective Yukawa coupling of $\tilde{\Omega}$ to the three light down-type quarks d, s, and b, is given by $d_i(G^\dagger MC)_{ij}d_j^c\ (\tilde{\Omega}/\langle\Omega\rangle)$. Remarkably, the effective Yukawa coupling matrix here, $G^\dagger MC/\langle\Omega\rangle$, which we will call Y_d, can be written simply in terms of the master matrix A. Using eq. (4.4), one obtains

$$\begin{aligned} Y_d\langle\Omega\rangle \cong G^\dagger MC &\cong (-m_d T^\dagger D^2 M^{-1})M(-TA) \\ &= m_d T^\dagger D^2 TA \\ &= m_d T^\dagger (I + TT^\dagger)^{-1} TA \\ &= m_d T^\dagger T (I + T^\dagger T)^{-1} A \\ &= m_d (A^{-2} - I)A^3 = m_d(A - AA^\dagger A). \end{aligned} \tag{4.9}$$

In going from line 3 to line 4, we have used the fact that $(I+TT^\dagger)^{-1}T = T(I+T^\dagger T)^{-1}$, as can easily be seen by expanding out the expressions is parentheses as power series. In the last line, we have used the fact that $A \equiv (I+T^\dagger T)^{-1/2}$, and that A is hermitian. Let us rewrite this expression in terms of the triangular matrix A_Δ, since that is the matrix whose elements are known. Since $A = \mathcal{D}A_\Delta\mathcal{U}$, we hav

$$\begin{aligned} Y_d\langle\Omega\rangle &= m_d(A - AA^\dagger A) \\ &= m_d[\mathcal{D}A_\Delta\mathcal{U} - (\mathcal{D}A_\Delta\mathcal{U})(\mathcal{U}^\dagger A_\Delta^\dagger \mathcal{D}^*)(\mathcal{D}A_\Delta\mathcal{U})] \\ &= m_d\mathcal{D}A_\Delta\ [I - A_\Delta^\dagger|\mathcal{D}|^2 A_\Delta]\ \mathcal{U}. \end{aligned} \tag{4.10}$$

The factor $\mathcal{U}$ on the right will be absorbed by the re-definition of the right-handed down-quark fields. Doing this re-definition, and using the fact that $m_d\mathcal{D}A_\Delta = \overline{m}_d A_\Delta \equiv \overline{M}_d$, the Yukawa coupling matrix takes the form

$$Y_d\langle\Omega\rangle = \overline{M}_d[I - A_\Delta^\dagger|\mathcal{D}|^2 A_\Delta]. \tag{4.11}$$

The mass matrix $\overline{M}_d$ is diagonalized by a bi-unitary transformation to give $V_{dL}^\dagger \overline{M}_d V_{dR} = M_d^{phys} = \mathrm{diag}(m_d, m_s, m_b)$. As we have shown in the previous chapters, the matrix V_{dL} should be the CKM matrix, while the matrix V_{dR} differs from the identity matrix by terms of order $O(m_s^2/m_b^2)$, $O(m_d m_s/m_b^2)$, and $O(m_d^2/m_b^2)$, which can be neglected. It is clear then that in the physical basis of the down quarks

$$Y_d^{phys} \cong \frac{1}{\langle\Omega\rangle} M_d^{phys}[I - A_\Delta^\dagger |\mathcal{D}|^2 A_\Delta]. \tag{4.12}$$

Obviously, only the second term in the brackets leads to flavor changing. Let us parametrize the unknown matrix $\mathcal{D}$ as $\mathrm{diag}(\delta, \epsilon, \zeta)$. The flavor-changing Yukawa coupling matrix of the $\tilde{\Omega}$ to the physical down-type quarks is of the form $d_i (Y_d^{FC})_{ij} d_j^c \tilde{\Omega}$, wher

$$\begin{aligned} Y_d^{FC} &= \tfrac{-1}{\langle\Omega\rangle} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ ce^{-i\theta} & a & 1 \end{pmatrix} \begin{pmatrix} |\delta|^2 & 0 & 0 \\ 0 & |\epsilon|^2 & 0 \\ 0 & 0 & |\zeta|^2 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \\ &= \tfrac{-1}{\langle\Omega\rangle} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \begin{pmatrix} \Delta_{dd} & \Delta_{ds} & \Delta_{db} \\ \Delta_{sd} & \Delta_{ss} & \Delta_{sb} \\ \Delta_{bd} & \Delta_{bs} & \Delta_{bb} \end{pmatrix}, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \Delta_{ds} &= \Delta_{sd} = |\delta|^2 b, \\ \Delta_{db} &= \Delta_{bd}^* = |\delta|^2 ce^{i\theta}, \\ \Delta_{sb} &= \Delta_{bs}^* = |\epsilon|^2 a + |\delta|^2 bce^{i\theta}. \end{aligned} \tag{4.14}$$

Note that the flavor-changing (i.e. off-diagonal) elements of Y_d^{FC} depend only on two unknown combinations of parameters: $|\delta|^2/\langle\Omega\rangle$ and $|\epsilon|^2/\langle\Omega\rangle$. Note also that Δ_{ds} and Δ_{sd} are real in the physical basis of the quarks, so that the ϵ_K parameter of the $K^0 - \overline{K^0}$ system, does not put constraints on flavor changing coming from the singlet scalar exchange.

The charged-lepton sector is identical except for a left-right transposition. So writing the flavor-changing Yukawa coupling matrix of the $\tilde{\Omega}$ to the physical charged leptons as $\ell_i^+ (Y_\ell^{FC})_{ij} \ell_j^- \tilde{\Omega}$, one finds

$$\begin{aligned} Y_\ell^{FC} &= \tfrac{1}{\langle\Omega\rangle} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ ce^{-i\theta} & a & 1 \end{pmatrix} \begin{pmatrix} |\delta|^2 & 0 & 0 \\ 0 & |\epsilon|^2 & 0 \\ 0 & 0 & |\zeta|^2 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \\ &= \tfrac{1}{\langle\Omega\rangle} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \begin{pmatrix} \Delta_{ee} & \Delta_{e\mu} & \Delta_{e\tau} \\ \Delta_{\mu e} & \Delta_{\mu\mu} & \Delta_{\mu\tau} \\ \Delta_{\tau e} & \Delta_{\tau\mu} & \Delta_{\tau\tau} \end{pmatrix}, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \Delta_{e\mu} &= \Delta_{\mu e} = |\delta|^2 b, \\ \Delta_{e\tau} &= \Delta_{\tau e}^* = |\delta|^2 ce^{i\theta}, \\ \Delta_{\mu\tau} &= \Delta_{\tau\mu}^* = |\epsilon|^2 a + |\delta|^2 bce^{i\theta}. \end{aligned} \tag{4.16}$$

The flavor-changing Yukawa couplings come into the processes $\ell_1 \to \ell_2\gamma$ through-two loop diagrams, as shown in [31, 32]. The specific diagrams that dominate in this model have the vector-like fermions running around the loop that gives an effective $\tilde{\Omega}$-photon-photon coupling. The resulting branching ratios for the flavor-changing lepton decays can be expressed in terms of the quantities given in eq. (4.16) as follows [33]:

$$BR(\ell_1 \to \ell_2\gamma) \cong 24 \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{v}{\langle\Omega\rangle}\right)^4 |\Delta_{\ell_1\ell_2}|^2. \tag{4.17}$$

One prediction is that

$$\frac{BR(\tau \to e\gamma)}{BR(\mu \to e\gamma)} \cong \left|\frac{\Delta_{e\tau}}{\Delta_{e\mu}}\right|^2 = \left|\frac{c}{b}\right|^2 = \left(\frac{m_b}{m_s}\right)^2 \left|\frac{V_{ub}}{V_{us}}\right|^2 \approx 1. \tag{4.18}$$

If one assumes that the expression for $\Delta_{\mu\tau}$ in eq. (4.16) is dominated by the $|\delta|^2$ term, then one would also have the prediction

$$BR(\tau \to \mu\gamma) \cong |c|^2 BR(\mu \to e\gamma) \cong 16 \cdot BR(\mu \to e\gamma). \tag{4.19}$$

Given the present limit [34] that $BR(\mu \to e\gamma) < 5.7 \times 10^{-13}$, this would gives a prediction that $BR(\tau \to \mu\gamma) < 10^{-11}$. This is well below even what is expected to be observable at a super-c-τ factory [35]. On the other hand, the branching ratio for this decay can be much larger if $\Delta_{\mu\tau}$ in eq. (4.16) is dominated by the $|\epsilon|^2$ term. As we will show below, there is an approximate theoretical bound that $|\epsilon|^2 < 1/2$. This would give

$$BR(\tau \to \mu\gamma) \leq 1.5 \times 10^{-9} \left(\frac{1 \text{ TeV}}{\langle\Omega\rangle}\right)^4. \tag{4.20}$$

The flavor-changing processes involving quarks do not get large enough contributions from the exchange of the singlet scalar $\tilde{\Omega}$ to stand out from Standard Model contributions. For instance, the coefficient of $(\overline{s}d)(\overline{s}d)$ operators is found from eqs. (4.13)-(4.14) to be of order $\frac{m_s^2}{\langle\Omega\rangle^2 M_{\tilde{\Omega}}^2} b^2|\delta|^4 < 10^{-15}(\text{GeV})^{-2}\left(\frac{1 \text{ TeV}}{M_*}\right)^4$, where we have used an upper bound on $|\delta|^2$ that is derived below. (From the first line of eq. (4.24) one finds that $|\delta|^2$ must be less than $(1+b^2)^{-1} \sim 1/17$.)

Let us now consider the parameters δ, ϵ, ζ. While the matrix $\mathcal{D} = \text{diag}(\delta, \epsilon, \zeta)$ is not known a priori, it is nevertheless possible to derive strict upper bounds on the parameters $|\delta|$, $|\epsilon|$, and $|\zeta|$ from the properties of the master matrix A. From the fact that $A \equiv (I + T^\dagger T)^{-1/2}$ and that $A = \mathcal{D}A_\Delta\mathcal{U}$, one has that

$$\begin{gathered} A\, A^\dagger = \mathcal{D}A_\Delta A_\Delta^\dagger \mathcal{D}^* = (I + T^\dagger T)^{-1} \\ (\mathcal{D}A_\Delta A_\Delta^\dagger \mathcal{D}^*)^{-1} - I = T^\dagger T. \end{gathered} \tag{4.21}$$

Computing the matrix on the left side of the above equation, one obtains

$$\begin{bmatrix} 1/|\delta|^2 & -b/(\delta^*\epsilon) & (ab - ce^{i\theta})/(\delta^*\zeta) \\ -b/(\delta\epsilon^*) & (1+b^2)/|\epsilon|^2 & -(a + ab^2 - bce^{i\theta})/(\epsilon^*\zeta) \\ (ab - ce^{-i\theta})/(\delta\zeta^*) & -(a + ab^2 - bce^{-i\theta})/(\epsilon\zeta^*) & (1 + a^2 + |ab - ce^{i\theta}|^2)/|\zeta|^2 \end{bmatrix} - I = T^\dagger T. \tag{4.22}$$

For any matrix T, there is an inequality that must be satisfied by the elements of $T^\dagger T$. namely

$$|(T^\dagger T)_{ij}|^2 \leq (T^\dagger T)_{ii}(T^\dagger T)_{jj}, \quad \forall\ i, j. \tag{4.23}$$

This is obvious if we write $T_{ij} = (\vec{t}_{(j)})_i$, where $\vec{t}_{(i)}, i = 1, 2, 3$, are three complex vectors. Then the inequality is just seen to be the statement that $|\vec{t}^*_{(i)} \cdot \vec{t}_{(j)}| \leq |\vec{t}_{(i)}|\ |\vec{t}_{(j)}|$. From this inequality with $(i, j) = (1, 2)$, $(1, 3)$, and $(2, 3)$, respectively, one obtains after a little algebra

$$\begin{aligned}
(1+b^2)|\delta|^2 + |\epsilon|^2 &\leq 1, \\
(1+a^2+|ab-ce^{i\theta}|^2)|\delta|^2 + |\zeta|^2 &\leq 1+a^2, \\
(1+a^2+|ab-ce^{i\theta}|^2)|\epsilon|^2 + (1+b^2)|\zeta|^2 &\leq 1+b^2+c^2.
\end{aligned} \tag{4.24}$$

using the values of a, b, c and θ given in eq. (2.13), the third equation of eq. (4.24) gives an upper bound on $|\epsilon|^2$ of approximately $1/2$.

In this chapter, we have assumed that the scale M_* of $\langle\Omega\rangle$ is in the low TeV range, otherwise the flavor-changing effects from exchanges of $\tilde{\Omega}$ would be hopelessly small. But then one must run the Yukawa couplings Y'_{AB} and y'_{Aj} shown in the last line of eq. (4.1) from the GUT scale down to the scale M_*. If these ran differently for the leptons and quarks, it would make the matrices Δ and M in eq. (4.2) different for quarks and leptons, and thus also make the master matrix A different for quarks and leptons. That could destroy the predictivity of the model. If one considered only gluon loops in the running there is no problem, as the effect would be to increase Δ and M by the same factor for quarks relative to leptons. This factor would cancel in the ratio $T = M^{-1}\Delta$, and therefore also in $A = [I + T^\dagger T]^{-1/2}$. However, the gluon loops could do the following: they could increase the Yukawa couplings Y'_{AB} and y'_{Aj} for quarks so much that the effect of these Yukawas on their own running could be much more significant for quarks than for leptons. That would make the forms of the matrices Δ and M — and therefore the form of A — different for quarks and leptons.

There are two ways to avoid this problem. One is that all the Yukawas Y'_{AB} and y'_{Ai} remained small for the whole range from M_{GUT} to M_*. This has a drawback, however. If these Yukawa couplings Y'_{AB} are small compared to 1, then the VEV $\langle\Omega\rangle$ would have to be large compared to a TeV to make the vector-like fermions in $\mathbf{5}' + \overline{\mathbf{5}}'$

heavy enough not to be seen. That would suppress flavor-changing effects from $\tilde{\Omega}$ exchange.

A cleaner way to avoid the problem is to assume the following two conditions: (a) The Yukawa couplings y'_{Ai} that generate the mass matrix Δ are small compared to 1, and the VEV of the Higgs fields $\mathbf{1}'_{Ai}$ correspondingly large compared to a TeV. (That would have the additional advantage of making flavor changing from the exchange of these scalars negligible.) (b) The Yukawa coupling matrix Y'_{AB} is proportional to the identity matrix, which could be the result of a flavor symmetry that acted on the vector-like families. Then even if gluon loops drove Y'_{AB} to be of order 1 at low scales, that would not affect the form of Y'_{AB}.

Another theoretical issue raised by M_* being near the weak scale is that the spontaneous breaking of the family symmetry group G_F would cause cosmological domain walls. This breaking is done by the VEVs $\langle \mathbf{1}_{Hi} \rangle$. To avoid overclosing the universe, these domain walls would have to be "inflated away". One simple possibility is that G_F is actually broken at a scale much higher than M_* but only induces a VEV for $\mathbf{1}_{Hi}$ that is of order M_*. For example, consider the terms $\mathcal{L}(\sigma) = -\frac{1}{2}M^2\sigma^2 + \overline{\psi}\psi\sigma$, where the scalar field σ and fermion bilinear $\overline{\psi}\psi$ are odd under a Z_2 and M is of order the GUT scale. Let the fermion bilinear get a condensate $\langle\overline{\psi}\psi\rangle = \Lambda^3$, where $\Lambda \sim (M^2 M_*)^{1/3}$, which is many orders of magnitude bigger than M_*. The Z_2 will be broken at the scale Λ, whereas $\langle\sigma\rangle = \Lambda^3/M^2 \sim M_*$.

4.4 Conclusions

The model of flavor symmetry and flavor violation proposed in [28] has the virtue that it is (i) conceptually simple, (ii) explains some of the qualitative features of the quark and lepton spectrum (*e.g.* the MNS angles being much larger than the CKM angles), and (c) is highly predictive. As such, it can provide a kind of "benchmark" for seeing how large various kinds of flavor-changing processes might be expected to be.

The model is of the "lopsided" type [5, 6, 7, 8, 9, 10, 11, 12, 13], which tends to give relatively large flavor-changing effects. In models with symmetric mass matrices, which are very common in the literature, off-diagonal Yukawa couplings Y_{ij} are typically proportional to $\sqrt{m_i m_j}/v$. In lopsided models, however, Y_{ij} and Y_{ji} are very different in magnitude from each other, being proportional to m_i/v and m_j/v. This is the reason for the name “lopsided", and why the flavor-changing effects tend to be relatively large.

It is likely, then, that the flavor-changing Yukawa couplings given in eqs. (4.13)-(4.16) (with the bounds in eq. (4.24)) are typical of what would expect for a new scalar field. We see that if the scale of new physics M_* is of order 1 TeV, there is good hope of eventually seeing the processes $\tau \to \mu\gamma$, $\tau \to e\gamma$, and $\mu \to e\gamma$. One also sees from this model, that observing such processes can confirm or rule out specific models of the origin of flavor and flavor violation.

Chapter 5

MODEL OF QUARK AND LEPTON MIXING AND MASS HIERARCHY

5.1 Introduction

It is shown that an idea proposed in the paper by Babu and Barr in 1996 that relates in a qualitatively correct way the inter-family mass hierarchies of the up quarks, down quarks, charged leptons, and neutrinos, can be combined with the idea which relates quark mixing and neutrino mixing discussed in the chapter 2. In this chapter, we will propose a possible way combining these two ideas. In the resulting model, the entire flavor structure of the quarks and leptons can be expressed in terms of two master matrices: a diagonal matrix H that gives the inter-family mass ratios, and an off-diagonal matrix A that controls all flavor mixing.

The flavor problem has two aspects: (a) explaining the pattern of mixing angles for the quarks and the leptons and (b) explaining the pattern of inter-family mass hierarchies for the up-type quarks, down-type quarks, charged leptons, and neutrinos. We will show in this chapter that an idea for explaining the mass hierarchies proposed in 1996 by K.S. Babu and S.M. Barr [5] can be successfully combined with the idea discussed in chapter 2 [28] for explaining the mixing angles. We shall refer to these as the "BB idea" and the "BC idea" respectively. The two ideas are actually complementary, and by combining them a model emerges that is simpler and more explanatory than either by itself.

The BB idea was based on the observation that the inter-family mass hierarchy is strongest for the up quarks (u, c, t), is intermediate for the down quarks (d, s, b) and charged leptons (e, μ, τ), and seemingly is weakest for the neutrinos(ν_1, ν_2, ν_3). It was noted in [5] that this can be explained if the mass matrices of these four types of fermions have the forms $M_u = H\, m_u H$, $M_d = H\, m_d$, $M_\ell = m_\ell$ and $M_\nu = m_\nu$, where H is is a diagonal matrix with a hierarchy among its elements, and where m_u, m_d, m_ℓ and m_ν are matrices with no special form, i.e. with all elements of the same order. Moreover, it was shown how these insertions of the hierarchy matrix H can arise in a simple and natural way from mixing between the three chiral families of usual fermions and extra vector-like fermions in $10' + \overline{10}'$ multiplets.

The BC idea was based on the observation that inter-family mixing is weaker for the left-handed quarks than for the left-handed leptons, and that this can be explained if the fermion mass matrices have the form $M_u = m_u$, $M = m_d A$, $M_\ell = A^T m_\ell$, and $M_\nu = A^T m_\nu A$, where A is a "master matrix" that controls all inter-family mixing, and where the matrices m_u, m_d, m_ℓ, and m_ν are diagonal and hierarchical and therefore have no mixing. Moreover, it was shown in [28] how these insertions of the master matrix A can arise in a simple and natural way from mixing between the three chiral families of usual fermions and extra vector-like fermions in $5' + \overline{5}'$ multiplets.

These two ideas are clearly similar in a number of respects. In the BB idea one matrix H controls all the inter-family mass hierarchies, while in the BC idea one matrix A controls all inter-family mixing. In both ideas, these master matrices arise from the mixing of the three chiral families with vector-like fermions. And the forms of the fermion mass matrices that arise in both schemes are products of diagonal, hierarchical matrices with non-diagonal, non-hierarchical matrices.

In section 5.2, we shall briefly review the BB idea. In section 5.3, we will show how BB idea can be combined with BC idea. In section 5.4, we deal with the question of introducing breaking of $SU(5)$ into the fermion mass matrices so as to avoid the well-known minimal $SU(5)$ relations $m_d = m_e$, $m_s = m_\mu$ and $m_b = m_\tau$ at the unification scale. In section 5.5, we show that the relative magnitudes of the various inter-family mass hierarchies (of the up quarks, down quarks, charged leptons, and neutrinos) come out to be of the right magnitude in the combined scheme without parameters having to be tuned to unnatural values.

5.2 Brief Review of The BB Idea

The BB idea was that every fermion 10 multiplet in a Yukawa term is accompanied by a factor in the mass matrix of a hierarchical, diagonal matrix H, which one can write as $H = \mathrm{diag}(\alpha, \beta, 1)\, h$, where $\alpha \ll \beta \ll 1$. This can happen as the result of the mixing of the 10 multiplets in the usual three chiral families, which we denote by $\mathbf{10}^U_i + \overline{\mathbf{5}}^U_i$, with extra vector-like 10 multiplets, which we denote by $\mathbf{10}^E{}_i + \overline{\mathbf{10}}^E{}_i$ $(i = 1, 2, 3)$. Let the "underlying" Yukawa terms that give electroweak-breaking quark and lepton masses be of the form

$$(\mathbf{10}^U_i Y^u_{ij} \mathbf{10}^U_j \mathbf{5}_H + (\mathbf{10}^U_i Y_{ij} \overline{\mathbf{5}}^U_j)\overline{\mathbf{5}}_H + (\mathbf{10}^U_i y_{ij} \overline{\mathbf{5}}^U_j)\overline{\mathbf{45}}_H + (\overline{\mathbf{5}}^U_i Y^\nu_{ij} \overline{\mathbf{5}}^U_j)\mathbf{5}_H \mathbf{5}_H / M_R. \tag{5.1}$$

The role of the term with the $\overline{\mathbf{45}}_H$ of Higgs fields is to give different contributions to the mass matrices of the down quarks and charged leptons [23, 14] and thus avoid the "bad" predictions of minimal SU(5) that $m_e = m_d$, and $m_\mu = m_s$ at the GUT scale. Suppose that $\mathbf{10}^U_i$ and $\mathbf{10}^E{}_i$ mix in a family-diagonal way to produce a light linear combination $\mathbf{10}^{\mathbf{L}}{}_i$ that contains Standard Model fermions and an orthogonal linear combination $\mathbf{10}^H_i$ that is superheavy. Then one can write

$$\mathbf{10}^U_i = \cos\theta_i \mathbf{10}^{\mathbf{L}}{}_i + \sin\theta_i \mathbf{10}^H_i. \tag{5.2}$$

Substituting this into eq. (5.1), one obtains for the effective Yukawa terms of the Standard Model fermions

$$(\mathbf{10}^{\mathbf{L}}{}_i \cos\theta_i Y^u_{ij} \cos\theta_j \mathbf{10}^{\mathbf{L}}{}_j)\mathbf{5}_H + (\mathbf{10}^{\mathbf{L}}{}_i \cos\theta_i Y_{ij} \overline{\mathbf{5}}^U_j)\overline{\mathbf{5}}_H + (\mathbf{10}^{\mathbf{L}}{}_i \cos\theta_i y_{ij} \overline{\mathbf{5}}^U_j)\overline{\mathbf{45}}_H + (\overline{\mathbf{5}}^U_i Y^\nu_{ij} \overline{\mathbf{5}}^U_j)\mathbf{5}_H \mathbf{5}_H / M_R. \tag{5.3}$$

Therefore, the effective quark and lepton mass terms of the Standard Model quarks and leptons can be written

$$\begin{aligned} M_u &= H\, m_u\, H, \\ M_d &= H\, m_d, \\ M_\ell &= m_\ell\, H, \\ M_\nu &= m_\nu, \end{aligned} \tag{5.4}$$

where

$$H = \begin{pmatrix} \cos\theta_1 & 0 & 0 \\ 0 & \cos\theta_2 & 0 \\ 0 & 0 & \cos\theta_3 \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} h, \tag{5.5}$$

and where $(m_u)_{ij} = Y^u_{ij} v_5$, $(m_d)_{ij} = Y_{ij} v_{\overline{5}} + y_{ij} v_{\overline{45}}$, $(m_\ell)_{ij} = Y_{ij} v_{\overline{5}} - 3y_{ij} v_{\overline{45}}$, and $(m_\nu)_{ij} = Y^\nu_{ij}(v_5^2/M_R)$. These four "underlying" mass matrices m_u ,m_d, m_ℓ, and m_ν are not assumed to have any special form, and therefore for each of them one expects all the elements to be roughly of the same order. From eqs. (5.4) and (5.5) one has

$$\begin{aligned} M_u &\sim \begin{pmatrix} \alpha^2 & \alpha\beta & \alpha \\ \alpha\beta & \beta^2 & \beta \\ \alpha & \beta & 1 \end{pmatrix} \mu_u, \quad M_\nu \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \mu_\nu \\ M_d &\sim \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ 1 & 1 & 1 \end{pmatrix} \mu_d, \qquad M_\ell \sim \begin{pmatrix} \alpha & \beta & 1 \\ \alpha & \beta & 1 \\ \alpha & \beta & 1 \end{pmatrix} \mu_\ell, \end{aligned} \tag{5.6}$$

where "$\sim$" means that the various elements are of the given order of magnitude. This obviously gives

$$\begin{aligned} m_u : m_c : m_t &\sim \alpha^2 : \beta^2 : 1 \\ m_d : m_s : m_b &\sim \alpha : \beta : 1 \\ m_e : m_\mu : m_\tau &\sim \alpha : \beta : 1 \\ m_{\nu_1} : m_{\nu_2} : m_{\nu_3} &\sim 1 : 1 : 1. \end{aligned} \tag{5.7}$$

This reproduces well, in a qualitative way, the strengths of the inter-family mass hierarchies of the different types of fermions. Also from inspection of eq. (5.6) it is apparent that

$$U_{MNS} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad V_{CKM} \sim \begin{pmatrix} 1 & \alpha/\beta & \alpha \\ \alpha/\beta & 1 & \beta \\ \alpha & \beta & 1 \end{pmatrix}. \tag{5.8}$$

This gives $O(1)$ MNS mixing angles and small CKM mixing angles, with $|V_{ub}| \sim |V_{us}V_{cb}|$, which also is qualitatively correct. On the other hand, since there are no constraints on the forms of the four underlying 3×3 mass matrices m_u, m_d, m_ℓ, and m_ν, the BB idea in this form has many free parameters and can only make qualitative post-dictions rather than precise quantitative predictions.

5.3 Combining The BB and BC Ideas

In the BB idea, all the inter-family mass hierarchies come from the single matrix H, while in the BC idea all the inter-family mixing comes from the single matrix A. The question naturally arises whether these two ideas can be combined in such a way that the whole flavor structure can be accounted for with only the matrices A and H, thereby producing a more predictive and explanatory model. The answer is yes, as we shall now show by describing a specific model that does this.

The fermion content of the model consists of the following SU(5) multiplets:

$$(\mathbf{10}_i^U + \overline{\mathbf{5}}_i^U)_{i=1,2,3} \quad + \quad (\mathbf{10}_A^E + \overline{\mathbf{10}}_A^E)_{A=1,2,3} \quad + \quad (\overline{\mathbf{5}}_m^E + \mathbf{5}_m^E)_{m=1,2,\ldots N}. \tag{5.9}$$

Yukawa terms involving only $\mathbf{10}_i^U$ and $\overline{\mathbf{5}}_i^U$ will give rise to "underlying" mass matrices that get multiplied by factors of the matrices H and A. In order for H and A to account for *all* the flavor structure, both the hierarchies among the masses and the pattern of mixing angles, the underlying mass matrices should have a trivial flavor structure, *i.e.* they should be proportional to the identity matrix.. This can be the case if there is an SO(3) family symmetry under which the $\mathbf{10}_i^U$ and $\overline{\mathbf{5}}_i^U$ transform as triplets. The underlying Yukawa terms would then have the form

$$Y_u(\mathbf{10}_i^U\ \mathbf{10}_i^U)\ \mathbf{5}_H \quad + \quad Y_d(\mathbf{10}_i^U\ \overline{\mathbf{5}}_i^U)\ \overline{\mathbf{5}}_H \quad + \quad Y_\nu(\overline{\mathbf{5}}_i^U\ \overline{\mathbf{5}}_i^U)\ \mathbf{5}_H\mathbf{5}_H/M_R. \tag{5.10}$$

Note that unlike eq. (5.1) there is no term here with the $\overline{\mathbf{45}}_H$ of Higgs fields. Since all the underlying Yukawa terms must be flavor-independent, due to the SO(3) symmetry, adding a term with the $\overline{\mathbf{45}}_H$ in eq. (5.10) would still leave the down quark and charged lepton mass matrices proportional to each other at the GUT scale. Therefore, the group-theoretical factors needed to avoid the "bad" minimal SU(5) relation $M_d = M_\ell^T$ must appear in either the H or A matrices. In the model we are describing, they will appear in the H matrix, as will be seen.

The matrix A arises, in exactly the manner explained earlier, from the mixing of the $\overline{\mathbf{5}}_i^U$ with the "extra" $\overline{\mathbf{5}}_m^E$, which are assumed not to transform under any flavor symmetry. Let there be at least two Standard-Models-singlet Higgs fields that are triplets under SO(3), denoted by $\mathbf{1}_H^{ni}$, where n labels the Higgs triplet and i is the SO(3) index. Then one can write the following mass and Yukawa terms for the fermion 5 multiplets:

$$\begin{aligned} & M_{mn}(\mathbf{5}_m^E\ \overline{\mathbf{5}}_n^E)\ +\ y_{mn}(\mathbf{5}_m^E\ \overline{\mathbf{5}}_i^U)\ \langle \mathbf{1}_H^{ni}\rangle \\ & = \mathbf{5}'_m(M_{mn}\overline{\mathbf{5}}'_m\ +\ \Delta_{mi}\overline{\mathbf{5}}_i^0), \end{aligned} \tag{5.11}$$

where $\Delta_{mi} = \sum_n y_{mn}\langle \mathbf{1}_H^{ni}\rangle$. We assume that the matrices M and Δ are superheavy and of the same order. (For example, they may both be of order the GUT scale.) These terms will make N linear combinations of the $\overline{\mathbf{5}}$ fields superheavy and leave three linear combinations light. These light linear combinations, which contain Standard Model quarks and leptons, will be denoted $\overline{\mathbf{5}}_i^L$. The superheavy combinations will be denoted by $\overline{\mathbf{5}}_m^H$.

It is easily seen that if $A \equiv [I + T^\dagger T]^{-1/2}$ and $B \equiv [I + T^\dagger T]^{-1/2} T^\dagger$, where $T = M^{-1}\Delta$, then $\overline{\mathbf{5}}^U = A\overline{\mathbf{5}}^L + B\overline{\mathbf{5}}^H$. Exactly as in the BC model, when substituted into eq. (5.10), this leads to factors of A in the effective mass matrices of the Standard Model quarks and leptons.

The factors of H in those matrices arise, as in the BB scheme, from the mixing of $\mathbf{10}_i^U$ with the $\mathbf{10}_A^E$. In order for H to come out diagonal, the $\mathbf{10}_A^E + \overline{\mathbf{10}}_A^E$ must transform under a flavor symmetry. A simple possibility is an abelian symmetry $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$, such that $\mathbf{10}'_A$ and $\overline{\mathbf{10}}'_A$ are odd under $Z_2^{(B)}$ if $A = B$ and even otherwise. Let there be three Standard-Model-singlet Higgs fields $\mathbf{1}_H^{Ai}$, which are triplets under SO(3). Then the following mass and Yukawa terms of the fermion 10 multiplets are allowed

$$\overline{\mathbf{10}}_A^E\,(Y_A \mathbf{1}_H + y_A \mathbf{24}_H)\,\mathbf{10}_A^E\ +\ \overline{\mathbf{10}}_A^E\left(Y'_A \mathbf{1}_H^{Ai} + y'_A \mathbf{24}_H^{Ai}\right)\mathbf{10}_i^U. \tag{5.12}$$

The role of the adjoint Higgs fields $\mathbf{24}_H$ and $\mathbf{24}_H^{Ai}$ is to introduce SU(5) breaking into the quark and lepton mass matrices, through H, and thus avoid the "bad" minimal SU(5) prediction that the down quark masses equal the charged lepton masses at the GUT scale. It is notationally simpler, however, to explain the mixing of the 10 multiplets without considering the effects of the adjoint fields in eq. (5.12), so we will first discuss the unrealistic case where the VEVs of the $\mathbf{24}_H$ are set to zero (which we will call the "minimal model") and then later discuss the realistic case where their VEVs are non-zero.

If certain coefficients in the Higgs potential are positive then the VEVs of $\mathbf{1}_H^{Ai}$ will be orthogonal to each other in SO(3) space: $\sum_i \langle \mathbf{1}_H^{Ai} \rangle \langle \mathbf{1}_H^{Bi} \rangle = c_A \delta_{AB}$. (In particular, if the coefficients of the terms $\left(\sum_{i=1}^{3} \mathbf{1}_H^{Ai} \mathbf{1}_H^{Bi} \right)^2$ are positive it will ensure this orthogonality.) Without loss of generality, one can then choose a basis in SO(3) space such that the axes are aligned with the VEVs of the three singlet VEVs. That is, so that $\langle \mathbf{1}_H^{Ai} \rangle = s_A \delta^{Ai}$. Defining, $Y_A \langle \mathbf{1}_H \rangle \equiv M_A$ and $Y'_A \langle \mathbf{1}_H^{Ai} \rangle \equiv \Delta_A \delta^{Ai}$, eq. (5.12) with adjoint VEVs set to zero gives

$$\overline{\mathbf{10}}_A^E \left(M_A \mathbf{10}_A^E \; + \; \Delta_A \delta^{Ai} \mathbf{10}_i^U \right). \tag{5.13}$$

The three linear combinations of 10 multiplets appearing with the parentheses in eq. (5.13) are superheavy and will be denoted $\mathbf{10}_A^H$, whereas the three linear combinations $(-\Delta_A \mathbf{10}_A^E + M_A \delta^{Ai} \mathbf{10}_i^U)$ that are orthogonal to them contain Standard Model fermions and will be denoted $\mathbf{10^L}_i$. This gives

$$\mathbf{10}_i^U = \cos\theta_i \; \mathbf{10^L}_i \; + \; \sin\theta_i \; \delta^{Ai} \; \mathbf{10}_A^H, \tag{5.14}$$

where $\cos\theta_i \equiv \delta^{Ai} M_A / \sqrt{|M_A|^2 + |\Delta_A|^2}$ and $\sin\theta_i \equiv \delta^{Ai} \; \Delta_A / \sqrt{|M_A|^2 + |\Delta_A|^2}$. Substituting this into eq. (5.10), one finds that every factor of $\mathbf{10}_i$ in the effective Yukawa couplings of the Standard Model fermions is accompanied by a factor of $\cos\theta_i$, as in eq. (5.3). We will assume a hierarchical pattern $|\Delta_1/M_1| \gg |\Delta_2/M_2| \gg 1 \gg |\Delta_3/M_3|$. Then we can define a matrix H by

$$H \equiv \begin{pmatrix} \cos\theta_1 & 0 & 0 \\ 0 & \cos\theta_2 & 0 \\ 0 & 0 & \cos\theta_3 \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \tag{5.15}$$

where $\alpha \ll \beta \ll \gamma \cong 1$. Substituting $\overline{\mathbf{5}}^U = A\overline{\mathbf{5}}^L + B\overline{\mathbf{5}}^H$ and eq. (5.14) into eq. (5.10) and using eq. (5.15), the effective mass matrices of the Standard Model quarks and leptons can then be written

$$\begin{aligned} &M_u = (H^2)\ \mu_u \\ &M_d = (H\ A)\ \mu_d \quad \longrightarrow \quad M_d = (H\ \mathcal{D})\ A_\Delta\ \mu_d, \\ &M_\ell = (A^T\ H)\ \mu_d \quad \longrightarrow \quad M_\ell = A_\Delta^T\ (\mathcal{D}\ H)\ \mu_d, \\ &M_\nu = (A^T\ A)\ \mu_\nu \quad \longrightarrow \quad M_\nu = A_\Delta^T\ (\mathcal{D}^2)\ A_\Delta\ \mu_\nu, \end{aligned} \tag{5.16}$$

This is the basic result of the model. Other than certain overall mass scales (μ_u, μ_d, and μ_ν) all the flavor structure of the quarks and leptons is controlled by two matrices: a mixing matrix A and a hierarchy matrix H. In going to the last expressions in each line of eq. (5.16), we have used $A = \mathcal{D}A_\Delta\mathcal{U}$ and absorbed $\mathcal{U}$ by field redefinitions (as explained previously). We write the matrix $\mathcal{D}$ as $\mathcal{D} = \text{diag}(\delta, \epsilon, 1)d$ and absorb the factors of d into redefined mass scales μ_d' and μ_ν'. One therefore ends up with the following result (for the "minimal" version of the model):

$$\begin{aligned} M_u &= \begin{pmatrix} |\alpha|^2 & 0 & 0 \\ 0 & |\beta|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mu_u, \\ M_d &= M_\ell^T = \begin{pmatrix} |\alpha\delta| & 0 & 0 \\ 0 & |\beta\epsilon| & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mu_d', \\ M_\nu &= \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ ce^{i\theta} & a & 1 \end{pmatrix} \begin{pmatrix} \delta^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mu_\nu'. \end{aligned} \tag{5.17}$$

Of course, the form obtained for M_ν is the same as before. The parameters called $pe^{i\theta_p}$ and $qe^{i\theta_q}$are here called ϵ^2 and δ^2in this chapter. It should be noted that in eq. (5.17), the phases of δ, ϵ, α, and β do not matter for the matrices M_u, M_d, and M_ℓ, as they can be absorbed into the fermion fields. But for the neutrino mass matrix M_ν the phases of δ and ϵ do make a difference, and have to take definite values to fit the neutrino masses and mixing angles.

One easily sees from eq. (5.17) that in this "minimal model" one has, to very good approximation, the following "postdictions":

$$\begin{aligned} m_u : m_c : m_t &= |\alpha|^2 : |\beta|^2 : 1, \\ m_d : m_s : m_b = m_e : m_\mu : m_\tau &= |\alpha\delta| : |\beta\epsilon| : 1, \\ q^2 : p^2 : 1 &= |\delta|^2 : |\epsilon|^2 : 1. \end{aligned} \tag{5.18}$$

From fitting the neutrino masses and mixing angles [], one can determine $|\epsilon| = \sqrt{p} \cong \sqrt{0.1525} = \frac{1}{2.56}$ and $|\delta| = \sqrt{q} \cong \sqrt{0.0141} = \frac{1}{8.44}$. And one can obtain the values of $|\alpha|$ and $|\beta|$ directly from the up quark mass ratios: $|\beta| = \sqrt{m_c/m_t} = \frac{1}{17.8}$ and $|\alpha| = \sqrt{m_u/m_t} = \frac{1}{393}$. (We take the fermion masses here and in the following equation to be the masses at 2×10^{16} GeV as run up to that scale using the MSSM renormalization group equations with $\tan\beta = 10$ [36].) From these values one has the following result:

$$\begin{aligned} &\text{minimal model hierarchy} & |\alpha\delta| : |\beta\epsilon| : 1 &= \tfrac{1}{3{,}579} : \tfrac{1}{49.2} : 1 \\ &\text{actual lepton ratios} & m_e : m_\mu : m_\tau &= \tfrac{1}{3{,}679} : \tfrac{1}{17.5} : 1 \\ &\text{actual quark ratios} & m_d : m_s : m_b &= \tfrac{1}{1142} : \tfrac{1}{60.1} : 1. \end{aligned} \tag{5.19}$$

One sees that the minimal model works surprisingly well, in fact better than in the BB idea taken by itself, where the inter-family mass ratios of the charged leptons and of the down quarks are $\alpha : \beta : 1$, as shown in eq. (5.7). (That would give $m_e/m_\tau \sim m_d/m_b \sim \alpha \sim \frac{1}{393}$, which is off by an order of magnitude for the electron.) Thus the factors of δ and ϵ, which come from combining the BB and BC ideas, give more realistic down quark and charged lepton mass hierarchies.

The combined model we are describing (so far in a minimal form) is more explanatory than the BC model. In the BC model the inter-family mass hierarchies of the up quarks, down quarks, charged leptons, and neutrinos are completely unrelated, being determined by four diagonal matrices whose elements are free parameters. In the combined model, these hierarchies are all related, and related in a way that we have just seen is qualitatively correct. The 12 parameters in the four hierarchical diagonal matrices of the BC model are replaced by just 7 parameters in the minimal model: $|\alpha|$, $|\beta|$, $|\delta|$, $|\epsilon|$, μ_u, μ'_d, and μ'_ν. This would be a huge increase in predictivity, but of course it is *too* predictive, since the minimal model gives the "bad" minimal SU(5) prediction that the charged lepton masses are equal to the down quark masses at the GUT scale. To cure this problem requires that group-theoretical factors reflecting the breaking of SU(5) appear in the fermion mass matrices. The simplest way for this to happen is through the matrix H as a result of the adjoint Higgs fields in eq. (5.12) getting non-zero VEVs. We shall now look at this in detail.

5.4 The Group-Theoretical Factors that Distinguish M_d from M_ℓ

As can be seen from eq. (5.19) [and below from eg. (5.21)], the group-theoretical factors must enhance the muon mass and the d quark mass by about a factor of about 3 or 4, while having little effect on the other quark and lepton masses. Again using the results of [] with quark and lepton masses run up to 2×10^{16} GeV, using the MSSM renormalization group equations, and normalizing those masses to the b quark mass, one has

$$\begin{aligned} (m_d, m_s, m_b)/m_b &= \left(\tfrac{1}{1142}, \tfrac{1}{60.14}, 1\right), \\ (m_e, m_\mu, m_\tau)/m_b &= \left(\tfrac{1}{2{,}967}, \tfrac{1}{14.1}, 1.24\right). \end{aligned} \tag{5.20}$$

Then using the values of $|\alpha\beta|$ and $|\beta\varepsilon|$ given in eq. (5.19), one has

$$\begin{aligned} \left(\tfrac{m_d}{|\alpha\delta|}, \tfrac{m_s}{|\beta\epsilon|}, m_b\right)/m_b &= (3.13, 0.817, 1), \\ \left(\tfrac{m_e}{|\alpha\delta|}, \tfrac{m_\mu}{|\beta\epsilon|}, m_\tau\right)/m_b &= (1.21, 3.49, 1.24). \end{aligned} \tag{5.21}$$

The ratios given in eq. (5.21), which are all predicted to be equal to 1 in the minimal model, must be accounted for by the group-theoretical factors.

Seemingly, the simplest way to do this is through the coupling of adjoint Higgs fields to the 10 multiplets of fermions, as shown in eq. (5.12). Let us first just consider the effect of the VEV of the $\mathbf{24}_H$, which couples as $\overline{\mathbf{10}}^E_A(y_A \mathbf{24}_H)\mathbf{10}^E_A$. If we define κ_A by $\frac{y_A\langle \mathbf{24}_H\rangle}{Y_A\langle \mathbf{1}_H\rangle} = \kappa_A Y_f/2$, where f stands for the fermion type u, u^c, d, or ℓ^c, and Y_f is the weak hypercharge of f, then the effect is that in eq. (5.13), M_A gets replaced by $M_A(1+\kappa_A Y_f/2)$. Assuming that $|\Delta_1/M_1| \gg |\Delta_2/M_2| \gg |\Delta_3/M_3| \sim 1$, then the angles defined after eq. (5.14) are different for different fermion types and given approximately by

$$\begin{aligned} \cos\theta_1^f &\cong \left|\tfrac{M_1}{\Delta_1}(1+\kappa_1 Y_f/2)\right| \\ \ll \cos\theta_2^f &\cong \left|\tfrac{M_2}{\Delta_2}(1+\kappa_2 Y_f/2)\right| \\ \ll \cos\theta_3^f &\cong \left[1+\left|\tfrac{\Delta_3}{M_3}\right|^2 (1+\kappa_3 Y_f/2)^{-2}\right|^{-1/2} \cong 1, \end{aligned} \tag{5.22}$$

where $Y_f/2$ is the weak hypercharge of the fermion of type f. Then the matrix H defined in eq. (5.15) is replaced by matrices H_f, which are different for different types of fermion in the 10 multiplets:

$$H_f \equiv \begin{pmatrix} \cos\theta_1^f & 0 & 0 \\ 0 & \cos\theta_2^f & 0 \\ 0 & 0 & \cos\theta_3^f \end{pmatrix}, \tag{5.23}$$

and the fermion mass matrices have the forms

$$\begin{aligned} &M_u = (H_u H_{u^c})\,\mu_u \\ &M_d = (H_d\, A)\,\mu_d \quad \longrightarrow \quad M_d = (H_d\, \mathcal{D})\, A_\Delta\, \mu_d, \\ &M_\ell = (A^T\, H_{\ell^c})\,\mu_d \quad \longrightarrow \quad M_\ell = A_\Delta^T\, (\mathcal{D}\, H_{\ell^c})\,\mu_d, \\ &M_\nu = (A^T\, A)\,\mu_\nu \quad \longrightarrow \quad M_\nu = A_\Delta^T\,(\mathcal{D}^2)\, A_\Delta\,\mu_\nu, \end{aligned} \tag{5.24}$$

If we consider the masses of the charged fermions of the second and third families, there are four mass ratios ($\frac{m_c}{m_t}$, $\frac{m_s}{m_b}$, $\frac{m_\mu}{m_\tau}$, and $\frac{m_\tau}{m_b}$) that must be fit using the parameters in eq. (5.22), and there are four such parameters, namely $|\Delta_3/M_3|, \kappa_3, |\Delta_2/M_2|$, and κ_2.

Consider first the ratio m_τ/m_b. As is well-known this is predicted in minimal SU(5) to be 1 at the GUT scale, as is also the case in the minimal version of the present model. In reality, however, this ratio is not exactly 1, though it is near to 1 (especially in the MSSM). In fact, for $\tan\beta = 10$ it is 1.24 at the GUT scale as shown in eq. (5.20). With the group-theoretic factors of eq. (5.22) one sees that it is given by

$$1.24 = \left(\frac{m_\tau}{m_b}\right)_{M_{GUT}} = \frac{\cos\theta_3^{\ell c}}{\cos\theta_3^d} = \sqrt{\frac{1+\left|\frac{\Delta_3}{M_3}\right|^2\left(1+\frac{1}{6}\kappa_3\right)^{-2}}{1+\left|\frac{\Delta_3}{M_3}\right|^2\left(1+\kappa_3\right)^{-2}}}, \tag{5.25}$$

which is indeed close to but not exactly 1, for $\Delta_3/M_3 < 1$. We can also write (putting in the values given in eq. (5.21)):

$$0.817 = \frac{m_s/m_b}{\epsilon\sqrt{m_c/m_t}} = \sqrt{\frac{1+\frac{1}{6}\kappa_2}{1-\frac{2}{3}\kappa_2}}\left(\frac{1+|\frac{\Delta_3}{M_3}|^2(1+\frac{1}{6}\kappa_3)^{-2}}{1+|\frac{\Delta_3}{M_3}|^2(1-\frac{2}{3}\kappa_3)^{-2}}\right)^{1/4}, \tag{5.26}$$

and

$$3.49 = \frac{m_\mu/m_b}{\epsilon\sqrt{m_c/m_t}} = \frac{1+\kappa_2}{\sqrt{(1+\frac{1}{6}\kappa_2)(1-\frac{2}{3}\kappa_2)}}\left(\frac{1+|\frac{\Delta_3}{M_3}|^2(1+\frac{1}{6}\kappa_3)^{-2}}{1+|\frac{\Delta_3}{M_3}|^2(1-\frac{2}{3}\kappa_3)^{-2}}\right)^{1/4}. \tag{5.27}$$

eqs. (5.25) to (5.27) contain three equations with three unknowns κ_2, κ_3, and $|\Delta_3/M_3|$. They are solved by the values $\kappa_2 = 11.2$, $\kappa_3 = -2$, and $|\Delta_3/M_3| = 0.86$. The remaining ratio m_c/m_t can then be fit by the choice $|\Delta_2/M_2| = 110$.

Fitting the first family masses is more involved. There are three additional masses to be fit (m_e, m_d, and m_u), but the expressions in eqs. (5.22) have only two additional parameters (κ_1 and $|\Delta_1/M_1|$. Indeed, it turns out that there is no fit. It is for this reason that one must include the effect of the term containing $\mathbf{24}_H^{Ai}$ in eq. (5.12). Actually, only one such adjoint Higgs field is required to obtain a good fit, namely $\mathbf{24}_H^{1i}$. However, as the expressions are somewhat complicated looking, we do not show them.

One sees, then, that introducing the group-theoretic factors required to break the well-known minimal SU(5) mass degeneracies means that the model ends up with as many free parameters as there are in the BC model of [28]. Thus combining that model with the BB idea leads to no increase in the number of precise quantitative predictions. However, there is a gain in explanatory power, in that the inter-family mass hierarchies of the different types of fermions are related to each other in a way that is qualitatively correct.

5.5 The Typical Values of δ and ϵ

We now turn to a discussion of the values of δ and ϵ, the elements of the diagonal matrix $\mathcal{D}$. It is a nontrivial condition for the viability of the model that the same values of $|\delta|$ and $|\epsilon|$ give realistic results both for the neutrino properties and for the mass hierarchies of the down quarks and charged leptons. As we have seen, the model clears this hurdle. The fit to the neutrino properties obtained in [28] gives $|\delta| \cong \frac{1}{8.44}$ and $|\epsilon| \cong \frac{1}{2.56}$, and these values also give realistic mass hierarchies, as shown in eq. (5.18).

The question arises whether these are natural values for $|\delta|$ and $|\epsilon|$ to have. Why should there be any hierarchy in the elements of $\mathcal{D}$? And why should that hierarchy be parallel to the hierarchy in H, with the diagonal elements increasing from the first to the third family? And why should they have these particular values? It turns out that the values of $|\delta|$ and $|\epsilon|$ needed for good fits are indeed natural, in the sense that they lie in the middle of the range of values that are most "likely" given the values of the elements of the triangular matrix A_Δ, as we will now show.

The matrix $\mathcal{D} = \mathrm{diag}(\delta, \epsilon, 1)$ arises from bringing the matrix A to the form $A = \mathcal{D}A_\Delta\mathcal{U}$, as previously explained. The matrix A, in turn, is defined by $A \equiv (I + T^\dagger T)^{-1/2}$, where $T = M^{-1}\Delta$, and M and Δ are the matrices appearing in eq. (5.11). It is natural to assume that the matrices M and Δ are both roughly of order the grand unification scale, but there is no symmetry reason why M and Δ should have any special form. Consequently, the matrix T has no reason to have any special form either.

Suppose that the elements of T are treated as random complex variables all of which have the same probability distribution. For each choice of T, one can compute the matrix A, and from that determine the matrices $\mathcal{D}$ and A_Δ. Not surprisingly, one finds that the elements of $\mathcal{D}$ are correlated with those of A_Δ. In fact, simple arguments show that if the elements of A_Δ that we have called a and b are large, then typically $|\delta| \sim 1/ab$ and $|\epsilon| \sim 1/a$. Since fitting the CKM angles gives $a \sim 2$ and $b \sim 4$, the most likely values are $|\delta| \sim 1/8$ and $|\epsilon| \sim 1/2$.

This is confirmed by a numerical search treating the elements of T as random variables. We have randomly generated one million matrices T whose elements are given by $T_{ij} = 10^{r_{ij}} e^{i\theta_{ij}}$, with $-1 < r_{ij} < +1$ and $0 < \theta_{ij} < 2\pi$ with uniform probability distribution. We compute the matrices A_Δ and $\mathcal{D}$ for each randomly generated T, and require that the parameters in A_Δ (*i.e.* a, b, c, and θ) agree with the values in eq. (2.13) within experimental limits. For those that meet this requirement, we plot the values of $|\epsilon|^{-1}$ and $|\delta|^{-1}$ in the Figure 5.1. One sees that there indeed tends to be a mild hierarchy $|\delta| < |\epsilon| < 1$. The dark cross in the Figure 5.1 represents the values that give the best fit to the neutrino properties according to [28]: $(|\epsilon|^{-1}, |\delta|^{-1}) = (2.56, 8.44)$. It is apparent from the Figure 5.1 that these lie in the most probable range.

5.6 Conclusions

We have shown that an idea proposed to explain the relative strengths of the interfamily mass hierarchies of different types of fermions [5] can be combined with an idea proposed to explain the relative strengths of interfamily mixing angles [28]. The former idea was based on all interfamily mass hierarchies coming from a single master hierarchy matrix H, the latter was based on all interfamily mixing angles coming from a master mixing matrix A. In both ideas, the master matrix arose from the mixing of the three usual fermion families of the standard model with extra vector-like fermions.

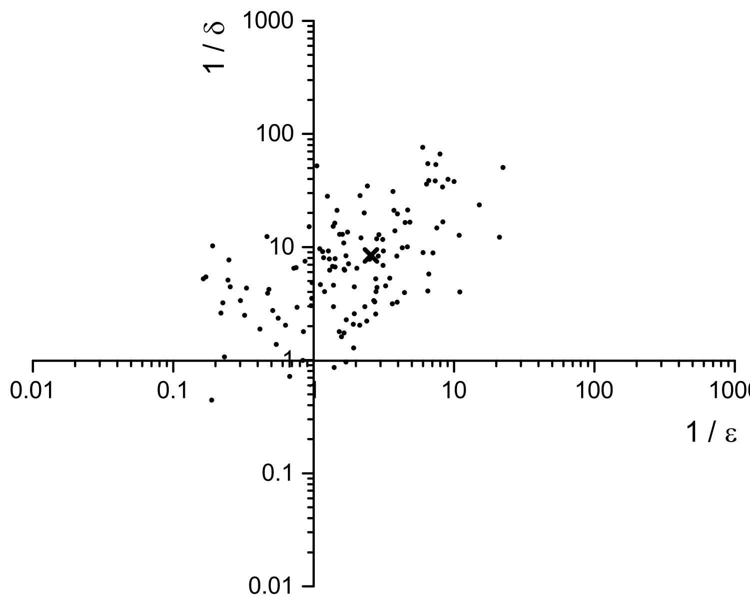

Figure 5.1: The values of $(|\epsilon|^{-1}, |\delta|^{-1})$ that come from randomly generated matrices T that give realistic A_Δ. The dark cross represents the values that give the best fit to neutrino properties: $(|\epsilon|^{-1}, |\delta|^{-1}) = (2.56, 8.44)$.

As we have shown in this paper, the two ideas dovetail together quite naturally. Moreover, we have shown that a certain conceptual simplification arises from the combination of the two ideas: in the combined model, all the standard model mass matrices can be expressed in terms of only the two master matrices, with some group-theoretic factors (analogous the well-known Georgi-Jarlskog factors [14]) to avoid the well-known bad minimal SU(5) relations between the charged lepton masses and down quark masses. If it were not for the need to introduce these group-theoretic factors, the combined model would have several fewer free parameters than the model of [28] and many fewer than that of [5]. The way that we introduced the group-theoretical factors in this paper made the number of free parameters to be the same as in [28]. There may be a simpler way to introduce these group-theoretic factors, leading to an even more predictive model, a possibility that deserves further study.

We have also shown that the typical interfamily mass hierarchies that arise in the combined model are qualitatively correct. Specifically, given the mass hierarchy among the up quarks $\begin{pmatrix} u, & c, & t \end{pmatrix}$ and the known CKM mixing parameters, the strengths of all the other interfamily mass hierarchies, namely those of $\begin{pmatrix} d, & s, & b \end{pmatrix}$, $\begin{pmatrix} e, & \mu, & \tau \end{pmatrix}$ and $\begin{pmatrix} \nu_{1,} & \nu_2, & \nu_3 \end{pmatrix}$, are predicted up to group-theoretical factors of order 1, and come out to be of the right magnitude. This is a very nontrivial and unexpected result, which perhaps lends some credibility to the basic approach proposed here.

Chapter 6

COGENERATION OF DARK MATTER AND BARYONS BY NON-STANDARD-MODEL SPHALERONS IN UNIFIED MODELS

6.1 Introduction

In this chapter, we will discuss an asymmetric dark matter model with a new sphaleron process besides the one we have already explained in section 1.13. The central idea of the model is that sphalerons of a new gauge interaction can convert a primordial asymmetry in B or L into a dark matter asymmetry. From the equilibrium conditions for the sphalerons of both the electroweak and the new interactions, one can compute the ratios of B, L, and X, where X is the dark matter number, thus determining the mass of the dark matter particle fairly precisely. Such a scenario can arise naturally in the context of unification with larger groups. An illustrative model embeddable in $SU(6) \times SU(2) \subset E_6$ is described in detail as well as an equally simple model based on $SU(7)$.

The fact that the number densities of dark matter and ordinary baryonic matter are comparable [37] has suggested to many authors that they may have a common origin, that is that the dark matter and baryonic matter may have been generated by the same processes, or that one of them may have been generated from the other. This idea is sometimes called "cogeneration" of dark matter and ordinary matter. There is a rapidly growing literature studying various ways that this might have happened [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49].

The first papers to propose this possibility [50, 38] were based on the idea that primordial asymmetries in baryon and lepton number (B, L) were partially converted into an asymmetry in some other global quantum number (call it X) by sphaleron processes [51, 52, 53] when the temperature of the universe was above the weak interaction scale. Assuming X to be conserved (or nearly so) at low temperatures, the lightest particles carrying this quantum number would be stable and could play the role of dark matter. What would result from such a scenario is “asymmetric dark matter" [54, 50].

A very interesting idea first proposed in [55] is that a primordial asymmetry in B or L (or both) is partly converted into an X asymmetry (and thus a dark matter asymmetry) by sphalerons of some new non-abelian gauge interaction. In this paper we point out that this mechanism arises very naturally in grand unified models. In a previous paper [50], it was noted by one of us that grand unified models with groups larger than $SU(5)$ provide a natural context for the emergence of dark matter. The larger fermion multiplets of such models typically contain fermions that are Standard Model singlets, which could play the role of dark matter. Unified models can also have accidentally conserved global charges (analogous to $B-L$ in $SU(5)$ models) that could be the charge X carried by dark matter. It was also noted in [50] that larger unification groups can have additional non-abelian subgroups whose sphalerons could convert B and L asymmetries into an X asymmetry. Here we show that simple models can indeed be constructed that realize this possibility. Most of this paper is devoted to an example based on an $SU(5) \times SU(2)$ that is embeddable in E_6. At the end of the paper we note that a similar and equally simple model can be constructed based on $SU(7)$. These models exploit all the features of grand unification favorable to the genesis of dark matter that were emphasized in [50].

In the section 6.2, we will introduce a model in which includes a new gauge group $SU(2)_*$ and the new interaction whose sphalerons are responsible for cogenerating the dark matter which is different from the electroweak group $SU(2)_L$. The particle contents of the model are also introduced. In the section 6.3, we will show that the present ratio of the number densities of dark matter particles and baryons, n_{DM}/n_B, can be calculated thermodynamically from just the particle content of the model and is independent of the nature of the primordial asymmetry. This allows one to determine the mass of the dark matter particle, which is given simply by $m_{DM} = m_p \frac{\Omega_{DM}}{\Omega_B} \frac{n_B}{n_{DM}}$. In the section 6.4, we will explain why a massless fermion S and a very light scalar field

σ are needed in the model to make the dark matter purely asymmetric. We will also discuss possible ways to detect the dark matter particles in collider experiments.

6.2 The Model

The model we propose is based on the gauge group $G_{SM} \times SU(2)_*$, which can be embedded in larger groups in the following way:

$$E_6 \supset SU(6) \times SU(2) \supset SU(5) \times SU(2) \supset G_{SM} \times SU(2). \tag{6.1}$$

We will denote the $SU(2)$ factor in eq. (6.1) by $SU(2)_*$ in order to distinguish it from the electrweak $SU(2)$, which we will call $SU(2)_L$. The particle content of the model is exactly what would arise from such an embedding. In particular, each family of fermions consists of the 27 particles that make up the fundamental representation of E_6

$$\begin{aligned} \mathbf{27} &\longrightarrow (15,1) + (\overline{6},2) \longrightarrow (10,1) + (5,1) + (\overline{5},2) + (1,2) \\ &= \left(\ell^c, \begin{bmatrix} u \\ d \end{bmatrix}, u^c\right) + \left(\begin{bmatrix} \overline{\ell} \\ \overline{\nu} \end{bmatrix}, \overline{d^c}\right) + \left(\begin{bmatrix} \ell_I \\ \nu_I \end{bmatrix}, d^c_I\right) + (\chi_I), \end{aligned} \tag{6.2}$$

where $I = 1, 2$, and the decomposition in eq. (6.2) corresponds to the sequence of groups in eq. (6.1). The index I in eq. (6.2) stands there and through the paper for the index of the extra $SU(2)_*$ group. Note that each family automatically contains particles, denoted χ_I, that are singlets under G_{SM} but non-singlet under $SU(2)_*$ and thus able to play the role of dark matter, illustrating the point made in [50].

While eqs. (6.1) and (6.2) show that our model is naturally unified in a larger group, this is not essential to the mechanism of cogeneration. Henceforth in this paper we will discuss the model as if its gauge group is just $G_{SM} \times SU(2)_*$ without any assumption about whether this is unified at some high scale. Nevertheless, it is convenient as a notational "shorthand" to refer to fermions and scalars by the $SU(5) \times SU(2)_*$ multiplets in which they *would* be contained if the model were further unified, and we shall often do this.

In $SU(5) \times SU(2)_*$ language, then, each family consists of $(10,1)+(5,1)+(\overline{5},2)+(1,2)$. Besides the $(1,2) = \chi_I$ already mentioned, there are other non-Standard Model fermions contained in each family, namely the half of the fermions in the $(\overline{5},2)$ and the fermions in $(5,1)$. This is a vector-like $5+\overline{5}$ pair, which we call "extra" vector-like fermions, as in previous chapters of this thesis.

The spontaneous breaking of $SU(2)_*$, at a scale M_*, is accomplished by the vacuum expectation value (VEV) of a $(1,2)$ multiplet of Higgs fields that we shall denote Ω_I. This Higgs field also gives mass to the "extra fermions" by means of the following Yukawa coupling

$$(\overline{5},2)(5,1)\langle(1,2)_\Omega\rangle \tag{6.3}$$

It was said above that the fermions χ_I that transform as $(1,2)$ play the role of dark matter. But more precisely, there are three families of these $SU(2)_*$ doublets, or altogether six flavors of them, and it is the lightest of them that is stable and composes the dark matter. To give these six fermions mass we introduce six partners for them that are singlets under all the gauge groups. We denote these by χ_a^c, $a = 1,..,6$. The Yukawa terms that give them mass are

$$Y_a(\chi_I \chi_a^c)\langle\Omega_I\rangle. \tag{6.4}$$

The value of the scale M_* at which $SU(2)_*$ is broken by $\langle\Omega_I\rangle$ does not matter very much as far as the mechanism for generating a dark matter asymmetry is concerned. It should certainly be large enough that the $SU(2)_*$ gauge bosons and the "extra" fermions in $(5,1)$ and $(\overline{5},2)$ would not already have been detected. On the other hand, the dark matter particles, which will later be seen to have mass around 1 GeV, obtain mass from the VEV of Ω_I. Therefore, the larger the VEV of Ω_I is, the smaller must be its Yukawa coupling to the dark matter particles. We know that some Yukawa couplings in nature are very small (those of e, u, and d are of order 10^{-5}). If one does not wish Yukawa couplings to be smaller than 10^{-5}, say, one would need M_* to be less than about 100 TeV. We imagine, therefore, that M_* is somewhere between 1 TeV and 100 TeV. Moreover, as will be discussed later, if M_* is larger than about 100 TeV, the $SU(2)_*$

gauge interactions will be too slow to keep the "dark sector" of particles in thermal equilibrium with the Standard Model particles long enough to avoid problems with primordial nucleosynthesis (The energy trapped in massless particles of the dark sector can cause the universe too expand too rapidly in the era of primordial nucleosynthesis).

It should be noted that since the Ω_I is in a pseudo-real representation of the gauge group and since we will not give it any global charge that would distinguish from its conjugate $\tilde{\Omega}_I \equiv i\sigma_2\Omega_I^*$, the symmetries of the model allow $\tilde{\Omega}_I$ to couple in the same ways that Ω_I can. For example, there are both $(\chi_I\chi_a^c)\langle\Omega_I\rangle$ and $(\chi_I\chi_a^c)\langle\tilde{\Omega}_I\rangle$ Yukawa terms, and similarly there are both $hh_I'\Omega_I$ and $hh_I'\tilde{\Omega}_I$ terms in the Higgs potential. (These facts imply that the chemical potential of the Ω fields zero, which is relevant to our later discussion.)

To break the electroweak gauge group and give mass to all the Standard Model quarks and leptons, there must be more than one $SU(2)_L$ doublet of Higgs fields. The masses of the up quarks come from a Higgs doublet, which we shall denote h, that would be contained in $(5,1)$ of $SU(5)\times SU(2)_*$. In that language, it has Yukawa couplings of the type $(10,1)(10,1)\langle(5,1)_h\rangle$, which contains in particular $uu^c\langle h\rangle$. The down quarks and charged leptons obtain mass from a pair of Higgs doublets, which we denote h_I', that would be contained in $(\overline{5},2)$ of $SU(5)\times SU(2)_*$. These have Yukawa couplings of the form $(10,1)(\overline{5},2)\langle(\overline{5},2)_h\rangle$, which contains in particular $dd_I^c\langle h_I'\rangle$ and $\ell^c\ell_I\langle h_I'\rangle$. The neutrinos can obtain mass from the dimension-5 effective operator $\nu_I\nu_J\langle h_I'\rangle\langle h_J'\rangle/M_R$. It does not matter for our purposes whether this operator arises from the Type I or Type II see-saw mechanism.

Finally, two more types of particle are contained in the model: some number (p) of gauge singlet fermions that will be denoted S and a gauge singlet boson that will be denoted σ. These will play the role, as will be seen, of enabling the dark matter particles to annihilate with their antiparticles efficiently enough to leave only "asymmetric dark matter." It means that we need the annihilation rate still be bigger than the Hubble expasion rate even when $T\sim 1$ GeV. (i.e. $\Gamma_{DM}\geq H$). This puts a constraint on the mass and coupling of the scalar σ, which will be discussed in the section 6.4.

The complete fermion and scalar content of the model is displayed in the Table 6.1. In the last two columns of the Table 6.1, we give the charges of the fields under two global symmetries, $U(1)_X$ and $U(1)_W$. The charge X is the crucial one for the model. It is the asymmetry in X that is responsible for the existence of stable dark matter. The charge W plays the role of constraining the couplings of the singlet fields S and σ that are responsible for the annihilation of dark matter particles with their anti-particles. In particular, the global $U(1)_W$ invariance means that these fields interact only by the Yukawa term $y(\chi^c S)\sigma$. This term allows the annihilation process $\chi^c + \overline{\chi^c} \longrightarrow S + \overline{S}$ to occur by the exchange of a σ boson in the t channel. The σ boson is assumed to have no vacuum expectation value, and therefore the S fermions are massless. In this way, essentially all the dark matter anti-particles annihilate with dark matter particles into massless particles, whose energy is red-shifted away as the universe expands, leaving only the dark matter particle excess, i.e. the "asymmetric dark matter." The global symmetries $U(1)_X$ and $U(1)_W$ can arise as accidental symmetries of the low energy theory even if $G_{SM} \times SU(2)_*$ is unified in a larger group, as will be discussed later.

6.3 The Genesis of The Dark Matter Asymmetry

Now that the particle content and couplings of the model have been defined, we turn to the process by which the dark matter asymmetry is generated. The sphalerons of $SU(2)_*$ create one each of every left-handed fermion that is a doublet of $SU(2)_*$, namely (for each family) the three colors of d_I^c, the leptons ν_I and ℓ_I, and the X-bearing particles χ_I. Thus, for the $SU(2)_*$ sphaleron processes $\Delta X = \Delta B = \frac{1}{2}\Delta L$. (We follow the loose but common practice of referring to processes that involve the anomaly of some group G as "sphaleron processes" or "sphalerons" even if they happen at a temperature far above the scale at which G is broken rather than through tunneling.) The sphalerons of the electroweak $SU(2)_L$ give $\Delta X = 0$ and $\Delta B = \Delta L$. All other processes at low energy conserve B, L, and X. (There might be grand unified interactions that

	name	$G_{SM} \times SU(2)_*$	$SU(5) \times SU(2)_*$	$SU(6) \times SU(2)_*$	E_6	X	W
$3\times$	$\begin{pmatrix} u \\ d \end{pmatrix}$	$(3, 2, +\frac{1}{6}; 1)$	$(10, 1)$	$(15, 1)$	27	0	0
$3\times$	u^c	$(\overline{3}, 1, -\frac{2}{3}; 1)$	$(10, 1)$	$(15, 1)$	27	0	0
$3\times$	ℓ^c	$(1, 1, +1; 1)$	$(10, 1)$	$(15, 1)$	27	0	0
$3\times$	$\begin{pmatrix} \overline{\ell} \\ \overline{\nu} \end{pmatrix}$	$(1, 2, +\frac{1}{2}; 1)$	$(5, 1)$	$(15, 1)$	27	0	0
$3\times$	$\overline{d^c}$	$(3, 1, -\frac{1}{3}; 1)$	$(5, 1)$	$(15, 1)$	27	0	0
$3\times$	$\begin{pmatrix} \ell_{1,2} \\ \nu_{1,2} \end{pmatrix}$	$(1, 2, -\frac{1}{2}; 2)$	$(\overline{5}, 2)$	$(\overline{6}, 2)$	27	0	0
$3\times$	$d^c_{1,2}$	$(\overline{3}, 1, +\frac{1}{3}; 2)$	$(\overline{5}, 2)$	$(\overline{6}, 2)$	27	0	0
$3\times$	$\chi_{1,2}$	$(1, 1, 0; 2)$	$(1, 2)$	$(\overline{6}, 2)$	27	$+1$	0
	$\chi^c_{1,\ldots,6}$	$(1, 1, 0; 1)$	$(1, 1)$	$(1, 1)$	1	-1	0
$p\times$	S	$(1, 1, 0; 1)$	$(1, 1)$	$(1, 1)$	1	0	$+1$
	$\Omega_{1,2}$	$(1, 1, 0; 2)$	$(1, 2)$	$(\overline{6}, 2)$	27	0	0
	h	$(1, 2, +\frac{1}{2}; 1)$	$(5, 1)$	$(15, 1)$	27	0	0
	$h'_{1,2}$	$(1, 2, -\frac{1}{2}; 2)$	$(\overline{5}, 2)$	$(\overline{6}, 2)$	27	0	0
	σ	$(1, 1, 0; 1)$	$(1, 1)$	$(1, 1)$	1	$+1$	-1

Table 6.1: The fermion and scalar content of the model.

violate these quantum numbers, and such interactions might have played a role in generating a primordial asymmetry in one or more of them. But when the temperature is far below the grand unification scale, we can neglect these interactions.)

There are four cosmological eras that need to be considered: (a) The era when some primordial asymmetry of B, L, or X (or some combination of them) was generated. This could have been by means of grand unified interactions; but in any case we assume that it happened when the temperature was much higher than the scale M_* at which the $SU(2)_*$ interactions are broken. It does not matter for us which of the many mechanisms that have been proposed for baryogenesis or leptogenesis is responsible for this, or what the relative values were of the asymmetries in B, L, and X that were produced in this primordial era. (b) The era after the primordial asymmetries were generated, but when the temperature is still greater than T_*, where T_* is the temperature below which the $SU(2)_*$ sphalerons processes effectively cease. (c) The era when $T_W < T < T_*$, where T_W is the temperature below which the $SU(2)_L$ sphaleron processes effectively cease. (T_W has been estimated to be about 200 GeV [52, 53].) And (d), the era when $T < T_W$.

In era (b), when both kinds of sphaleron processes ($SU(2)_*$ and $SU(2)_L$) are active, the ratios of X, B, and L are determined by thermodynamics. The point is that the requirement of equilibrium for the two kinds of sphaleron processes gives two conditions on the two independent ratios of these quantum numbers. At the end of era (b), when T falls below T_* and the $SU(2)_*$ sphaleron processes effectively cease, the ratio of X to $B-L$ is frozen, because all other processes conserve both $B-L$ and X.

In the next era, when $T_W < T < T_*$, the $SU(2)_L$ sphaleron processes continue to violate B and L, though conserving $B-L$, and the ratio of B to L changes to a new value that can be computed from the requirement that the $SU(2)_L$ sphalerons are in equilibrium. The ratio of B to L becomes frozen when the temperature falls below T_W, since after that point all processes conserve both B and L. Consequently, from that point on, down to the present, X, B, and L remain in constant ratios to each other.

Finally, when the temperature falls below the mass of the lightest X-bearing particle (which is the dark matter particle), virtually all the particles with non-zero X annihilate with their anti-particles into massless S fermions, except the residue that cannot annihilate due to the asymmetry in X.

We now turn to the thermodynamic calculation of the ratios of X, B, and L, which parallels the calculations in [56, 50]. We start with era (b) when there is already some primordial asymetry and when $T > T_*$. We assume that in this era the $SU(2)_*$ symmetry may be treated as unbroken. Thus the particles within an irreducible multiplet of $G_{SM} \times SU(2)_*$ all have equal chemical potentials, and the chemical potentials of the gauge bosons vanish. Finally, we assume that the scattering processes involving the Yukawa interactions and scalar self-interactions are in equilibrium. For the processes in thermal equilibrium, the sum of initial state chemical potentials is equal to the sum of final state chemical potentials. This gives relations among the chemical potentials that allow one to write all of them in terms of just five, namely μ_Q, μ_L, μ_χ, μ_h, and μ_σ, which are respectively the chemical potentials of the quark doublets (u, d), the lepton doublets (ν_I, ℓ_I), the χ_I, and the scalar fields h and σ. In particular, we have

$$\begin{aligned}
&\mu_\Omega = \mu_{\tilde{\Omega}} \quad \Rightarrow \quad \mu_\Omega = 0, \\
&\mu_L + \mu_{\overline{L}} + \mu_\Omega = 0 \quad \Rightarrow \quad \mu_{\overline{L}} = -\mu_L, \\
&\mu_h + \mu_{h'} + \mu_\Omega = 0 \quad \Rightarrow \quad \mu_{h'} = -\mu_h, \\
&\mu_{u^c} = -\mu_Q - \mu_h, \\
&\mu_{d^c} = -\mu_Q - \mu_{h'}, \\
&\mu_{\ell^c} = -\mu_L - \mu_{h'}, \\
&\mu_{d^c} + \mu_{\overline{d^c}} + \mu_\Omega = 0 \quad \Rightarrow \quad \mu_{\overline{d^c}} = -\mu_{d^c} = \mu_Q - \mu_h, \\
&\mu_\chi + \mu_{\chi^c} + \mu_\Omega = 0 \quad \Rightarrow \quad \mu_{\chi^c} = -\mu_\chi, \\
&\mu_{\chi^c} + \mu_S + \mu_\sigma = 0 \quad \Rightarrow \quad \mu_S = \mu_\chi - \mu_\sigma.
\end{aligned} \tag{6.5}$$

These relationships come from the scattering processes allowed by certain Yukawa interactions and scalar interactions. The eighth line is from the Yukawa interaction $(\chi_I \chi_a^c)\Omega_I$. The first line comes from the fact that both the terms $(\chi_I \chi_a^c)\Omega_I$ and $(\chi_I \chi_a^c)\tilde{\Omega}_I$ exist. The second and seventh lines are from the interactions $(\overline{5}, 2)(5, 1)(1, 2)_\Omega$. The third line is from the interaction $(\overline{5}, 2)_{h'}(5, 1)_h(1, 2)_\Omega$. The fourth line is from the interaction $(10, 1)(10, 1)(5, 1)_h$. The fifth and sixth lines are from the interaction $(10, 1)(\overline{5}, 2)(\overline{5}, 2)_{h'}$. Finally, the last line can be derived from the interaction term $\chi^c S \sigma$.

The next step is to realize that the electric charge Q and the global charge W are conserved by all interactions, and therefore the conditions $Q = 0$ and $W = 0$ must be satisfied. These two conditions will allow us to solve for the chemical potentials of the scalars, μ_h and μ_σ, in terms of those of the fermions, μ_Q, μ_L, and μ_χ. In computing the density of Q and W, we assume that all the particles of the model have masses small compared to M_* and thus to T. (When $T \sim T_*$, the mass of Ω may perhaps not be negligible compared to T, but this will not matter for what follows since $\mu_\Omega = 0$.) Moreover, when the number density of particles of type i, which we call n_i, is low [], it will be linearly related to its chemical potential,

$$n_i \propto \frac{\mu_i}{T}. \tag{6.6}$$

We will show the derivation of eq. (6.6) in the appendix. The condition for electric charge to vanish is then

$$
\begin{aligned}
0 &= Q \propto 6\mu_L(-1) + 3\mu_{\ell^c}(+1) + 3\mu_{\overline{L}}(+1) + \\
&\quad 9\mu_Q(+\tfrac{2}{3}) + 9\mu_{u^c}(-\tfrac{2}{3}) + 9\mu_Q(-\tfrac{1}{3}) + 18\mu_{d^c}(+\tfrac{1}{3}) + 9\mu_{\overline{d^c}}(-\tfrac{1}{3}) + \\
&\quad (b(0)/f(0))[\mu_h(+1) + 2\mu_{h'}(-1)] \\
\Rightarrow \quad 0 &= -12\mu_L + 24\mu_h \quad \Rightarrow \quad \mu_h = \tfrac{1}{2}\mu_L
\end{aligned}
\tag{6.7}
$$

where in each term the first number is the number of states of that type, the number in parentheses is the charge of that type, $f(x) \equiv \frac{1}{4\pi^2}\int_0^\infty y^2 dy[\cosh^2(\frac{1}{2}\sqrt{y^2+x^2})]^{-1}$ and $b(x) \equiv \frac{1}{4\pi^2}\int_0^\infty y^2 dy[\sinh^2(\frac{1}{2}\sqrt{y^2+x^2})]^{-1}$ are integrals over the Fermi and Bose distribution functions and $x = m/T$. Since we are assuming that the particle masses are small compared to T_*, we have that $b(x)/f(x) \cong b(0)/f(0) = 2$. In obtaining the last line of eq. (6.7), we have used the relations given in eq. (6.5). In a similar way we have, from the vanishing of W

$$
\begin{aligned}
0 &= W \propto p\mu_S(+1) + (b(0)/f(0))\mu_\sigma(-1) \quad \Rightarrow \quad \mu_S = \tfrac{2}{p}\mu_\sigma \\
&\Rightarrow \quad \mu_\chi - \mu_\sigma = \tfrac{2}{p}\mu_\sigma \quad \Rightarrow \quad \mu_\sigma = \left(\tfrac{p}{p+2}\right)\mu_\chi,
\end{aligned}
\tag{6.8}
$$

where to get the last line of eq. (6.8), we have used the last relation in eq. (6.5). We remind the reader that the integer p in eq. (6.8) is the the number of massless S fields. (See Table 6.1) The minimal model would therefore simply have $p = 1$.

The final step in analyzing era (b), is to use the equilibrium conditions for the two types of sphalerons to relate the chemical potentials of the fermions, μ_Q, μ_L, and μ_χ. For the sphaleron processes of $SU(2)_L$, as we have already explained in the section 1.13, only the left-handed doublet fermions will be produced. Therefore, the condition for equilibrium of the $SU(2)_L$ sphalerons is simply

$$
0 = 9\mu_Q + 6\mu_L + 3\mu_{\overline{L}} \quad \Rightarrow \quad \mu_Q = -\frac{1}{3}\mu_L, \tag{6.9}
$$

where the number in front of each term is the number of fermion doublets of that type produced by a sphaleron, and where we have used $\mu_{\overline{L}} = -\mu_L$ from eq. (6.7). For the $SU(2)_*$ sphalerons, for the similar reason, the equilibrium condition is

$$\begin{aligned} 0 &= 6\mu_L + 9\mu_{d^c} + 3\mu_\chi = 6\mu_L + 9(-\mu_Q + \mu_h) + 3\mu_\chi \\ \Rightarrow \quad 0 &= \tfrac{21}{2}\mu_L - 9\mu_Q + 3\mu_\chi \quad \Rightarrow \quad 0 = \tfrac{27}{2}\mu_L + 3\mu_\chi \\ \Rightarrow \quad \mu_\chi &= -\tfrac{9}{2}\mu_L, \end{aligned} \tag{6.10}$$

where in the middle steps in eq. (6.10) we have used eqs. (6.7) and (6.9) to eliminate μ_h and μ_Q.

So, finally, we have from eqs. (6.7) - (6.10) all the chemical potentials in terms of just one, μ_L. We are now in a position to compute the ratio of X to $B-L$ at the end of era (b). Again assuming that the particles that carry B, L, and X are light compared to T_*, one has

$$\begin{aligned} B &\propto \tfrac{1}{3}\left(18\mu_Q - 9\mu_{u^c} - 18\mu_{d^c} + 9\mu_{\overline{d^c}}\right) = 18\mu_Q - 6\mu_h = -9\mu_L, \\ L &\propto 12\mu_L - 6\mu_{\overline{L}} - 3\mu_{\ell^c} = 21\mu_L - 3\mu_h = \tfrac{39}{2}\mu_L, \\ X &\propto 6\mu_\chi - 6\mu_{\chi^c} + (b(0)/f(0))\mu_\sigma = 12\mu_\chi + 2\mu_\sigma = -9\left(\tfrac{7p+12}{p+2}\right)\mu_L. \end{aligned} \tag{6.11}$$

Consequently,

$$\frac{X}{B-L} = \frac{6}{19}\left(\frac{7p+12}{p+2}\right), \tag{6.12}$$

which, by a very strange coincidence, is simply equal to 2 in the minimal case, where $p = 1$. This is the ratio of X to $B-L$ that exists also at the present era.

In order to obtain the present ratio of X to B, which is our aim, we need to consider what happened in era (c), when the present ratio of B to L was established. We assume that in era (c), where $T > T_W > M_W$, the electroweak symmetry is unbroken and therefore the chemical potential of the W bosons vanishes and the chemical potentials are equal for all particles within any Standard Model multiplet.

In era (c), we no longer have to consider the quantum number X or the chemical potentials of the X-bearing particles, as they do not affect the ratio of B to L. The important chemical potentials are μ_Q, μ_L, and μ_h. The chemical potentials of the other quarks and leptons are given in terms of these by the relations in eq. (6.5), which are still valid in era (c) for the particles of the Standard Model. Eq. (6.9), which gives the relation arising from the equilibrium of $SU(2)_L$ sphaleron processes, is also still valid.

The strategy is the same as the calculation done in era (b), but simpler. The first step is to use the condition that the universe has $Q = 0$ to derive a formula for μ_h in terms of μ_Q and μ_L. This relation is different from that for era (b), given in eq. (6.7), because in era (b) the charge density included the contributions from the "extra" quarks and leptons in $(5, 1)$ and $(\overline{5}, 2)$, namely the $\overline{\ell}$, $\overline{\nu}$, $\overline{d^c}$ and the half of the $\ell_{1,2}$, $\nu_{1,2}$ and $d^c_{1,2}$ with which they mate to obtain mass. Those fermions are light compared to T in era (b); but in era (c) (or at least near the end of that era) we can neglect them because we assume that they are heavy compared to the electroweak scale and thus highly Boltzmann suppressed.

Therefore the only particles that one must consider in computing the electric charge density are all the fermions of the Standard Model and the three electroweak Higgs doublets h, and $h'_{1,2}$. The relations given in eq. (6.5) that involve only the chemical potentials of these are still valid. All the Standard Model fermions may be treated as massless (since we are assuming that $SU(2)_L$ is unbroken in this era). However, the masses of h and $h'_{1,2}$ must be taken into account. We therefore define the quantity $c_h \equiv [b(m_h/T_W) + b(m_{h'_1}/T_W) + b(m_{h'_2}/T_W)]/b(0)$.

Given all this, one has

$$\begin{aligned}
0 \;=\; Q &\propto 3\mu_L(-1) + 3\mu_{\ell^c}(+1) \\
&\quad 9\mu_Q(+\tfrac{2}{3}) + 9\mu_{u^c}(-\tfrac{2}{3}) + 9\mu_Q(-\tfrac{1}{3}) + 9\mu_{d^c}(+\tfrac{1}{3}) + \\
&\quad (b(0)/f(0))c_h\mu_h(+1) + \qquad (6.13)\\
\Rightarrow \quad 0 &= -6\mu_L + 6\mu_Q + (12 + 2c_h)\mu_h \\
\Rightarrow \quad 0 &= -8\mu_L + (12 + 2c_h)\mu_h \quad \Rightarrow \quad \mu_h = \tfrac{4}{6+c_h}\mu_L,
\end{aligned}$$

where we have used the $SU(2)_L$ sphaleron equilibrium condition $\mu_Q = -\frac{1}{3}\mu_L$, given in eq. (6.9). Now that we have both μ_h and μ_Q in terms of μ_L, we may compute the ratio of B to L. Again, this gives a result different from eq. (6.9), because of the different relation between temperature and mass that holds in era (c). One obtains

$$\begin{aligned}
B &\propto \tfrac{1}{3}\left(18\mu_Q - 9\mu_{u^c} - 9\mu_{d^c}\right) = 12\mu_Q = -4\mu_L \\
L &\propto 6\mu_L - 3\mu_{\ell^c} = 9\mu_L - 3\mu_h = 9\mu_L - 3\tfrac{4}{6+c_h}\mu_L = \tfrac{42+9c_h}{6+c_h}\mu_L,
\end{aligned} \qquad (6.14)$$

Therefore, when T falls below T_W, the ratio L/B is frozen at

$$\frac{L}{B} = -\frac{3}{4}\left(\frac{14 + 3c_h}{6 + c_h}\right). \qquad (6.15)$$

Combining this with eq. (6.12) gives

$$\frac{X}{B} = \frac{6}{19}\left(\frac{7p + 12}{p + 2}\right)\left(\frac{66 + 13c_h}{4(6 + c_h)}\right), \qquad (6.16)$$

which for the minimal case $p = 1$ reduces to

$$\frac{X}{B} = \frac{66 + 13c_h}{2(6 + c_h)}. \qquad (6.17)$$

For the allowed range $0 < c_h < 3$ this varies between 5.5 and 5.833. If, as seems reasonable, one assumes that one linear combination of the three electroweak Higgs doublets (the “Standard Model Higgs doublet”) is much lighter than the others, one would expect $c_h \cong 1$, giving $X/B \cong 5.64$. A value of $X/B \approx 5.6$ implies that the dark matter particle has a mass close to 1 GeV. [Since the Standard Model Higgs doublet will get the VEV, in the section 1.1.1, we know the mass parameter in the Higgs potential $\mu = \sqrt{v^2\lambda}$ and $\lambda \leq 1$. This means μ can't be too large but other linear comninations could be. Therefore, only one will contribute to c_h.]

Besides the dark matter particle itself, there are five other flavors of $\chi(\chi^c)$ particles. These are, by definition, heavier than the dark matter particle and will all have decayed or annihilated by the time the temperature reaches 1 GeV. It is important that the energy released in these decays and annihilations does not get trapped in the dark sector (i.e. the sector of χ, χ^c, σ, and S), as otherwise the thermal energy of the massless S particles at the time of primordial nucleosynthesis might cause the universe to expand too fast, leading to an excessive primordial Helium abundance. However, as long as $M_* < 100$ TeV, the particles of the dark sector are kept in thermal contact with the Standard Model particles by $SU(2)_*$ gauge interactions and do not “overheat".

6.4 Asymmetric Dark Matter Particle and Possible Ways to Detect

In order for the dark matter to be almost purely asymmetric, there must be an efficient mechanism for dark matter particles and their antiparticles to annihilate into massless particles. This is why we introduced the massless S fermion(s) and the scalar σ. Given the Yukawa coupling $y(\chi^c S)\sigma$, which was mentioned earlier, the exchange of a σ allows the annihilation process $\chi^c + \overline{\chi^c} \longrightarrow S + \overline{S}$. In order to have the density of dark matter anti-particles very small compared to the density of dark matter particles, m_σ/y must be less than about 10 GeV. Of course, this involves fine-tuning in the absence of supersymmetry or some other symmetry or mechanism that would make such a small scalar mass natural.

In computing the ratios of B, L, and X above, we made certain assumptions about the $SU(2)_L$ and $SU(2)_*$ dynamics. In particular, we assumed that at the temperature when the anomalous processes of one of these interactions become cosmologically negligible, the interaction in question may be treated as still unbroken. It is possible to make other assumptions []. The result for the B, L, X ratios would not greatly change. But to get an exact result one would need to understand the sphaleron dynamics and the details of the $SU(2)_L$ and $SU(2)_*$ phase transitions well.

The whole scenario depends on there being a global charge X that is conserved except for the $SU(2)_*$ anomaly (and possibly GUT-scale interactions). The question is why there should be such a global $U(1)_X$ and whether it is compatible with grand unification. The answer is that it can arise as an accidental symmetry of the low-energy theory. And, despite appearances, this can easily happen even in a grand unified version of this model. For example, consider an embedding of the model into $SU(5) \times SU(2)_*$. Suppose that all the Yukawa couplings allowed by $SU(5) \times SU(2)_*$ exist, except for $(\overline{5}, 2)(1, 2)(5, 1)_h$. (In other words, the following Yukawa couplings exist: $(10, 1)(10, 1)(5, 1)_h$, $(10, 1)(\overline{5}, 2)(\overline{5}, 2)_h$, $(5, 1)(\overline{5}, 2)(1, 2)_h$, and $(1, 2)(1, 1)(1, 2)_h$. This is easily ensured by a discrete global symmetry that commutes with $SU(5) \times SU(2)_*$. For instance, one can have a Z_N symmetry under which $(10, 1) \rightarrow \omega(10, 1)$, $(5, 1)_h \rightarrow \omega^{*2}(5, 1)_h$, $(\overline{5}, 2)_h \rightarrow \omega^*(\overline{5}, 2)_h$, with all other multiplets transforming trivially. It is easy to show that with the coupling $(\overline{5}, 2)(1, 2)(5, 1)_h$ missing, the global $U(1)_X$ shown in the Table 5.1 arises as an accidental symmetry of the low-energy theory.

If the model like the one we have discussed is correct, how would one observe dark matter in the laboratory? It would be very difficult to produce or detect it directly, since it interacts with the Standard Model particles only by the $SU(2)_*$ gauge interactions, which are much more feeble than the weak interactions, because broken at a much higher scale. On the other hand, the "extra" quarks and leptons that are in $(5, 1)$ and $(\overline{5}, 2)$ could be directly produced in accelerators through their Standard Model interactions. These then could decay into a combination of Standard Model particles and the dark particles $\chi(\chi^c)$ by means of their $SU(2)_*$ gauge interactions. Each "extra" fermion in $(\overline{5}, 2)$ is a partner in an $SU(2)_*$ doublet with a Standard model fermion, to which it can be converted by emitting an $SU(2)_*$ gauge boson. That boson, in turn, can decay into $\chi + \overline{\chi}$). The "extra" fermions can also decay by ordinary charged weak interactions entirely into Standard Model particles. The point is that the "extra" fermions mix slightly with the Standard Model fermions of the same color and charge. For example, the $\overline{d^c}$ mix with the left-handed d, s, and b quarks with mixing angles that are of order $m_{d,s,b}/m_{\overline{d^c}}$, and similarly for the "extra" leptons.

The model described above is an illustration of a general idea that could be implemented in other ways. For example, one can construct an $SU(7)$ unified model that is in many ways quite similar to this. The fermions can be placed in three families, each consisting of $21 + \overline{7} + \overline{7} + \overline{7}$, which is the simplest way to incorporate a family in $SU(7)$. Under the $SU(5) \times SU(2)_*$ subgroup, each family decomposes into $(10, 1) + (5, 2) + (1, 1) + 3 \times (\overline{5}, 1) + 3 \times (1, 2)$. As in the model described earlier in this paper there would be $(1, 2)$ fermions, which could be the dark matter, and "extra fermions" in $5 + \overline{5}$ of the $SU(5)$. A difference with the model described earlier, which would be phenomenologically significant, is that the extra vector-like fermions in the $SU(7)$ model would not be paired in $SU(2)_*$ doublets with ordinary Standard Model quarks and leptons. *Both* components of each $(5, 2)$ get large mass with $(\overline{5}, 1)$ multiplets. Nevertheless, there would be mixing between the "extra fermions" and the Standard Model (SM) fermions. As in the model described earlier, those mixing angles would be of order the ratio of the masses of the SM fermions and extra fermions. The result would be that a heavier extra fermion would predominantly decay into a lighter extra fermion plus a dark matter pair, as its decays into a SM fermion plus dark matter pair would be suppressed by these small mixing angles. The lightest extra quark (or lepton) would have no choice, however, but to decay into SM quarks (or leptons). This would predominantly happen through the weak interactions, since, as in the model described earlier, the $\overline{d^c}$ would mix slightly with the left-handed d, s, and b, and similarly for the leptons.

6.5 Conclusions

We have shown that it is possible to construct simple unified models in which the sphalerons of a new interaction convert asymmetries of B and L into a dark matter asymmetry. Since there are two kinds of sphaleron process involved, the equilibrium conditions allow one to compute the ratios of B, L, and X (the dark matter number) independently of the nature of the primordial asymmetry, e.g. whether it was an asymmetry in B or in L. Since one can compute the ratio of X to B in such models, one obtains a prediction for the mass of the dark matter particle. The dark matter particles in the scenario we describe does not have Standard Model gauge interactions and so would not be easily detectable in a direct way. However, such models generically give rise to extra vectorlike pairs of quarks and leptons that transform like $5 + \overline{5}$ of $SU(5)$. These could be directly produced, and decay into Standard Model fermions plus dark matter particle-antiparticle pairs. The phenomenology of such models remains to be explored.

These models predict the number density of dark matter *a priori* but not their mass, leaving mass of the dark matter particle to be inferred from the measured dark matter density. It would be interesting to see if a model could be constructed which predicts *a priori* both the number density and mass of the dark matter particles.

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