ALGEBRAIC METHODS IN GRAPH THEORY

by

Weiqiang Li

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

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by

Weiqiang Li

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Louis F. Rossi, Ph.D.
Chair of the Department of Mathematics

Approved: ____

George H. Watson, Ph.D.
Dean of the College of Arts and Sciences

Approved: ____

James G. Richards, Ph.D.
Vice Provost for Graduate and Professional Education

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Dr. Sebastian M. Cioabă, Ph.D.
Professor in charge of dissertation

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Dr. Felix Lazebnik, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Dr. Qing Xiang, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Dr. William J. Martin, Ph.D.
Member of dissertation committee

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ABSTRACT

The algebraic methods have been very successful in understanding the structural properties of graphs. In general, we can use the eigenvalues of the adjacency matrix of a graph to study various properties of graphs. In this thesis, we obtain the whole spectrum of a family of graphs called Wenger graphs $W_m(q)$. We also study the a conjecture of Brouwer, concerning the second connectivity of strongly regular graphs. Finally, we compute the extendability of matchings for many strongly regular graphs and many distance-regular graphs.

Chapter 1

INTRODUCTION

1.1 Introduction

In mathematics and computer science, graph theory studies the combinatorial objects called graphs. They are mathematical structures used to model pairwise relations between objects. A *graph* in this context is made up of *vertices* or *nodes* and lines called *edges* that connect them.

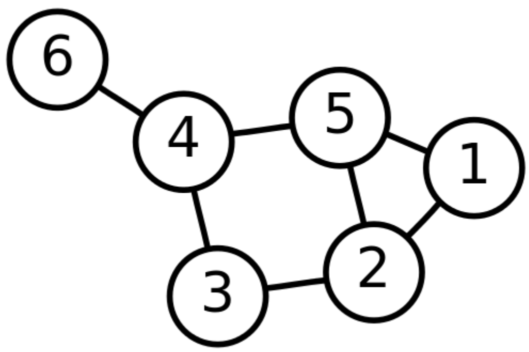

Figure 1.1: An example of graph with 6 vertices and 7 edges.

Using algebraic properties of matrices associated to graphs, we can study the combinatorial properties of graphs. For example, spectral graph theory makes use of Laplacian matrices and adjacency matrices of graphs. Many important combinatorial properties of graphs could be revealed by examining of the eigenvalues and eigenvectors of those matrices. Such properties include connectivity and edge expansion, which will be discussed later in this chapter.

In this thesis, we will focus on adjacency matrices. Adjacency matrices are useful tools in graph theory, both practically and theoretically. The Perron-Frobenius eigenvector of the adjacency matrix of the web graph gives rise to a ranking of all the vertices. It is this very vector that makes the founders of Google billionaires. The second largest eigenvalue of a graph gives information about diameter and expansion. The construction of sparse graphs with large spectral gap (the difference between the first and the second eigenvalues) is a central task in graph theory. Fix an integer k, the k-regular graphs with "largest" spectral gap are Ramanujan graphs. The construction of such graphs helps to build a sparse and highly connected network in internet, thus providing high speed communication and lowering the cost of building the network. Using Ramanujan graphs, many companies (e.g. Akamai Technologies) have constructed Content Distribution Network, one of the most important types of networks nowadays. Such network allows companies (e.g. Microsoft) to provide higher download speed for their software. Theoretically, eigenvalues also help us to understand the structure of graphs. The smallest eigenvalue gives information about independence number and chromatic number. Eigenvalues interlacing gives information about substructures of graphs. The fact that eigenvalue multiplicities must be integral provides strong restrictions. All the eigenvalues together provide a useful graph invariant.

Before we discuss the relations between eigenvalues and graph properties, we first provide some background in graph theory. For undefined notations, see Bollobás [11]. In the most common sense of the term, a *simple* graph is an ordered pair $\Gamma = (V, E)$ comprising a set V of vertices or nodes together with a set E of edges or lines, which are 2-element subsets of V (i.e., an edge is related with two vertices, and the relation is represented as an unordered pair of the vertices with respect to the particular edge). In this thesis, we only consider simple graphs. Assume that $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. We say v_i and v_j are *adjacent* if and only if $e = \{v_i, v_j\} \in E$. In this case, we write $v_i \sim v_j$. For simplicity, an edge $\{v_i, v_j\} \in E$ will be denoted v_iv_j, where v_i and v_j are the endpoints of the edge. If a vertex v is an endpoint of an edge e, we say v is incident with e. I included several important examples of graphs here:

1. The *complete graph* K_n is the graph on n vertices such that every two vertices are joined by an edge.

2. The *complete bipartite graph* $K_{m,n}$ is the graph with vertex set is the union of 2 partite sets $V = X \cup Y$, where $X \cap Y = \emptyset$, $|X| = m$ and $|Y| = n$. Two vertices u and v are adjacent iff they are in different partite sets.

3. The *complete multipartite graph* $K_{m \times n}$ is the graph with vertex set is the union of m partite sets $V = X_1 \cup \ldots \cup X_m$, where $|X_i| = n$. Two vertices u and v are adjacent iff they are in different partite sets.

4. The *path* P_n is the graph with vertex set $[n] := \{1, \ldots, n\}$ such that $i \sim j$ if and only if $|i - j| = 1$.

5. The *cycle* C_n is the graph obtained by adding one edge $\{1, n\}$ to P_n.

6. The *line graph* $L(\Gamma)$ of Γ is the graph with the edge set of Γ as vertex set, where two vertices are adjacent if the corresponding edges of Γ have an endpoint in common.

Here are some general notations which will be use in this thesis. The *degree* of a vertex v is the number of edges incident with v. A graph Γ is called *regular* of degree (or valency) k, when every vertex has degree k. The *neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices that are adjacent to v. A *subgraph* H of a graph Γ is a graph with $V(H) \subseteq V(\Gamma)$ and $E(H) \subseteq E(\Gamma)$. A graph H is an *induced subgraph* of Γ if $E(H) = \{uv \in E(\Gamma) \mid u, v \in V(H)\}$. If H is an induced subgraph of Γ with vertex set $S = V(H)$, the subgraph induced by S is denoted $\Gamma[S]$. The *complement* of a graph Γ, denoted $\bar{\Gamma}$, is the graph with $V(\bar{\Gamma}) = V(\Gamma)$ and $uv \in E(\bar{\Gamma})$ if and only if $uv \notin E(\Gamma)$. A *clique* in a graph Γ is a set that induces a complete graph. An *independent set*, or *coclique*, in a graph is a set that induces an empty graph, the complement of a complete graph. The cardinality of the largest independent set of a graph Γ is denoted $\alpha(\Gamma)$. A *walk* of length l between two vertices u and v, is a sequence of vertices $u = u_0, u_1, \ldots, u_l = v$, such that for $0 \leq i \leq k-1$, u_i is adjacent to u_{i+1}. If all these vertices are distinct, the walk is a *path*. If there exists a path between any two vertices, the graph is *connected*. Otherwise, the graph is *disconnected*. A *component* of Γ is a maximal connected subgraph. For a connected graph Γ and any two vertices $u, v \in V(\Gamma)$, the *distance* $d(u, v)$ between u and v is the minimum length of any path connecting them. The *diameter* D is defined to be $\max_{u,v \in V(\Gamma)} d(u, v)$. Note that if Γ is disconnected, we defined the diameter to be infinity.

I will need the following important properties of graphs:

1. A graph Γ is called *k-connected* if deleting any $k-1$ vertices does not disconnect the graph. The connectivity of a graph is defined to be the largest k, such that the graph is k-connected.

2. Similarly, a graph Γ is called *k-edge-connected* if deleting any $k-1$ edges does

not disconnect the graph. The edge-connectivity of a graph is defined to be the largest k, such that the graph is k-edge-connected.

3. A graph Γ is called *bipartite* when its vertex set can be partitioned into two disjoint parts X, Y such that both X and Y are independent sets.

1.2 Adjacency Matrices

Let Γ be a finite simple graph. The adjacency matrix of Γ is the 0-1 matrix A indexed by the vertex set $V(\Gamma)$ of Γ, where $A_{xy} = 1$ when there is an edge from x to y in Γ and $A_{xy} = 0$ otherwise.

Suppose Γ is a simple graph with n vertices. Since A is real and symmetric, all its eigenvalues are real. The spectrum of Γ is defined as the multiset of the eigenvalues of its adjacency matrix. We write the eigenvalues in decreasing order, that is $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$.

There are a lot of things we can say about a graph by looking at its spectrum. We will present some useful facts which will be used in the later sections.

If Γ is regular of degree k, then its adjacency matrix A has row sums k. We can write it in matrix form, $A\vec{1} = k\vec{1}$, where $\vec{1}$ is a column vector with all entries equal to 1. Actually, k is the largest eigenvalue of Γ, and the multiplicity of k equals the number of components of Γ. So, $k \geq \theta_2 \geq \cdots \geq \theta_n$.(See [19, Chapter 3])

If Γ is bipartite, then its adjacency matrix is of the form $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. It follows that the spectrum of a bipartite graph is symmetric with respect to 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue θ, then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with eigenvalue $-\theta$. The converse is also true. Actually, we have the following theorem.

Theorem 1.2.1. *[19, Proposition 3.4.1]*

(i) A graph Γ is bipartite if and only if for each eigenvalue θ of Γ, also $-\theta$ is an eigenvalue, with the same multiplicity.

(ii) If Γ is connected with largest eigenvalue θ_1, then Γ is bipartite if and only if $-\theta_1$ is an eigenvalue of Γ.

Another interesting result is that we can give an upper bound for the diameter in terms of the number of distinct eigenvalues:

Theorem 1.2.2. *[19, Proposition 1.3.3] Let Γ be a connected graph with l distinct eigenvalues, then the diameter of Γ is at most $l-1$.*

1.3 Interlacing

A powerful tool used in spectral graph theory is eigenvalue interlacing (for more details, see [58]). Consider two sequences of real numbers: $\theta_1 \geq \cdots \geq \theta_n$, and $\eta_1 \geq \cdots \geq \eta_m$ with $m < n$. The second sequence is said to *interlace* the first one if for $i = 1, \ldots, m$,

$$\theta_i \geq \eta_i \geq \theta_{n-m+i}.$$

The interlacing is *tight* if there exists an integer k, $0 \leq k \leq m$, such that

$$\theta_i = \eta_i \text{ for } 1 \leq i \leq k \text{ and } \theta_{n-m+i} = \eta_i \text{ for } k+1 \leq i \leq m$$

Theorem 1.3.1 (Haemers [58])**.** *Let S be a real $n \times m$ matrix such that $S^T S = I$. Let A be a real symmetric matrix of order n with eigenvalues $\theta_1 \geq \ldots \geq \theta_n$. Define $B = S^T AS$ and let B have eigenvalues $\eta_1 \geq \ldots \geq \eta_m$ and respective eigenvectors $v_1, \ldots, v_m$.*

(i) The eigenvalues of B interlace those of A.

(ii) *If* $\eta_i = \theta_i$ *or* $\eta_i = \theta_{n-m+i}$ *for some* $i \in [1, m]$*, then* B *has a* η_i*-eigenvector* v *such that* Sv *is a* η_i*-eigenvector of* A*.*

(iii) *If for some integer* l*,* $\eta_i = \theta_i$*, for* $i = 1, \ldots, l$ *(or* $\eta_i = \theta_{n-m+i}$ *for* $i = l, \ldots, m$*), then* Sv_i *is a* η_i*-eigenvector of* A *for* $i = 1, \ldots, l$ *(respectively* $i = l, \ldots, m$*).*

(iv) *If the interlacing is tight, then* $SB = AS$*.*

If we take $S = [I, 0]^T$, then B is just a principal submatrix of A and we have the following consequence.

Theorem 1.3.2 (Cauchy Interlacing)**.** *[63, p.185] If* B *is a principal submatrix of a symmetric matrix* A*, then the eigenvalues of* B *interlace the eigenvalues of* A*.*

This result implies that if H is an induced subgraph of a graph Γ, then the eigenvalues of H interlace the eigenvalues of Γ. In order to get more eigenvalue interlacing results, we need to introduce another important matrix.

Consider a partition $V(\Gamma) = V_1 \cup \ldots \cup V_s$ of the vertex set of Γ into s non-empty subsets. For $1 \leq i, j \leq s$, let $b_{i,j}$ denote the average number of neighbors in V_j of the vertices in V_i. The *quotient matrix* of this partition is the $s \times s$ matrix whose (i, j)-th entry equals $b_{i,j}$. The partition is called *equitable* if for each $1 \leq i, j \leq s$, any vertex $v \in V_i$ has exactly $b_{i,j}$ neighbors in V_j. Another consequence of Theorem 1.3.1 is that the eigenvalues of a quotient matrix interlace the eigenvalues of the adjacency matrix of a graph.

Theorem 1.3.3. *For a graph* Γ*, let* B *be the quotient matrix of a partition of* $V(\Gamma)$*. Then the eigenvalues of* B *interlace those of* A*. Furthermore, if the interlacing is tight, then the partition is equitable.*

A consequence of Theorem 1.3.3 is the famous Hoffman Ratio bound. This bound will be used in Chapters 4 and 5.

Theorem 1.3.4 (Hoffman Ratio bound)**.** *(See [19, Theorem 3.5.2]) If Γ is a connected, k-regular graph, then*

$$\alpha(\Gamma) \leq n\frac{-\theta_n}{k-\theta_n},$$

and if an independent set C meets this bound, then every vertex not in C is adjacent to precisely $-\theta_n$ vertices of C.

Proof. We can partition the vertex set $V(\Gamma)$ into C and $V(\Gamma)\backslash C$. The corresponding quotient matrix of A is

$$B = \begin{bmatrix} 0 & k \\ \frac{k\alpha}{n-\alpha} & k - \frac{k\alpha}{n-\alpha} \end{bmatrix},$$

where $\alpha = \alpha(\Gamma)$. The matrix B has eigenvalues $\eta_1 = k = \theta_1$ (the row sum) and $\eta_2 = -k\alpha/(n-\alpha)$ (the matrix trace). By interlacing, $\theta_n \leq \eta_2 = -k\alpha/(n-\alpha)$. This gives the required inequality. The second part of the theorem follows from the fact that when equality happens, the interlacing is tight. Thus, the partition is equitable. □

1.4 Spectral Gap and Edge Expansion

Assume that Γ is k-regular. One of the most well-studied eigenvalue of Γ is the second eigenvalue θ_2. (See the survey by Hoory, Linial and Wigderson [66]) If the spectral gap $k-\theta_2$ is large, then the graph has good connectivity, expansion and randomness properties. In this section, we discuss the relation between the spectral gap and edge expansion.

Let A and B be vertex-disjoint subset of $V(\Gamma)$. Let $E(A,B)$ denote the set of edges with one endpoint in A and the other one in B, and $e(A,B)$ be the size of $E(A,B)$. In particular, $E(A,A^c)$ is a cut-set of Γ, where $A^c := V(\Gamma) \setminus A$.

Theorem 1.4.1 ([18, 85]). *Let Γ be a connected k-regular graph with v vertices. If A is a vertex subset of size a, then*

$$e(A, A^c) \geq \frac{(k - \theta_2)a(v - a)}{v}.$$

The *edge expansion constant* $h(\Gamma)$ (a.k.a. *isoperimetric constant* or *Cheeger number*) of a graph Γ is defined as the minimum of $\frac{e(S, V \setminus S)}{|S|}$, where the minimum is taken over all non-empty S with $|S| \leq |V(\Gamma)|/2$.

Theorem 1.4.2 (Alon and Milman [3]). *Let Γ be a regular graph of degree k, not K_n with $n \leq 3$. Then*

$$\frac{1}{2}(k - \theta_2) \leq h(\Gamma) \leq \sqrt{k^2 - \theta_2^2}.$$

1.5 Groups, Characters and Finite Fields

Abstract algebra plays an important role in graph theory because many important families of graphs can be constructed using well-understood algebraic structures. On the other hand, the spectrum of many graphs can be computed by character sums. In this section, I will review some basic facts of abstract algebra. All these results can be found in many algebra books, for example, see Isaacs [68] or Dummit and Foote [47].

A *group* G is a set together with a binary operation $*$ (say) satisfying the following axioms:

1. $a, b \in G$ implies $a * b \in G$ (closure);
2. $a, b, c \in G$ implies $(a * b) * c = a * (b * c)$ (associativity);
3. there is an element called the *identity* $e \in G$ such that $a * e = e * a = a$ for all $a \in G$ (identity element);

4. for any $a \in G$, there is a $b \in G$ so that $a * b = b * a = e$ (inverses); we write a^{-1} to denote the inverse of a.

If in addition to this, $a * b = b * a$ for all $a, b \in G$, we say that G is *abelian* or *commutative*. When G is finite, we call the size of G the *order* of G. To indicate $a * b$ we sometimes drop the $*$ and simply write ab with no cause for confusion. In this case, we assume that the group is multiplicative and we use the symbol 1 to denote the identity element. If we want to emphasize that the groups we are dealing with are additive, we will write the group operation as the symbol $+$. And the symbol 0 will be the identity element.

Here are some examples of groups which will be used later:

1. $\mathbb{Z}$ under addition.

2. $\mathbb{C}^*$, non-zero complex numbers under multiplication.

3. $\mathbb{Z}/n\mathbb{Z}$ under addition consists of residue classes modulo n. This is a finite abelian group of order n.

4. $(\mathbb{Z}/p\mathbb{Z})^*$ is the set of coprime residue classes mod p, with p prime. This is a finite abelian group of order $p - 1$.

Theorem 1.5.1. *If G is a finite abelian group of order n, then $g^n = 1$ for any element $g \in G$.*

A *character* χ of an abelian group G is a map

$$\chi : G \to \mathbb{C}^*,$$

such that $\chi(ab) = \chi(a)\chi(b)$. It is an example of a *homomorphism*. The character that sends every element to the element 1 is called the *trivial character*. It is easy to check the following facts about characters,

1. $\chi(1) = 1$;

2. $\chi(a^{-1}) = \chi(a)^{-1}$;

3. If G is a finite group of order n, then $\chi(g)$ must be an n-th root of unity.

If G is a finite abelian group of order n, then there are exactly n distinct characters. In this case, the set of characters forms a group under multiplication of characters, where the product of two characters χ and ψ is defined in the following way:

$$(\chi\psi)(a) := \chi(a)\psi(a),$$

where χ and ψ are characters from G to $\mathbb{C}^*$. We call this the character group of G and denote it by $\hat{G}$. The identity element of $\hat{G}$ is the trivial character. The character inverse to χ is χ^{-1} defined by

$$\chi^{-1}(a) = \chi(a)^{-1}.$$

A *field* F is a set together with two binary operations $+$ and $\circ$ satisfying the following three conditions:

1. the algebraic structure $(F, +)$ is an abelian group.

2. the algebraic structure $(F^*, \circ)$ is an abelian group, where $F^* = F \setminus \{0\}$ and 0 is the identity element in $(F, +)$.

3. the operation $\circ$ distributes over $+$, i.e. for all $x, y, z \in F$, we have $x \circ (y + z) = (x \circ y) + (x \circ z)$.

Well-known examples are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ under addition and multiplication.

A *finite field* is a field with a finite number of elements. We have already seen that $\mathbb{Z}/p\mathbb{Z}$ is a finite field when p is a prime number. Actually, we have the following two theorems for finite fields:

Theorem 1.5.2. *The number of elements in a finite field is a power of a prime.*

Theorem 1.5.3. *Let $n \geq 1$ be an integer and p be a prime. Then there exists a unique finite field with p^n elements.*

From now on, we will use $\mathbb{F}_q$ to denote the finite field with q elements, where q is a prime power. As $\mathbb{F}_q$ has two operations, each operation will give rise to a character. We will just mention the additive characters since will need to use them in Chapter 2.

Let $\omega = e^{2\pi i/p}$ be a complex p-th root of unity, where i is the imaginary number. For $x \in \mathbb{F}_q$, the trace of x is defined as $tr(x) = \sum_{k=0}^{e-1} x^{p^k}$. Since $(\mathbb{F}_q, +)$ has q characters, we can index each character by an element of $\mathbb{F}_q$. Let ψ_j be one of those q characters of $(\mathbb{F}_q, +)$, where $j \in \mathbb{F}_q$. When applying ψ_j to an element $a \in \mathbb{F}_q$, we have $\psi_j(a) = \omega^{tr(ja)}$.

1.6 Cayley Graphs

There is a simple procedure for constructing k-regular graphs. It proceeds as follows. Let G be an abelian group and S a k-element subset of G. We require that S is *symmetric* in the sense that $s \in S$ implies $s^{-1} \in S$. The *Cayley graph* on G with generating set S is the graph Γ with vertex set G and edge set $E = \{(x, y) \mid x^{-1}y \in S\}$. We denote this graph by $\mathrm{Cay}(G, S)$. Note that $\mathrm{Cay}(G, S)$ is regular with degree $|S|$. One immediate example is the cycle C_n, which is $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{-1, 1\})$, where the group is under addition.

Suppose that χ is a character of G. It is easy to check that $(\chi(g))_{g\in G}$ is an eigenvector of the adjacency matrix of $\mathrm{Cay}(G,S)$. The eigenvalues of the Cayley graph of an abelian group are easily determined as follows.

Theorem 1.6.1. *Let G be a finite abelian group and S a symmetric subset of G of size k. Then the eigenvalues of the adjacency matrix of Cay(G,S) are given by*

$$\theta_\chi = \sum_{s\in S} \chi(s),$$

where χ ranges over all the characters of G.

There is a generalization of the above theorem to non-abelian groups. This is essentially contained in Babai [5] and Diaconis and Shahshahani [46]. To state the theorem, we need to use the concept of an irreducible character. For details of irreducible characters, see Sagan [94].

Theorem 1.6.2. *Let G be a finite group and S a symmetric subset which is stable under conjugation. Then the eigenvalues of the adjacency matrix of Cay(G,S) are given by*

$$\theta_\chi = \frac{1}{\chi(1)} \sum_{s\in S} \chi(s)$$

where χ ranges over all irreducible characters of G. Moreover, the multiplicity of θ_χ is $\chi(1)^2$.

Chapter 2

SPECTRUM OF WENGER GRAPHS

In this chapter, we will compute the whole spectrum of a family of graph $W_m(q)$ which are defined by a family of equations over finite fields. It turns out that for certain parameter sets, those graphs have large spectral gap, and thus have large edge expansion. Most of the results of this chapter have appeared in Cioabă, Lazebnik and Li [35].

2.1 Introduction

Let $q = p^e$, where p is a prime and $e \geq 1$ is an integer. For $m \geq 1$, let P and L be two copies of the $(m+1)$-dimensional vector spaces over the finite field $\mathbb{F}_q$. We call the elements of P *points* and the elements of L *lines*. If $a \in \mathbb{F}_q^{m+1}$, then we write $(a) \in P$ and $[a] \in L$. Consider the bipartite graph $W_m(q)$ with partite sets P and L defined as follows: a point $(p) = (p_1, p_2, \ldots, p_{m+1}) \in P$ is adjacent to a line $[l] = [l_1, l_2, \ldots, l_{m+1}] \in L$ if and only if the following m equalities hold:

$$l_2 + p_2 = l_1 p_1 \tag{2.1}$$

$$l_3 + p_3 = l_2 p_1 \tag{2.2}$$

$$\vdots \tag{2.3}$$

$$l_{m+1} + p_{m+1} = l_m p_1. \tag{2.4}$$

The graph $W_m(q)$ has $2q^{m+1}$ vertices, is q-regular and has q^{m+2} edges.

In [105], Wenger introduced a family of p-regular bipartite graphs $H_k(p)$ as follows. For every $k \geq 2$, and every prime p, the partite sets of $H_k(p)$ are two copies of integer sequences $\{0, 1, \ldots, p-1\}^k$, with vertices $a = (a_0, a_1, \ldots, a_{k-1})$ and $b = (b_0, b_1, \ldots, b_{k-1})$ forming an edge if

$$b_j \equiv a_j + a_{j+1} b_{k-1} \pmod{p} \text{ for all } j = 0, \ldots, k-2.$$

The introduction and study of these graphs were motivated by an extremal graph theory problem of determining the largest number of edges in a graph of order n containing no cycle of length $2k$. This parameter also known as the Turán number of the cycle C_{2k}, is denoted by $\mathrm{ex}(n, C_{2k})$. Bondy and Simonovits [12] showed that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$, $n \to \infty$. Lower bounds of magnitude $n^{1+1/k}$ were known (and still are) for $k = 2, 3, 5$ only, and the graphs $H_k(p)$, $k = 2, 3, 5$, provided new and simpler examples of such magnitude extremal graphs. For many results on $\mathrm{ex}(n, C_{2k})$, see Verstraëte [101], Pikhurko [93] and references therein.

In [71], Lazebnik and Ustimenko, using a construction based on a certain Lie algebra, arrived at a family of bipartite graphs $H'_n(q)$, $n \geq 3$, q is a prime power, whose partite sets were two copies of $\mathbb{F}_q^{n-1}$, with vertices $(p) = (p_2, p_3, \ldots, p_n)$ and $[l] = [l_1, l_3, \ldots, l_n]$ forming an edge if

$$l_k - p_k = l_1 p_{k-1} \text{ for all } k = 3, \ldots, n.$$

It is easy to see that for all $k \geq 2$ and prime p, graphs $H_k(p)$ and $H'_{k+1}(p)$ are isomorphic, and the map

$$\phi : (a_0, a_1, \ldots, a_{k-1}) \mapsto (a_{k-1}, a_{k-2}, \ldots, a_0),$$
$$(b_0, b_1, \ldots, b_{k-1}) \mapsto [b_{k-1}, b_{k-2}, \ldots, b_0],$$

provides an isomorphism from $H_k(p)$ to $H'_{k+1}(p)$. Hence, graphs $H'_n(q)$ can be viewed as generalizations of graphs $H_k(p)$. It is also easy to show that graphs $H'_{m+2}(q)$ and $W_m(q)$ are isomorphic: the function

$$\psi : (p_2, p_3, \ldots, p_{m+2}) \mapsto [p_2, p_3, \ldots, p_{m+2}],$$
$$[l_1, l_3, \ldots, l_{m+2}] \mapsto (-l_1, -l_3, \ldots, -l_{m+1}),$$

mapping points to lines and lines to points, is an isomorphism of $H'_{m+2}(q)$ to $W_m(q)$. Combining this isomorphism with the results in [71], we obtain that the graph $W_1(q)$ is isomorphic to an induced subgraph of the point-line incidence graph of the projective plane $PG(2, q)$, the graph $W_2(q)$ is isomorphic to an induced subgraph of the point-line incidence graph of the generalized quadrangle $Q(4, q)$, and $W_3(q)$ is a homomorphic image of an induced subgraph of the point-line incidence graph of the generalized hexagon $H(q)$.

We call the graphs $W_m(q)$ *Wenger graphs*. The representation of Wenger graphs as $W_m(q)$ graphs first appeared in Lazebnik and Viglione [73]. These authors suggested another useful representation of these graphs, where the right-hand sides of equations are represented as monomials of p_1 and l_1 only, see [102]. For this, define a bipartite graph $W'_m(q)$ with the same partite sets as $W_m(q)$, where $(p) = (p_1, p_2, \ldots, p_{m+1})$ and $[l] = [l_1, l_2, \ldots, l_{m+1}]$ are adjacent if

$$l_k + p_k = l_1 p_1^{k-1} \text{ for all } k = 2, \ldots, m+1. \tag{2.5}$$

The map

$$\omega : (p) \mapsto (p_1, p_2, p'_3, \ldots, p'_{m+1}), \text{ where } p'_k = p_k + \sum_{i=2}^{k-1} p_i p_1^{k-i}, \ k = 3, \ldots, m+1,$$
$$[l] \mapsto [l_1, l_2, \ldots, l_{m+1}],$$

defines an isomorphism from $W_m(q)$ and $W'_m(q)$.

It was shown in [71] that the automorphism group of $W_m(q)$ acts transitively on each of P and L, and on the set of edges of $W_m(q)$. In other words, the graphs $W_m(q)$ are point-, line-, and edge-transitive. A more detailed study, see [73], also showed that $W_1(q)$ is vertex-transitive for all q, and that $W_2(q)$ is vertex-transitive for even q. For all $m \geq 3$ and $q \geq 3$, and for $m = 2$ and all odd q, the graphs $W_m(q)$ are not vertex-transitive. Another result of [73] is that $W_m(q)$ is connected when $1 \leq m \leq q-1$, and disconnected when $m \geq q$, in which case it has q^{m-q+1} components, each isomorphic to $W_{q-1}(q)$. In [103], Viglione proved that when $1 \leq m \leq q-1$, the diameter of $W_m(q)$ is $2m+2$. Note that the statement about the number of components of $W_m(q)$ becomes apparent from the representation (2.5). Indeed, as $l_1 p_1^i = l_1 p_1^{i+q-1}$, all points and lines in a component have the property that their coordinates i and j, where $i \equiv j \mod (q-1)$, are equal. Hence, points (p), having $p_1 = \ldots = p_q = 0$, and at least one distinct coordinate p_i, $q+1 \leq i \leq m+1$, belong to different components. This shows that the number of components is at least q^{m-q+1}. As $W_{q-1}(q)$ is connected and $W_m(q)$ is edge-transitive, all components are isomorphic to $W_{q-1}(q)$. Hence, there are exactly q^{m-q+1} of them. A result of Watkins [104], and the edge-transitivity of $W_m(q)$ imply that the vertex connectivity (and consequently the edge connectivity) of $W_m(q)$ equals the degree of regularity q, for any $1 \leq m \leq q-1$.

Shao, He and Shan [96] proved that in $W_m(q)$, $q = p^e$, p prime, for $m \geq 2$, for any integer $l \neq 5, 4 \leq l \leq 2p$ and any vertex v, there is a cycle of length $2l$ passing through the vertex v. Note that the edge-transitivity of $W_m(q)$ implies the existence of a $2l$ cycle through any edge, a stronger statement. Li and Lih [74] used the Wenger graphs to determine the asymptotic behavior of the Ramsey number $r_n(C_{2k}) = \Theta(n^{k/(k-1)})$ when $k \in \{2, 3, 5\}$ and $n \to \infty$; the Ramsey number $r_n(G)$

equals the minimum integer N such that in any edge-coloring of the complete graph K_N with n colors, there is a monochromatic G. Representation (2.5) points to a relation of Wenger graphs with the moment curve $t \mapsto (1, t, t^2, t^3, ..., t^m)$, and, hence, with Vandermonde's determinant, which was explicitly used in [105]. This is also in the background of some geometric constructions by Mellinger and Mubayi [84] of magnitude extremal graphs without short even cycles.

In Section 2.2, we determine the spectrum of the graphs $W_m(q)$. Futorny and Ustimenko [52] considered applications of Wenger graphs in cryptography and coding theory, as well as some generalizations. They also conjectured that the second largest eigenvalue θ_2 of the adjacency matrix of Wenger graphs $W_m(q)$ is bounded from above by $2\sqrt{q}$. The results of this chapter confirm the conjecture for $m = 1$ and 2, or $m = 3$ and $q \geq 4$, and refute it in other cases. We wish to point out that for $m = 1$ and 2, or $m = 3$ and $q \geq 4$, the upper bound $2\sqrt{q}$ also follows from the known values of θ_2 for the point-line $(q+1)$-regular incidence graphs of the generalized polygons $PG(2, q)$, $Q(4, q)$ and $H(q)$ and eigenvalue interlacing (see Brouwer, Cohen and Neumaier [17]). In [75], Li, Lu and Wang showed that the graphs $W_m(q)$, $m = 1, 2$, are Ramanujan, by computing the eigenvalues of another family of graphs described by systems of linear equations in [72], $D(k, q)$, for $k = 2, 3$. Their result follows from the fact that $W_1(q) \simeq D(2, q)$, and $W_2(q) \simeq D(3, q)$. For more on Ramanujan graphs, see Lubotzky, Phillips and Sarnak [82], or Murty [87]. Our results also imply that for fixed m and large q, the Wenger graph $W_m(q)$ are expanders. For more details on expanders and their applications, see Hoory, Linial and Wigderson [66], and references therein.

2.2 Main Results

Theorem 2.2.1. *For all prime power q and $1 \le m \le q-1$, the distinct eigenvalues of $W_m(q)$ are*

$$\pm q,\ \pm\sqrt{mq},\ \pm\sqrt{(m-1)q},\ \cdots,\ \pm\sqrt{2q},\ \pm\sqrt{q},\ 0. \tag{2.6}$$

The multiplicity of the eigenvalue $\pm\sqrt{iq}$ of $W_m(q)$, $0 \le i \le m$, is

$$(q-1)\binom{q}{i}\sum_{d=i}^{m}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k}. \tag{2.7}$$

Proof. As the graph $W_m(q)$ is bipartite with partitions L and P, we can arrange the rows and the columns of an adjacency matrix A of $W_m(q)$ such that A has the following form:

$$A = \begin{array}{cc} & \begin{array}{cc} L & P \end{array} \\ \begin{array}{c} L \\ P \end{array} & \begin{pmatrix} 0 & N^T \\ N & 0 \end{pmatrix} \end{array} \tag{2.8}$$

which implies that

$$A^2 = \begin{pmatrix} N^TN & 0 \\ 0 & NN^T \end{pmatrix}. \tag{2.9}$$

As the matrices N^TN and NN^T have the same spectrum, we just need to compute the spectrum for one of these matrices. To determine the spectrum of N^TN, let H denote the point-graph of $W_m(q)$ on L. This means that the vertex set of H is L, and two distinct lines $[l]$ and $[l']$ of $W_m(q)$ are adjacent in H if there exists a point $(p) \in P$, such that $[l] \sim (p) \sim [l']$ in $W_m(q)$. More precisely, $[l]$ and $[l']$ are adjacent in H, if there exists $p_1 \in \mathbb{F}_q$ such that for all $i = 1, \ldots, m$, we have

$$l_1 \neq l_1' \text{ and } l_{i+1} - l_{i+1}' = p_1(l_i - l_i') \iff$$
$$l_1 \neq l_1' \text{ and } l_{i+1} - l_{i+1}' = p_1^i(l_1 - l_1').$$

This implies that H is actually the Cayley graph of the additive group of the vector space $\mathbb{F}_q^{m+1}$ with a generating set

$$S = \{(t, tu, \dots, tu^m) \mid t \in \mathbb{F}_q^*, u \in \mathbb{F}_q\}. \tag{2.10}$$

Let ω be a complex p-th root of unity. Recall that for $x \in \mathbb{F}_q$, the trace of x is defined as $tr(x) = \sum_{k=0}^{e-1} x^{p^k}$. The eigenvalues of H are indexed after the $(m+1)$-tuples $(w_1, \dots, w_{m+1}) \in \mathbb{F}_q^{m+1}$. By Theorem 1.6.1, they can be represented in the following form:

$$\begin{aligned}
\theta_{(w_1,\dots,w_{m+1})} &= \sum_{(t,tu,\dots,tu^m)\in S} \omega^{tr(tw_1)} \cdot \omega^{tr(tuw_2)} \cdot \dots \cdot \omega^{tr(tu^m w_{m+1})} \\
&= \sum_{t\in\mathbb{F}_q^*,\, u\in\mathbb{F}_q} \omega^{tr(tw_1+tuw_2+\dots+tu^m w_{m+1})} \\
&= \sum_{t\in\mathbb{F}_q^*,\, u\in\mathbb{F}_q} \omega^{tr(t(f(u)))} \text{ (where } f(u) := w_1 + w_2 u + \dots + w_{m+1} u^m) \\
&= \sum_{t\in\mathbb{F}_q^*,\, f(u)=0} \omega^{tr(t(f(u)))} + \sum_{t\in\mathbb{F}_q^*,\, f(u)\neq 0} \omega^{tr(t(f(u)))}.
\end{aligned}$$

Note that $\theta_{(w_1,\dots,w_{m+1})}$ here can be considered as a Jacobi sum. For more details regarding Jacobi sums, see Ireland and Rosen [67, Chapter 8]. As $\sum_{t\in\mathbb{F}_q^*} \omega^{tr(tx)} = q-1$ for $x = 0$, and $\sum_{t\in\mathbb{F}_q^*} \omega^{tr(tx)} = -1$ for every $x \in \mathbb{F}_q^*$, we obtain that

$$\theta_{(w_1,\dots,w_{m+1})} = |\{u \in \mathbb{F}_q \mid f(u) = 0\}|(q-1) - |\{u \in \mathbb{F}_q \mid f(u) \neq 0\}|. \tag{2.11}$$

Let B be the adjacency matrix of H. Then $N^T N = B + qI$. This implies that the eigenvalues of $W_m(q)$ can be written in the form

$$\pm\sqrt{\theta_{(w_1,\dots,w_{m+1})} + q},$$

where $(w_1, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$. Let $f(X) = w_1 + w_2X + \cdots + w_{m+1}X^m \in \mathbb{F}_q[X]$. We consider two cases.

1. $f = 0$. In this case, $|\{u \in \mathbb{F}_q \mid f(u) = 0\}| = q$, and $\theta_{(w_1,\ldots,w_{m+1})} = q(q-1)$. Thus, $W_m(q)$ has $\pm q$ as its eigenvalues.

2. $f \neq 0$. In this case, let $i = |\{u \in \mathbb{F}_q \mid f(u) = 0\}| \leq m$ as $1 \leq m \leq q-1$. This shows that $\theta_{(w_1,\ldots,w_{m+1})} = i(q-1) - (q-i) = iq - q$ and implies that $\pm\sqrt{\theta_{(w_1,\ldots,w_{m+1})} + q} = \pm\sqrt{iq}$ are eigenvalues of $W_m(q)$. Note that for any $0 \leq i \leq m$, there exists a polynomial f over $\mathbb{F}_q$ of degree at most $m \leq q-1$, which has exactly i distinct roots in $\mathbb{F}_q$. For such f, $|\{u \in \mathbb{F}_q \mid f(u) = 0\}| = i$, and, hence, there exists $(w_1, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, such that $\theta_{(w_1,\ldots,w_{m+1})} = iq - q$. Thus, $W_m(q)$ has $\pm\sqrt{iq}$ as its eigenvalues, for any $0 \leq i \leq m$, and the first statement of the theorem is proven.

The arguments above imply that the multiplicity of the eigenvalue $\pm\sqrt{iq}$ of $W_m(q)$ equals the number of polynomials of degree at most m (not necessarily monic) having exactly i distinct roots in $\mathbb{F}_q$. To calculate these multiplicities, we need the following lemma. Particular cases of the lemma were considered in Zsigmondy [108], and in Cohen [37]. The complete result appears in A. Knopfmacher and J. Knopfmacher [70].

Lemma 2.2.2 ([70]). *Let q be a prime power, and let d and i be integers such that $0 \leq i \leq d \leq q-1$. Then the number $b(q,d,i)$ of monic polynomials in $\mathbb{F}_q[X]$ of degree d, having exactly i distinct roots in $\mathbb{F}_q$ is given by*

$$b(q,d,i) = \binom{q}{i} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k}. \tag{2.12}$$

By Lemma 2.2.2, the number of polynomials of degree at most m in $\mathbb{F}_q[X]$ (not necessarily monic) having exactly i distinct roots in $\mathbb{F}_q$ is

$$\sum_{d=i}^{m}(q-1)\, b(q,d,i) = (q-1)\binom{q}{i}\sum_{d=i}^{m}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k}. \tag{2.13}$$

This concludes the proof the theorem. □

The previous result shows that $W_m(q)$ is connected and has $2m+3$ distinct eigenvalues, for any $1 \le m \le q-1$. By Theorem 1.2.2, the diameter of a graph is strictly less than the number of distinct eigenvalues. This implies that the diameter of Wenger graph is less or equal to $2m+2$. This is actually the exact value of the diameter of the Wenger graph as shown by Viglione [103].

Since the sum of multiplicities of all eigenvalues of the graph $W_m(q)$ is equal to its order, and remembering that the multiplicity of $\pm q$ is one when $1 \le m \le q-1$, we have a combinatorial proof of the following identity.

Corollary 2.2.3. *For every prime power q, and every m, $1 \le m \le q-1$,*

$$\sum_{i=0}^{m}\binom{q}{i}\sum_{d=i}^{m}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k} = \frac{q^{m+1}-1}{q-1}. \tag{2.14}$$

The identity (2.14) seem to hold for all integers $q \ge 3$, so a direct proof is desirable. Other identities can be obtained by taking the higher moments of the eigenvalues of $W_m(q)$.

As we discussed in the introduction, for $m \ge q$, the graph $W_m(q)$ has q^{m-q+1} components, each isomorphic to $W_{q-1}(q)$. This, together with Theorem 2.2.1, immediately implies the following.

Proposition 2.2.4. *For $m \ge q$, the distinct eigenvalues of $W_m(q)$ are*

$$\pm q,\ \pm\sqrt{(q-1)q},\ \pm\sqrt{(q-2)q},\ \cdots,\ \pm\sqrt{2q},\ \pm\sqrt{q},\ 0,$$

and the multiplicity of the eigenvalue $\pm\sqrt{iq}$, $0 \le i \le q-1$, *is*

$$(q-1)q^{m+1-q}\binom{q}{i}\sum_{d=i}^{q}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k}.$$

2.3 Remarks

In 1995, Lazebnik and Ustimenko [72] constructed an infinite family of graphs $D(k,q)$ of high density and without any cycle of length strictly less than $k+5$. Their construction is as follows:

Let q be a prime power, and let P and L be two copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. By $D(q)$ we denote a bipartite graph with the bi-partition (P, L) and edges defined as follows. We say that vertices $(p) \in (p_1, p_2, p_3, \ldots) \in P$ and $[l] \in [l_1, l_2, l_3, \ldots] \in L$ are adjacent if and only if the following relations on their coordinates hold:

$$\begin{aligned} l_2 - p_2 &= l_1 p_1 \\ l_3 - p_3 &= l_2 p_1 \\ l_{4i} - p_{4i} &= l_1 p_{4i-2} \\ l_{4i+1} - p_{4i+1} &= l_1 p_{4i-1} \\ l_{4i+2} - p_{4i+2} &= l_{4i} p_1 \\ l_{4i+3} - p_{4i+3} &= l_{4i+1} p_1 \\ &\text{for all } i = 1, 2, \ldots \end{aligned}$$

For each positive integer $k \ge 2$, let P_k and L_k be canonical projections of P and L onto their k initial coordinates. Imposing the first $k-1$ adjacency relations on vectors from P_k and L_k, we obtain a bipartite graph of order $2q^k$ with bi-partition (P_k, L_k). Denote this graph $D(k,q)$. For odd k, the girth of $D(k,q)$ is at least $k+5$.

Note that $D(k, q)$ is defined by more complicated equations. And the point-graph of $D(k, q)$ on L_k or the line-graph of $D(k, q)$ on P_k is not a Cayley graph. We can not apply the same method to compute the its spectrum as in the spectrum of Wenger graph. It will be interesting if one can compute the spectrum of those graphs.

Motivated by our computation of the spectrum of Wenger graphs, Cao, Lu, Wan, Wang and Wang [28] used techniques similar to Theorem 2.2.1 and computed the spectrum of Linearized Wenger graphs $L_m(q)$. The definition of these graphs is similar to the construction of Wenger graphs. The only difference is equation (2.5). A point $(p) = (p_1, p_2, \ldots, p_{m+1}) \in P$ is adjacent to a line $[l] = [l_1, l_2, \ldots, l_{m+1}] \in L$ if and only if

$$l_k + p_k = l_1 (p_1)^{p^{k-2}} \text{ for all } k = 2, \ldots, m+1. \tag{2.15}$$

Cao, Lu, Wan, Wang and Wang [28] also determined the diameter and girth of linearized Wenger graphs. However, they can not find an explicit formula for the eigenvalue multiplicities of the linearized Wenger graphs when $m < e$, where $q = p^e$.

Chapter 3

STRONGLY REGULAR GRAPHS

In this chapter, I will give some facts about strongly regular graphs which will be used in the next two chapters, see [19, 56] for more details.

3.1 Introduction

A graph Γ is said to be *strongly regular* with parameters (v, k, λ, μ) (shorthanded (v, k, λ, μ)-SRG from now on) if Γ is k-regular with v vertices, every adjacent pair of vertices have λ common neighbors, and every non-adjacent pair of vertices have μ common neighbors. The study of strongly regular graphs lies at the intersection of graph theory, algebra and finite geometry [22, 25, 26] and has applications in coding theory and computer science, among others [27, 97]. Note that if Γ is strongly regular with parameters (v, k, λ, μ), then $\bar{\Gamma}$ is also strongly regular with parameters $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$. It is easy to tell whether a strongly regular graph is connected by looking at its parameters.

Lemma 3.1.1. *Let Γ be a (v, k, λ, μ)-SRG which is not K_v or $\bar{K}_v$. The following are equivalent:*

1. *Γ is disconnected.*

2. *$\lambda = k - 1$.*

3. *$\mu = 0$.*

4. Γ *is a disjoint union of* m *copies of* K_{k+1}*, where* $m = v/(k+1)$*.*

A strongly regular graph is called *imprimitive* if it, or its complement, is disconnected, and *primitive* otherwise. Two examples of strongly regular graph are the cycle C_5 and the Petersen graph. Their parameters are $(5, 2, 0, 1)$ and $(10, 3, 0, 1)$, respectively. There are many constructions of strongly regular graphs. I will present three well-known families of strongly regular graphs in this section.

1. Let $q = 4t+1$ be a prime power. The *Paley graph* Paley(q) is a $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$-SRG. It is the Cayley graph $\mathrm{Cay}(\mathbb{F}_q, (\mathbb{F}_q^*)^2)$, where $\mathbb{F}_q$ is the finite field with order q and $(\mathbb{F}_q^*)^2$ is the set of non-zero squares in the field.

2. The *Lattice graph* L_n is a strongly regular graph with parameters $(n^2, 2(n-1), n-2, 2)$. It is the line graph of the complete bipartite graph $K_{n,n}$.

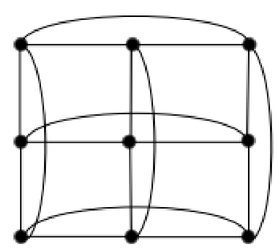

Figure 3.1: The Lattice graph L_3.

3. The *Triangular graph* $T(m)$ is a strongly regular graph with parameter $(\binom{m}{2}, 2m-4, m-2, 4)$. It is the line graph of the complete graph K_m. The vertices

of $T(m)$ are the 2-subsets of $\{1, \ldots, m\}$ and $\{u, v\} \sim \{x, y\}$ if and only if $|\{u, v\} \cap \{x, y\}| = 1$.

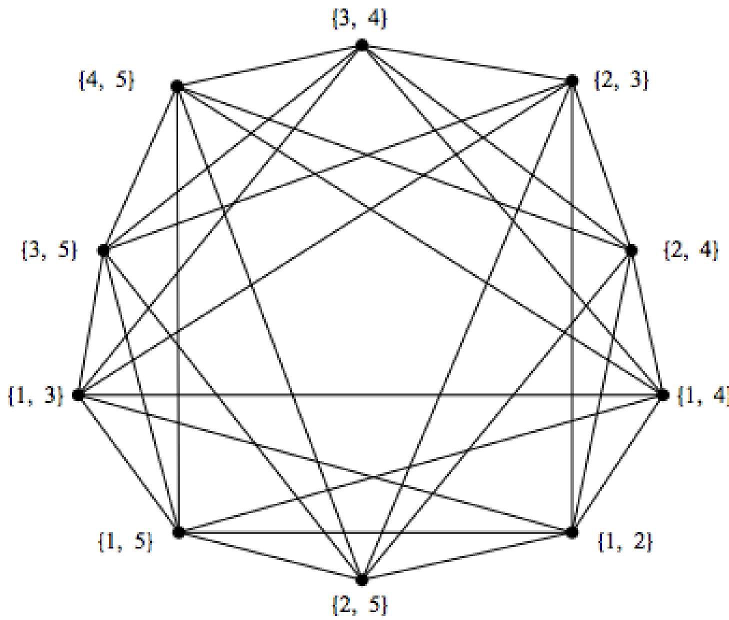

Figure 3.2: The Triangular graph $T(5)$.

3.2 Eigenvalues of Strongly Regular Graphs

Let A denote the adjacency matrix of a (v, k, λ, μ)-SRG. Then A satisfies the following matrix equation.

$$A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J,$$

where J is the all ones matrix. Consider an eigenvector $\vec{x}$ of A orthogonal to the all ones vector $\vec{1}$ (which is an eigenvector of A corresponding to k) with corresponding eigenvalue γ. Applying $\vec{x}$ to the above equation reduces to

$$\gamma^2 - (\lambda - \mu)\gamma - (k - \mu) = 0.$$

Any eigenvalue of A that is not equal to k must be a solution to the above equation, thus A has exactly three distinct eigenvalues. Let $k > \theta_2 > \theta_v$ be the distinct eigenvalues of Γ, then

$$\theta_2 = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$
$$\theta_v = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$

Let m_2 and m_v denote the multiplicity of the eigenvalues θ_2 and θ_v, respectively. Since k has multiplicity 1 and the trace of A is 0,

$$m_2 + m_v = v - 1 \quad \text{and} \quad m_2\theta_2 + m_v\theta_v = -k.$$

This yields

$$m_2 = \frac{(v-1)\theta_v + k}{\theta_v - \theta_2} \quad \text{and} \quad m_v = -\frac{(v-1)\theta_2 + k}{\theta_v - \theta_2}.$$

Substituting the values of θ_2 and θ_v,

$$m_2 = \frac{1}{2}\left(v - 1 - \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right)$$
$$m_v = \frac{1}{2}\left(v - 1 + \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right).$$

Strongly regular graphs with $m_2 = m_v$ are called *conference graphs*. Such graphs have parameters $(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$. The Paley graphs belong to this case, but there are many further examples. It is well-known that a strongly regular graph is either a conference graph or all its eigenvalues are integer.

3.3 Seidel's Classification of Strongly Regular Graphs

The following is Seidel's classification (see [95] or [18, Section 9.2]) of the strongly regular graphs with smallest eigenvalue $\theta_v = -2$, which will be used in Chapter 4 and 5.

Theorem 3.3.1. *Let Γ be a strongly regular graph with smallest eigenvalue -2. Then Γ is one of*

(i) the complete n-partite graph $K_{n\times 2}$, with parameters $(v, k, \lambda, \mu) = (2n, 2n - 2, 2n - 4, 2n - 2)$, $n \geq 2$,

(ii) the lattice graph $L(K_{n,n})$, $n \geq 3$,

(iii) the Shrikhande graph, with parameters $(v, k, \lambda, \mu) = (16, 6, 2, 2)$,

(iv) the triangular graph $T(n)$, $n \geq 5$,

(v) one of the three Chang graphs, with parameters $(v, k, \lambda, \mu) = (28, 12, 6, 4)$,

(vi) the Petersen graph, with parameters $(v, k, \lambda, \mu) = (10, 3, 0, 1)$,

(vii) the Clebsch graph, with parameters $(v, k, \lambda, \mu) = (16, 10, 6, 6)$,

(viii) the Schläfli graph, with parameters $(v, k, \lambda, \mu) = (27, 16, 10, 8)$.

Strongly regular graphs have high connectivity. In 1985, Brouwer and Mesner (see [23] or [18, Section 9.3]) used eigenvalue interlacing and Theorem 3.3.1 to compute the vertex-connectivity of any connected strongly regular graph.

Theorem 3.3.2 (Brouwer and Mesner [23]). *If Γ is a primitive strongly regular graph of valency k, then Γ is k-connected. Any disconnecting set of size k must be the neighborhood of some vertex.*

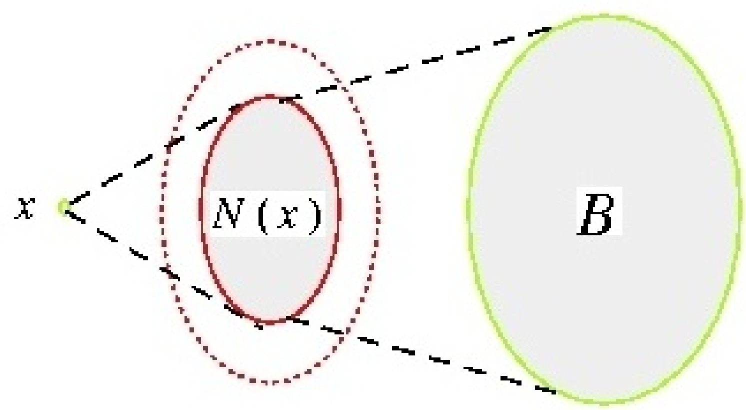

Figure 3.3: The minimum disconnecting set in a primitive strongly regular graph.

3.4 More Strongly Regular Graphs and Neumaier's Classification

A 2-$(n, K, 1)$-design or a Steiner K-system is a point-block incidence structure on n points, such that each block has K points and any two distinct points are contained in exactly one block. The block graph of such a Steiner system has as vertices the blocks of the design and two distinct blocks are adjacent if and only if they intersect. The block graph of a Steiner K-system is a strongly regular graph with parameters $\left(\frac{n(n-1)}{K(K-1)}, \frac{K(n-K)}{K-1}, (K-1)^2 + \frac{n-1}{K-1} - 2, K^2\right)$ (See [56, Section 10.3]).

An orthogonal array $OA(t, n)$ with parameters t and n is a $t \times n^2$ matrix with entries from the set $[n] = \{1, \ldots, n\}$ such that the n^2 ordered pairs defined by any two distinct rows of the matrix are all distinct. It is well known that an orthogonal $OA(t, n)$ is equivalent to the existence of $t - 2$ mutually orthogonal Latin squares. In this thesis, we use Godsil and Royle's notation $OA(t, n)$ from [56, Section 10.4] to denote orthogonal arrays. Note that in other books such as Brouwer and Haemers

[18] the orthogonal array $OA(t,n)$ is denoted by $OA(n,t)$. Given an orthogonal array $OA(t,n)$, one can define a graph Γ as follows: The vertices of Γ are the n^2 columns of the orthogonal array and two distinct columns are adjacent if they have the same entry in one coordinate position. The graph Γ is an $(n^2, t(n-1), n-2+(t-1)(t-2), t(t-1))$-SRG ([56, Section 10.4]). Any strongly regular graph with such parameters is called a Latin square graph (see [18, Section 9.1.12], [56, Section 10.4] or [76, Chapter 30]). When $t=2$ and $n \neq 4$, such a graph must be the line graph of $K_{n,n}$ which is also the graph associated with an orthogonal array $OA(2,n)$ (see [76, Problem 21F]).

In 1979, Neumaier [88] gave a classification strongly regular graphs with smallest eigenvalue $-m$, where $m \geq 2$ is a fixed integer, by showing the following theorem.

Theorem 3.4.1. *Let $m \geq 2$ be a fixed integer. Then with finitely many exceptions, the strongly regular graphs with smallest eigenvalue are of one of the following types:*

(a) Complete multipartite graphs with classes of size m,

(b) Block graphs of Steiner m-systems with parameters

$$\left(\frac{n(n-1)}{m(m-1)}, \frac{m(n-m)}{m-1}, (m-1)^2 + \frac{n-1}{m-1} - 2, m^2\right),$$

(c) Latin square graphs with parameters

$$(n^2, m(n-1), n-2+(m-1)(m-2), m(m-1)).$$

Note that this theorem generalized Theorem 3.3.1, but it did not tell us how many exceptions there are for each $m > 2$.

3.5 Strongly Regular Graphs from Copolar and Δ-spaces

A pair (P, L), where $L \subseteq 2^P$, is called a partial linear space if (i) every $l \in L$ contains at least two points in P, and (ii) for two distinct $p, q \in P$ there is at most one line $l \in L$ that contains both. We call the elements of P points and the elements in L lines. A point p is on the line l if $p \in l$. Also, two distinct points are collinear if there is a line that contains both points. A partial linear space (P, L) is called a *copolar space* (following Hall [59]) or *proper delta space* (according to Higman; see Hall [59] and the references therein) if for any point p and line l, $p \notin l$, p is collinear with none or all but one of the points of l. A more general notion is the notion of a Δ-space. A partial linear space (P, L) is called a Δ-*space* if for any point p and line l, $p \notin l$, p is collinear with none, all but one or all the points of l. We say a partial linear space (P, L) is of order (s, t) if every line contains exactly $s + 1$ points, and every point is in exactly $t + 1$ lines.

Assume that the point graph Γ of a Δ-space of order (s, t) (i.e. the graph with vertex point P where two points are adjacent if they are collinear) is strongly regular with parameters (v, k, λ, μ) with $k = s(t + 1)$. Hall [59] determined all the strongly regular graphs that appear as the point graph of a copolar space and these graphs are: the triangular graphs $T(m)$, the symplectic graphs $Sp(2r, q)$ over the field $\mathbb{F}_q$ for any q prime power, the strongly regular graphs constructed from the hyperbolic quadrics $O^+(2r, 2)$ and from the elliptic quadrics $O^-(2r, 2)$ over the field $\mathbb{F}_2$ respectively, and the complements of Moore graphs. At the end of this section, I will briefly describe $Sp(2r, q)$, $O^+(2r, 2)$ and $O^-(2r, 2)$ as I will need them in Chapter 4. See [31, Section 5,6,7] for more details.

Let q be a prime power and $r \geq 2$ be an integer. If x is a non-zero (column) vector in $\mathbb{F}_q^{2r}$, denote by $[x]$ the 1-dimensional vector subspace of $\mathbb{F}_q^{2r}$ that is spanned by x and denote by x^t the row vector that is the transpose of x. Let M be the $2r \times 2r$ block diagonal matrix whose diagonal blocks are $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The symplectic graph $Sp(2r, q)$ over $\mathbb{F}_q$ has vertex set formed by the 1-dimensional subspaces $[x]$ of $\mathbb{F}_q^{2r}$ with $[x] \sim [y]$ if and only if $x^t My \neq 0$. The symplectic graph $Sp(2r, q)$ is a $\left(\frac{q^{2r}-1}{q-1}, q^{2r-1}, q^{2r-2}(q-1), q^{2r-2}(q-1)\right)$-SRG. See [31, Section 5] for a proof.

The hyperbolic quadric graphs $O^+(2r, 2)$ is the subgraph of $Sp(2r, 2)$ induced by $V^+ := \{(x_1, \dots, x_{2r})^t \in \mathbb{F}_2^{2r} : x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} = 1\}$ (the complement of a hyperbolic quadric in $\mathbb{F}_2^{2r}$). The vertex $x := (x_1, \dots, x_{2r})^t$ is adjacent to $y := (y_1, \dots, y_{2r})^t$ if $x^t My = 1$, where M is the block diagonal matrix defined above. It is known [56, Chapter 10] that $O^+(2r, 2)$ is a $(2^{2r-1} - 2^{r-1}, 2^{2r-2} - 2^{r-1}, 2^{2r-3} - 2^{r-2}, 2^{2r-3} - 2^{r-1})$-SRG.

The elliptic quadric graph $O^-(2r, 2)$ is the subgraph of $Sp(2r, 2)$ induced by $V^- := \{x_1, \dots, x_{2r})^t \in \mathbb{F}_2^{2r} : x_1^2 + x_2^2 + x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} = 1\}$ (the complement of an elliptic quadric in $\mathbb{F}_2^{2r}$). The vertex $x := (x_1, \dots, x_{2r})^t$ is adjacent to $y := (y_1, \dots, y_{2r})^t$ if $x^t My = 1$, where M is the block diagonal matrix defined above. It is known [56, Chapter 10] that $O^-(2r, 2)$ is a $(2^{2r-1} + 2^{r-1}, 2^{2r-2} + 2^{r-1}, 2^{2r-3} + 2^{r-2}, 2^{2r-3} + 2^{r-1})$-SRG.

Chapter 4

DISCONNECTING STRONGLY REGULAR GRAPHS

In this chapter, we will study the minimum size of a subset of vertices of a connected strongly regular graph whose removal disconnects the graph into non-singleton components. Most of the results of this chapter have appeared in Cioabă, Kim and Koolen [31] and Cioabă, Koolen and Li [33].

4.1 Introduction

In 1996, Brouwer [14] conjectured that the minimum size of a disconnecting set of vertices whose removal disconnects a connected (v, k, λ, μ)-SRG into non-singleton components equals $2k - \lambda - 2$, which is the size of the neighborhood of an edge.

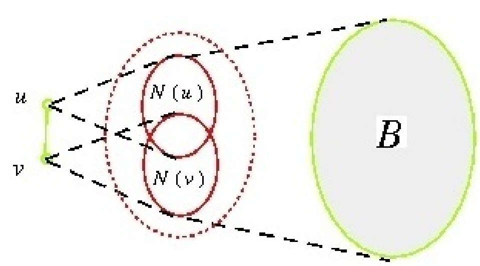

Figure 4.1: The disconnecting set in Brouwer's conjecture.

Cioabă, Kim and Koolen [31] showed that strongly regular graphs constructed from copolar spaces and from the more general spaces called Δ-spaces (See Section

3.5) are counterexamples to Brouwer's conjecture. Using J.I. Hall's characterization of finite reduced copolar spaces, they found that the triangular graphs $T(m)$, the symplectic graphs $Sp(2r, q)$ over the field $\mathbb{F}_q$ (for any q prime power), and the strongly regular graphs constructed from the hyperbolic quadrics $O^+(2r, 2)$ and from the elliptic quadrics $O^-(2r, 2)$ over the field $\mathbb{F}_2$, respectively, are counterexamples to Brouwer's Conjecture. We will just present their result on the triangular graphs $T(m)$.

Proposition 4.1.1 (Cioabă, Kim and Koolen [31]). *For $m \geq 6$, the minimum size of a disconnecting set of vertices whose removal disconnects $T(m)$ into non-singleton components equals $3m - 9$ and the only disconnecting sets of this size are formed (modulo a permutation of $[m]$) by the set of vertices adjacent to at least one of the vertices $\{1, 2\}, \{1, 3\}$ or $\{2, 3\}$.*

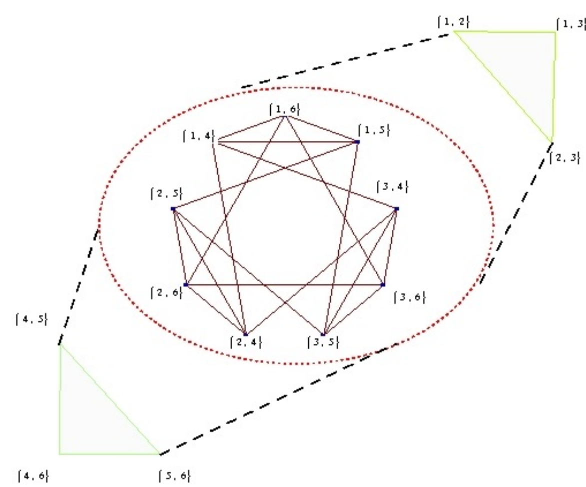

Figure 4.2: The Triangular graph $T(6)$ with its disconnecting set colored by red.

However, it seems that for many families of strongly regular graphs, Brouwer's Conjecture is true. Cioabă, Kim and Koolen proved that Brouwer's Conjecture is true for many families of strongly regular graphs including the conference graphs, the point graphs of generalized quadrangles $GQ(q,q)$, the lattice graphs, the Latin square graphs, the strongly regular graphs with smallest eigenvalue -2 (except the triangular graphs) and the primitive strongly regular graphs with at most 30 vertices except for a few cases. In this chapter, we extend several results from [31] and we show that Brouwer's Conjecture is true for any (v,k,λ,μ)-SRG with $\max(\lambda,\mu) \leq k/4$. This makes significant progress towards solving an open problem from [31] stating that Brouwer's Conjecture is true for any (v,k,λ,μ)-SRG with $\lambda < k/2$. In Sections 4.3 and 4.4, we prove that Brouwer's Conjecture is true for any block graph of a Steiner K-system when $K \in \{3,4\}$ and for any Latin square graph with parameters $(n^2, t(n-1), n-2+(t-1)(t-2), t(t-1))$, when $n \geq 2t \geq 6$. My results and Neumaier's characterization (Theorem 3.4.1) of strongly regular graphs with fixed minimum eigenvalue [88] enable us to verify the status of Brouwer's Conjecture for all but finitely many strongly regular graphs with minimum eigenvalue -3 or -4. In Section 4.5, we prove that the edge version of Brouwer's Conjecture is true for any primitive strongly regular graph; we show that the minimum number of edges whose removal disconnects a (v,k,λ,μ)-SRG into non-singletons, equals $2k-2$, which is the edge-neighborhood of an edge.

Here are some notations which will be used in this chapter. If X is a subset of vertices of a graph Γ, let $N(X) = \{y \notin X : y \sim x \text{ for some } x \in X\}$ denote the neighborhood of X. If Γ is a (v,k,λ,μ)-SRG, then $|N(\{u,v\})| = 2k-\lambda-2$ for every edge uv of Γ. We denote by $\kappa_2(\Gamma)$ the minimum size of a disconnecting set of Γ whose removal disconnects the graph into non-singleton components if such a set exists. This parameter has been studied for many families of graphs (see Boesch and

Tindell [10], Balbuena, Carmona, Fàbrega and Fiol [7] or Hamidoune, Lladó and Serra [60] for example). We say that a connected (v, k, λ, μ)-SRG Γ is OK if either it has no disconnecting set such that each component has as at least two vertices, or if $\kappa_2(\Gamma) = 2k - \lambda - 2$.

Let Γ be a connected graph. If S is a disconnecting set of Γ of minimum size such that the components of $\Gamma \setminus S$ are not singletons, then denote by A the vertex set of one of the components of $\Gamma \setminus S$ of minimum size. By our choice of A, $|B| \geq |A|$, where $B := V(\Gamma) \setminus (A \cup S)$. As S is a disconnecting set, $N(A) \subset S$ and consequently, $|S| \geq |N(A)|$. Note that it is possible for the disconnecting set S to contain a vertex y and its neighborhood $N(y)$ in which case $y \in S$, but $y \notin N(A)$ and thus, $S \neq N(A)$. In order to prove Brouwer's Conjecture is true for a (v, k, λ, μ)-SRG Γ with vertex set V and $v \geq 2k - \lambda + 3$, we will show that $|S| \geq 2k - \lambda - 2$ for any subset of vertices A with $3 \leq |A| \leq \frac{v}{2}$ having the property that A induces a connected subgraph of Γ. In some situations, we will be able to prove the stronger statement that $|N(A)| > 2k - \lambda - 2$. Throughout this chapter, S will be a disconnecting set of Γ, A will stand for a subset of vertices of Γ that induces a connected subgraph of $\Gamma \setminus S$ of smallest order and $B := V(\Gamma) \setminus (A \cup S)$. As before, $N(A) \subset S$ and thus, $|S| \geq |N(A)|$. Let $a = |A|, b = |B|$ and $s = |S|$. We will need the following results.

Lemma 4.1.2 (Lemma 2.3 [31])**.** *If Γ is a connected (v, k, λ, μ)-SRG, then*

$$|S| \geq \frac{4ab\mu}{(\lambda - \mu)^2 + 4(k - \mu)}. \tag{4.1}$$

Proposition 4.1.3. *Let Γ be a connected (v, k, λ, μ)-SRG and $c \geq 3$ a fixed integer. If $a \geq c$ and*

$$4(c-2)\left[(k-\mu)\left(k+c-1-\frac{\lambda(c-1)}{c-2}\right)-ck\right] > (\lambda-\mu)^2(2k-\lambda-2) \tag{4.2}$$

then $|S| > 2k - \lambda - 2$.

Proof. Let s denote the minimum size of a disconnecting set S whose removal leaves only non-singleton components. Assume that $s \leq 2k-\lambda-2$. This implies $a+b = v-s \geq v-(2k-\lambda-2) = v+\lambda+2-2k$. As $b \geq a \geq c$, we obtain $ab \geq c(v+\lambda+2-2k-c)$. This inequality, Lemma 4.1.2 and $v = 1+k+k(k-\lambda-1)/\mu$ imply

$$\begin{aligned}
s &\geq \frac{4ab\mu}{(\lambda-\mu)^2+4(k-\mu)} \geq \frac{4c(v+\lambda+2-2k-c)\mu}{(\lambda-\mu)^2+4(k-\mu)} \\
&= \frac{4c[k^2+(-\lambda-\mu-1)k+\mu(\lambda+3-c)]}{(\lambda-\mu)^2+4(k-\mu)} \\
&> 2k-\lambda-2
\end{aligned}$$

where the last inequality can be shown to be equivalent to our hypothesis (4.2) by a straightforward calculation. This contradiction finishes our proof. □

Lemma 4.1.4. *Let Γ be a connected (v,k,λ,μ)-SRG and let C be a clique with $q \geq 3$ vertices contained in Γ. If $k-2\lambda-1>0$, then $|N(C)| > 2k-\lambda-2$.*

Proof. If x, y and z are three distinct vertices of C, then x, y and z have at least $q-3$ common neighbors. Thus, by inclusion and exclusion, we obtain that

$$\begin{aligned}
|N(C)| &\geq |N(\{x,y,z\})| - (q-3) \\
&\geq 3(k-2) - 3(\lambda-1) + (q-3) - (q-3) \\
&= 2k-\lambda-2+(k-2\lambda-1) \\
&> 2k-\lambda-2.
\end{aligned}$$

□

We will also need the following *inproduct bound* (see [20, Lemma 2.2]).

Lemma 4.1.5 (Brouwer and Koolen [20])**.** *Among a set of k 01-vectors in $\mathbb{R}^n$, say $v_1, v_2, \ldots, v_k$, of length n and average weight w, there are two with inner product at least $w\left(kw/n-1\right)/(k-1) = \frac{w^2}{n} - \frac{w(n-w)}{n(k-1)}$.*

Proof. Let $M = [v_1|v_2|\dots|v_k]$. Denote $r_1, r_2, \dots, r_n$ the weights of the row vectors of M. Note that the average row weight is $\frac{1}{n}\sum_{i=1}^{n} r_i = kw/n$. The sum of all pairwise inner products of the column vectors is $\sum_{i=1}^{n}\binom{r_i}{2}$, since every pair of 1's in a row contribute 1 to the sum. This sum is at least $n\binom{kw/n}{2}$, since $\binom{x}{2}$ is a convex function of x. By the Pigeonhole principle, there are two column vectors with inner product at least

$$n\binom{kw/n}{2}\Big/\binom{k}{2} = \frac{w(\frac{kw}{n}-1)}{k-1}$$

□

4.2 Brouwer's Conjecture is True When $\max(\lambda, \mu) \leq k/4$

In this section, we prove that Brouwer's Conjecture is true for all connected (v, k, λ, μ)-SRGs with λ and μ relatively small.

Theorem 4.2.1. *If Γ is a connected (v, k, λ, μ)-SRG with $\max(\lambda, \mu) \leq k/4$, then $\kappa_2(\Gamma) = 2k - \lambda - 2$.*

Proof. Let Γ be a (v, k, λ, μ)-SRG with $\max(\lambda, \mu) \leq k/4$. Assume that S is a disconnecting set of vertices of size $s = |S| \leq 2k - \lambda - 3$ such that $V(\Gamma) \setminus S = A \cup B$ and there are no edges between A and B. Assume that each component of A and B has at least 3 vertices from now on. Let $s = |S|, a = |A| \geq 3$ and $b = |B| \geq 3$.

We first prove that $b + s \geq a + s \geq 9k/4$. Assume that $a + s < 9k/4$. As each component of A has at least 3 vertices, it means we can find three vertices x, y, z in A that induce a triangle or a path of length 2. If x, y, z induce a triangle, then $|N(\{x, y, z\})| \geq 3(k-2) - 3(\lambda - 1) = 3k - 3\lambda - 3$. If x, y, z induce a path of length 2, then $|N(\{x, y, z\})| \geq 3k - 4 - (2\lambda + \mu - 1) = 3k - 2\lambda - \mu - 3$. In either case, we obtain that $|N(\{x, y, z\})| \geq 3k - 3\max(\lambda, \mu) - 3$. As $\{x, y, z\} \cup N(\{x, y, z\}) \subset A \cup S$, we deduce that $9k/4 > a + s \geq 3k - 3\max(\lambda, \mu)$ which implies $\max(\lambda, \mu) > k/4$ contradicting our hypothesis. Thus, $b + s \geq a + s \geq 9k/4$.

For two disjoint subsets of vertices X and Y, denote by $e(X, Y)$ the number of edges between X and Y. Let $\theta_1(X)$ be the largest eigenvalue of the adjacency matrix of the subgraph induced by X. Denote by X^c the complement of X. Let $\alpha = \frac{e(A,A^c)}{a} = \frac{e(A,S)}{a}$ and $\beta = \frac{e(B,B^c)}{b} = \frac{e(B,S)}{b}$. We consider two cases depending on the values of α and β.

Case 1. $\max(\alpha, \beta) \leq 3k/4$.

We first prove that $\theta_2 \geq k/4$. Since $\alpha \leq 3k/4$, the average degree of the subgraph induced by A is $k - \alpha \geq k/4$. Similarly, as $\beta \leq 3k/4$, the average degree of the subgraph induced by B is $k - \beta \geq k/4$. Eigenvalue interlacing (Theorem 1.3.2) implies $\theta_2 \geq \min(\theta_1(A), \theta_1(B)) \geq \min(k - \alpha, k - \beta) \geq k/4$. As $\theta_2\theta_v = \mu - k > -k$ and $\theta_2 \geq k/4$, we deduce that $\theta_v > -4$. This implies Γ is a conference graph (for which we know the Brouwer's Conjecture is true as proved in [31]) or $\theta_v \geq -3$. The case $\theta_v = -2$ was solved completely in [31] so we may assume that $\theta_v = -3$. As $\lambda - \mu = \theta_2 + \theta_v$, we get $\lambda - \mu \geq k/4 - 3$. If $\mu \geq 4$, then $\lambda \geq k/4 + 1$, contradiction. Thus, we may assume $1 \leq \mu \leq 3$. Because $\theta_v = -3$, we get $\theta_2 = \frac{k-\mu}{3}$. This implies $\lambda = \mu + \theta_2 + \theta_v = \mu + \frac{k-\mu}{3} - 3 = \frac{k+2\mu-9}{3}$.

If $\mu = 1$, then $\lambda = \frac{k-7}{3} > \frac{k}{4}$ when $k > 28$. If $k \leq 28$ and $k \equiv 1 \pmod 3$, the possible parameter sets are: $(50, 7, 0, 1)$, $(91, 10, 1, 1)$, $(144, 13, 2, 1)$, $(209, 16, 3, 1)$, $(286, 19, 4, 1)$, $(375, 22, 5, 1)$, $(476, 25, 6, 1)$, $(589, 28, 7, 1)$. The only parameter sets with integer eigenvalue multiplicities are $(50, 7, 0, 1)$, $(209, 16, 3, 1)$ and $(375, 22, 5, 1)$. A strongly regular graph with $\mu = 1$ must satisfy the inequality $k \geq (\lambda + 1)(\lambda + 2)$ (see [6, 24]). This implies there are no strongly regular graphs with parameters $(209, 16, 3, 1)$ and $(375, 22, 5, 1)$. The parameters $(50, 7, 0, 1)$ correspond to the Hoffman-Singleton graph which is OK as proved in [31].

If $\mu = 2$, then $\lambda = \frac{k-5}{3} > \frac{k}{4}$ when $k > 20$. If $k \leq 20$ and $k \equiv 2 \pmod 3$, the feasible parameter sets are $(16, 5, 0, 2)$, $(33, 8, 1, 2)$, $(56, 11, 2, 2)$, $(85, 14, 3, 2)$, $(120, 17, 4, 2)$, $(161, 20, 5, 2)$. The only parameter sets with integer eigenvalue multiplicities are $(16, 5, 0, 2)$ and $(85, 14, 3, 2)$. The parameter set $(16, 5, 0, 2)$ corresponds to the Clebsch graph which is OK as proved in [31]. It is not known whether there exists a strongly regular graphs with the parameters $(85, 14, 3, 2)$. However, this parameter set satisfies Lemma 2.5 from [31] so if it exists, such a graph is OK.

If $\mu = 3$, then $\lambda = \frac{k-3}{3} > \frac{k}{4}$ when $k > 12$. If $k \leq 12$ and $k \equiv 0 \pmod 3$, the feasible parameter sets are $(6, 3, 0, 3)$, $(15, 6, 1, 3)$, $(28, 9, 2, 3)$ and $(45, 12, 3, 3)$. A $(6, 3, 0, 3)$-SRG is isomorphic to $K_{3,3}$ and a $(15, 6, 1, 3)$-SRG is isomorphic to the complement of the triangular graph $T(6)$. By [31, Proposition 10.2], both these graphs are OK. The other parameter set with integer eigenvalue multiplicities is $(45, 12, 3, 3)$. There are exactly 78 strongly regular graphs with parameters $(45, 12, 3, 3)$ (see [38]) and they are all OK by [31, Lemma 2.5].

Case 2. $\max(\alpha, \beta) > 3k/4$.

Assume $\alpha > 3k/4$; the case $\beta > 3k/4$ is similar (replace A by B, a by b and α by β in the analysis below) and will be omitted. Applying Lemma 4.1.5 to the characteristic vectors of the neighborhoods (restricted to S) of the vertices in A, we deduce that there exist two distinct vertices u and v in A such that $|N(u) \cap N(v) \cap S| \geq \frac{\alpha\left(\frac{a\alpha}{s}-1\right)}{a-1}$. As $a + s \geq 9k/4$ and $s \leq 2k - \lambda - 3$, we obtain that $a \geq k/4 + \lambda + 3$. Because $\alpha/s > \frac{3k}{4(2k-\lambda-3)}$, we get that $|N(u) \cap N(v) \cap S| \geq \frac{\alpha\left(\frac{a\alpha}{s}-1\right)}{a-1} > \frac{3k}{4} \cdot \frac{a \cdot \frac{3k}{4(2k-\lambda-3)}-1}{a-1}$. The right-hand side of the previous inequality is greater than $k/4$ if and only if $\frac{a \cdot \frac{3k}{4(2k-\lambda-3)}-1}{a-1} > \frac{1}{3}$. This is equivalent to $a \cdot \frac{k+4\lambda+12}{8k-4\lambda-12} > 2$. Since $a \geq k/4 + \lambda + 3$, the previous inequality is true whenever $(k/4 + \lambda + 3)(k + 4\lambda + 12) > 2(8k - 4\lambda - 12)$.

This is the same as

$$\left(\frac{k+4(\lambda+3)}{2}\right)^2 > 16\left(k - \frac{\lambda+3}{2}\right), \tag{4.3}$$

which is equivalent to

$$k + 4(\lambda+3) > 8\sqrt{k - \frac{\lambda+3}{2}}. \tag{4.4}$$

For $\lambda \geq 1$, this inequality is true as $k+4(\lambda+3) \geq 4\sqrt{k(\lambda+3)} \geq 8\sqrt{k} > 8\sqrt{k - \frac{\lambda+3}{2}}$.

For $\lambda = 0$, the inequality (4.4) is true for all k except when $8 \leq k \leq 32$. As $\mu \leq k/4$, this implies that $1 \leq \mu \leq 8$. We show that the condition

$$4(k-2\lambda)(k-\mu) > (\lambda-\mu)^2(2k-\lambda-3) \tag{4.5}$$

of Proposition 2.4 from [31] is satisfied in each of these cases, therefore showing that Γ is OK and finishing our proof. The inequality above is the same as $4k(k-\mu) > \mu^2(2k-3)$. As $k-\mu \geq 3\mu$, the previous inequality is true whenever $4k(3\mu) > \mu^2(2k-3)$ which is true when $6k > \mu(k-1)$. This last inequality is true whenever $1 \leq \mu \leq 6$.

If $\mu = 7$, then inequality (4.5) is $4k(k-7) > 49(2k-3)$ which holds when $k \geq 31$. As $k \geq 4\mu = 28$, the previous condition will be satisfied except when $k \in \{28, 29, 30\}$. But in this situation, the strongly regular graph does not exist. This is because $\theta_2 + \theta_v = \lambda - \mu = -7$ and $\theta_2\theta_v = \mu - k \in \{-21, -22, -23\}$ which is impossible as θ_2 and θ_v are integers.

If $\mu = 8$ and $k = 32$, the graph would have parameters $(157, 32, 0, 8)$. However, such a graph does not exist as θ_2 and θ_v would have to be integers satisfying $31 = k - \lambda - 1 = -(\theta_2+1)(\theta_v+1)$ and $\theta_2\theta_v = \mu - k = -24$ which is again impossible as θ_2 and θ_v are integers. This finishes our proof. □

We checked the tables with feasible parameters for strongly regular graphs on Brouwer's homepage [16]. The following parameters satisfy the condition $\max(\lambda, \mu) \leq$

$k/4$ and are not parameters of block graphs of Steiner systems or Latin square graphs with less than 200 vertices: $(45, 12, 3, 3)$, $(50, 7, 0, 1)$, $(56, 10, 0, 2)$, $(77, 16, 0, 4)$, $(85, 14, 3, 2)^?$, $(85, 20, 3, 5)$, $(96, 19, 2, 4)$, $(96, 20, 4, 4)$, $(99, 14, 1, 2)^?$, $(115, 18, 1, 3)^?$, $(125, 28, 3, 7)$, $(133, 24, 5, 4)^?$, $(133, 32, 6, 8)^?$, $(156, 30, 4, 6)$, $(162, 21, 0, 3)^?$, $(162, 23, 4, 3)^?$, $(165, 36, 3, 9)$, $(175, 30, 5, 5)$, $(176, 25, 0, 4)^?$, $(189, 48, 12, 12)^?$, $(196, 39, 2, 9)^?$. The existence of the graphs with "?" is unknown at this time.

4.3 Brouwer's Conjecture for the Block Graphs of Steiner Systems

Recall that a 2-$(n, K, 1)$-design or a Steiner K-system is a point-block incidence structure on n points, such that each block has K points and any two distinct points are contained in exactly one block. The block graph of such a Steiner system has as vertices the blocks of the design and two distinct blocks are adjacent if and only if they intersect. The block graph of a Steiner K-system is a strongly regular graph with parameters $\left(\frac{n(n-1)}{K(K-1)}, \frac{K(n-K)}{K-1}, (K-1)^2 + \frac{n-1}{K-1} - 2, K^2\right)$. When $K \geq 5$, Theorem 6.3.9 implies that when $n > 4K^2 + 5K + 24 + \frac{96}{K-4}$, the associated strongly regular graph satisfies Brouwer's Conjecture. In the next two sections, we improve this result when $K \in \{3, 4\}$.

4.3.1 Block graphs of Steiner triple systems

A 2-$(n, 3, 1)$-design is called a Steiner triple system of order n or STS(n). It is known that a STS(n) exists if and only if $n \equiv 1, 3 \bmod 6$. If $n = 7$, then the block graph of a STS(7) is the complete graph K_7. When $n \geq 9$, the block graph of a STS(n) is a strongly regular graph with parameters $\left(\frac{n(n-1)}{6}, \frac{3(n-3)}{2}, \frac{n+3}{2}, 9\right)$ and $2k - \lambda - 2 = \frac{6(n-3)}{2} - \frac{n+3}{2} - 2 = \frac{5n-25}{2}$.

In this section, we prove that any strongly regular graph that is the block graph of a Steiner triple system satisfies Brouwer's Conjecture.

Theorem 4.3.1. *For every $n \geq 7, n \equiv 1, 3 \pmod 6$, if Γ is the block graph of a STS(n), then $\kappa_2(\Gamma) = 2k - \lambda - 2 = \frac{5n-25}{2}$. Equality happens if and only if the disconnecting set is the neighborhood of an edge.*

Proof. We begin our proof with some small values of n. When $n = 7$, the block graph of a STS(7) is the complete graph K_7 which is OK. When $n = 9$, the block graph of a STS(9) has parameters $(12, 9, 6, 9)$ and by [31, Lemma 2.1], such a graph is OK. When $n = 13$, the block graph of a STS(13) has parameters $(26, 15, 8, 9)$ and by [31, Example 10.3], such a graph is OK.

Assume $n \geq 15$ for the rest of the proof. If $a = 3$, then we have two cases:

1. The set A induces a triangle.

   We show that the vertices of A have at least 3 common neighbors outside A. If the three blocks in A have non-empty intersection, we may assume they are $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$. These vertices have at least $\frac{n-7}{2} \geq 3$ common neighbors. If the three blocks in A have an empty intersection, we may assume they are $\{1, 2, 3\}$, $\{2, 4, 5\}$, $\{3, 5, 6\}$. These vertices have at least three common neighbors: $\{1, 5, *\}$, $\{3, 4, *\}$ and $\{2, 6, *\}$. By inclusion and exclusion, we get that $|S| \geq |N(A)| \geq 3(k-2) - 3(\lambda - 1) + 3 = 3n - 18 > \frac{5n-25}{2}$ as $n > 11$. This finishes the proof of the case when A induces a triangle.

2. The set A induces a path of length 2. We may assume that the vertices of A are $\{1, 2, 3\}$, $\{2, 4, 5\}$, $\{4, 6, 7\}$. These vertices have at least four common neighbors: $\{1, 4, *\}$, $\{3, 4, *\}$, $\{2, 6, *\}$, $\{2, 7, *\}$. By inclusion and exclusion, we get that $|S| \geq |N(A)| \geq k - 2 + 2(k-1) - 2\lambda - (\mu - 1) + 4 = \frac{7n-49}{2} > \frac{5n-25}{2}$ as $n > 12$.

This finishes the proof of the case $a = 3$. Assume that $a \geq 4$. When $n = 15$, Lemma 4.1.2 implies $|S| \geq ab \geq 4 \times (31 - |S|)$ which gives $|S| \geq 25$. Here, we want to show that $|S| \neq 25$. If $|S| = 25$, then $a + b = 10$ and $a \leq b$ imply that $a \in \{4, 5\}$. There are three cases:

1. The set A induces a clique of size 4. Then A contain three vertices x, y, z which induce a triangle. By previous discussion, $|S| \geq |N(A)| \geq |N(\{x, y, z\})| - 1 \geq 3(k-2) - 3(\lambda - 1) + 3 - 1 = 3n - 19 > \frac{5n-25}{2}$ as $n > 13$.

2. The set A induces a clique of size 5. We claim that there are three vertices x, y, z, whose blocks have non-empty intersection. Otherwise, we may assume A have three vertices of the forms $\{1, 2, 3\}$, $\{2, 4, 5\}$, $\{3, 5, 6\}$. A has two other vertices, one may be $\{1, 4, 6\}$, and the other one must be one of the following forms: $\{1, 5, *\}$, $\{3, 4, *\}$ and $\{2, 6, *\}$. There are three block with non-empty intersection, which is a contradiction. Assume that $x, y, z \in A$, and these three blocks have non-empty intersection. Then they have at least $\frac{n-7}{2} = 4$ common neighbors. By the same argument, $|S| \geq |N(A)| \geq |N(\{x, y, z\})| - 2 \geq 3(k-2) - 3(\lambda - 1) + 4 - 2 = 3n - 19 > \frac{5n-25}{2}$ as $n > 13$.

3. The set A does not induces a clique. As A induces a connected subgraph, we can find three vertices x, y, z, such that they induces a path of length 2. By previous discussion, $|S| \geq |N(A)| \geq |N(\{x, y, z\})| - 2 \geq k - 2 + 2(k-1) - 2\lambda - (\mu - 1) + 4 - 2 = \frac{7n-53}{2} > \frac{5n-25}{2}$ as $n > 14$.

When $n \in \{19, 21, 25, 27\}$, the parameters of Γ satisfy Proposition 4.1.3 (with $c = 4$), thus the disconnecting set is greater than $2k - \lambda - 2$. Assume that $n \geq 31$ from now on. Let p denote the number of points contained in the blocks corresponding to the vertices of A. We may assume that $p \leq n - p$. Otherwise, we can choose another component of $V\backslash S$ as our A. Obviously, $a \leq \frac{p(p-1)}{6}$. If A does not induce a clique, let x and y be two non-adjacent vertices of A. Then

$$\begin{aligned}|S| \geq |N(A)| \geq |N(\{x,y\})| - |A \setminus \{x,y\}| &= 2k - \mu - (a-2)\\ &= 2k - \mu - a + 2 = 3n - 16 - a.\end{aligned}$$

If $a < \frac{n-7}{2}$, then $|S| \geq 3n - 16 - a > \frac{5n-25}{2}$ and we are done. Otherwise, if $a \geq \frac{n-7}{2}$, then $\frac{p(p-1)}{6} \geq a \geq \frac{n-7}{2}$ which implies $p > \sqrt{3(n-7)}$. Now B is spanned by at most $n-p$ points and thus, $|B| \leq \frac{(n-p)(n-p-1)}{6}$. This implies that

$$|S| \geq \frac{n(n-1)}{6} - \frac{p(p-1)}{6} - \frac{(n-p)(n-p-1)}{6} = \frac{p(n-p)}{3}.$$

As $n \geq 31$, $p > \sqrt{3(n-7)} \geq \sqrt{72}$, so $p \geq 9$. Thus, $|S| \geq \frac{p(n-p)}{3} \geq 3(n-9) > \frac{5n-25}{2}$ which finishes the proof when A does not induce a clique. If A induces a clique, then $k - 2\lambda - 1 = \frac{n-17}{2} > 0$ and Lemma 4.1.4 imply that $|S| \geq |N(A)| > 2k - \lambda - 2$. This finishes our proof. $\square$

4.3.2 Block graphs of Steiner quadruple systems

A 2-$(n,4,1)$-design is called a Steiner quadruple system of order n or SQS(n). It is known that a SQS(n) exists if and only if $n \equiv 1,4 \bmod 12$. When $n = 13$, the block graph of a SQS(13) is the complete graph K_{13}. When $n \geq 16$, the block graph of a SQS(n) is a strongly regular graph with parameters $\left(\frac{n(n-1)}{12}, \frac{4(n-4)}{3}, \frac{n+20}{3}, 16\right)$ and $2k - \lambda - 2 = \frac{8(n-4)}{3} - \frac{n+20}{3} - 2 = \frac{7n-58}{3}$. In this section, we prove that any strongly regular graph that is the block graph of a Steiner quadruple system satisfies Brouwer's Conjecture.

Theorem 4.3.2. *For $n \geq 13, n \equiv 1,4 \pmod{12}$, if Γ is the block graph of a SQS(n), then $\kappa_2(\Gamma) = 2k - \lambda - 2 = \frac{7n-58}{3}$. Equality happens if and only if the disconnecting set is the neighborhood of an edge.*

Proof. When $n = 13$, the block graph of a SQS(13) is the complete graph K_{13} which is OK. When $n = 16$, the SRG is a complete multipartite graph with parameters $(20, 16, 12, 16)$. By [31, Lemma 2.1], this graph is OK. Assume that $25 \leq n \leq 126$. Suppose first that $3 \leq a \leq 5$. We have two possible cases:

1. The set A contains three vertices x, y and z that form a triangle. We consider two subcases depending on whether or not the blocks of x, y, z have nonempty intersection.

   (a) The three blocks of x, y and z intersect in one point. We may assume that $x = \{1, 2, 3, 4\}, y = \{1, 5, 6, 7\}, z = \{1, 8, 9, 10\}$. Then x, y and z have at least $\frac{n-10}{3} \geq 5$ common neighbors.

   (b) The three blocks of x, y and z have an empty intersection. For $1 \leq u \neq v \leq n$, denote by $\{u, v, *, *\}$ the block of the Steiner system containing u and v. We may assume that $x = \{1, 2, 3, 4\}, y = \{2, 5, 6, 7\}, z = \{1, 5, 8, 9\}$. Then x, y and z have at least 6 common neighbors: $\{1, 6, *, *\}$,$\{1, 7, *, *\}$, $\{2, 8, *, *\}$, $\{2, 9, *, *\}$, $\{3, 5, *, *\}$, $\{4, 5, *, *\}$.

   By inclusion and exclusion, we obtain that

$$\begin{aligned}|S| \geq |N(A)| &\geq |N(\{x, y, z\})| - (a - 3)\\ &\geq [3(k-2) - 3(\lambda - 1) + 5] - 2\\ &= 3n - 36 \geq \frac{7n - 58}{3}.\end{aligned}$$

   If the equality holds, then $n = 25, a = 5, b = 6$ and $s = 39$. However, Lemma 4.1.2 implies that

$$|S| \geq \frac{4 \times 5 \times 6 \times 16}{(15-16)^2 + 4 \times (28 - 16)} \approx 39.1836 \tag{4.6}$$

   which is a contradiction that finishes the proof of this subcase.

2. The subgraph induced by A contains no triangles. Let $x \sim y \sim z$ denote an induced path of length 2 that is contained in the subgraph induced by A. We may assume that $x = \{1,2,3,4\}, y = \{4,5,6,7\}, z = \{7,8,9,10\}$. These vertices have at least 6 common neighbors: $\{4,8,*,*\}$, $\{4,9,*,*\}$, $\{4,10,*,*\}$, $\{1,7,*,*\}$, $\{2,7,*,*\}$, $\{3,7,*,*\}$. By inclusion and exclusion, we get that

$$\begin{aligned} |S| \geq |N(A)| &\geq |N(\{x,y,z\})| - (a-3) \\ &\geq [k-2+2(k-1)-2\lambda-(\mu-1)+6]-2 \\ &= \frac{10n-133}{3} \geq \frac{7n-58}{3}. \end{aligned}$$

If the equality holds, then, $n = 25, a = 5, s = 39$ and $b = 6$. By the same argument as in (4.6), we obtain again a contradiction.

Suppose that $a \geq 6$ and $25 \leq n \leq 107$. The hypothesis of Proposition 4.1.3 (with $c = 6$) holds if $f(n) = -7n^3/27 + 326n^2/9 - 2848n/3 + 139168/27 > 0$. As the polynomial $f(n)$ approximately has roots $107.3212, 24.9760, 7.4171$, we obtain that $f(n) > 0$ if $25 \leq n \leq 107$. Thus, the disconnecting set is greater than $2k - \lambda - 2$.

Suppose that $a = 6$ and $108 \leq n \leq 126$. If A induces a clique, $k - 2\lambda - 1 = \frac{2n-59}{3} > 0$ and Lemma 4.1.4 imply that $|S| \geq |N(A)| > 2k - \lambda - 2$. If A does not induce a clique, by the same argument in case 2, we get that

$$\begin{aligned} |S| \geq |N(A)| &\geq |N(\{x,y,z\})| - (a-3) \\ &\geq [k-2+2(k-1)-2\lambda-(\mu-1)+6]-3 \\ &= \frac{10n-136}{3} > \frac{7n-58}{3}. \end{aligned}$$

Suppose that $a \geq 7$ and $108 \leq n \leq 126$. The hypothesis of Proposition 4.1.3 (with $c = 7$) holds if $f(n) = -7n^3/27 + 374n^2/9 - 1104n + 150112/27 > 0$. As the polynomial $f(n)$ approximately has roots $128.4292, 25.2414, 6.6152$, we obtain that $f(n) > 0$ if $108 \leq n \leq 126$. Thus, the disconnecting set is greater than $2k - \lambda - 2$.

Now, assume that $n \geq 127$. Let p denote the number of points contained in the blocks corresponding to the vertices of A. We may assume that $p \leq n-p$. Otherwise, we can choose some other component of $V \backslash S$ as our A. Obviously, $a \leq \frac{p(p-1)}{12}$. If A does not induce a clique, let x and y be two non-adjacent vertices of A. Then

$$\begin{aligned}|S| &\geq |N(A)| \geq |N(\{x,y\})| - |A \setminus \{x,y\}| = 2k - \mu - (a-2) \\ &= 2k - \mu - a + 2 = \frac{8n-74}{3} - a.\end{aligned}$$

If $a < \frac{n-16}{3}$, then $|S| \geq \frac{8n-74}{3} - a > \frac{7n-58}{3}$ and we are done. Otherwise, if $a \geq \frac{n-16}{3}$, then $\frac{p(p-1)}{12} \geq a \geq \frac{n-16}{3}$ which implies $p > \sqrt{2(n-16)}$. Now B is spanned by at most $n-p$ points and thus, $|B| \leq \frac{(n-p)(n-p-1)}{12}$. This implies

$$\begin{aligned}|S| &\geq \frac{n(n-1)}{12} - \frac{p(p-1)}{12} - \frac{(n-p)(n-p-1)}{12} \\ &= \frac{p(n-p)}{6} \geq \frac{\sqrt{2(n-16)}(n - \sqrt{2(n-16)})}{6}.\end{aligned}$$

Now $\frac{\sqrt{2(n-16)}(n-\sqrt{2(n-16)})}{6} > \frac{7n-58}{3}$ is equivalent to $f(n) = n^3 - 144n^2 + 2368n - 10952 > 0$. It is true when $n \geq 126$. Thus, $|S| > \frac{7n-58}{3}$ in this case. If A induces a clique, $k - 2\lambda - 1 = \frac{2n-59}{3} > 0$ and Lemma 4.1.4 imply that $|S| \geq |N(A)| > 2k - \lambda - 2$ which finishes our proof. $\square$

4.4 Brouwer's Conjecture for Latin Square Graphs

Recall that an orthogonal array $OA(t,n)$ with parameters t and n is a $t \times n^2$ matrix with entries from the set $[n] = \{1, \ldots, n\}$ such that the n^2 ordered pairs defined by any two distinct rows of the matrix are all distinct. It is well known that an orthogonal $OA(t,n)$ is equivalent to the existence of $t-2$ mutually orthogonal

Latin squares (see [56, Section 10.4]). Given an orthogonal array $OA(t,n)$, one can define a graph Γ as follows: The vertices of Γ are the n^2 columns of the orthogonal array and two distinct columns are adjacent if they have the same entry in one coordinate position. The graph Γ is an $(n^2, t(n-1), n-2+(t-1)(t-2), t(t-1))$-SRG.

When $t = 2$ and $n \neq 4$, such a graph must be the line graph of $K_{n,n}$ which is also the graph associated with an orthogonal array $OA(2,n)$ (see [76, Problem 21F]); this graph is OK by [31, Section 8]. When $t = 2$ and $n = 4$, there are two strongly regular graphs with parameters $(16,6,2,2)$, the line graph of $K_{4,4}$ and the Shrikhande graph (see [18, p. 125]); they are both OK by [31, Section 10].

The following theorem is the main result of this section and it shows that Latin square graphs with parameters $(n^2, t(n-1), n-2+(t-1)(t-2), t(t-1))$ satisfy Brouwer's Conjecture when $n \geq 2t \geq 6$. In particular, this will imply that strongly regular graphs obtained from orthogonal arrays $OA(t,n)$ satisfy Brouwer's Conjecture when $n \geq 2t \geq 6$.

Theorem 4.4.1. *Let $t \geq 3$ be an integer. For any $n \geq 2t$, if Γ is an $(n^2, t(n-1), n-2+(t-1)(t-2), t(t-1))$-SRG, then*

$$\kappa_2(\Gamma) = 2k - \lambda - 2 = (2t-1)n - t^2 + t - 2.$$

The only disconnecting set of size $(2t-1)n - t^2 + t - 2$ are of the form $N(\{uv\})$, where u and v are two adjacent vertices in Γ.

Proof. If A induces a clique in Γ, then the inequality $k - 2\lambda - 1 = (t-2)n - 2t^2 + 5t - 1 \geq (t-2)(2t) - 2t^2 + 5t - 1 = t - 1$ and Lemma 4.1.4 imply that $|S| > 2k - \lambda - 2$.

If A does not induce a clique in Γ, then A must contain an induced path of length 2. By inclusion and exclusion, we deduce that $a + s \geq 3k - 2\lambda - \mu =$

$(3t-2)n-3t^2+4t$. Assume by contradiction that $s \leq 2k-\lambda-2 = (2t-1)n-t^2+t-2$. Then $b \geq a \geq k-\lambda-\mu+2 \geq (t-1)(n-2t+1)+3$. Combining these inequalities with $a+b = v-s \geq n^2-(2t-1)n+t^2-t+2$, we get $ab \geq [(t-1)(n-2t+1)+3][n^2-(3t-2)n+3t^2-4t]$. By Lemma 4.1.2, we have

$$s \geq \frac{4t(t-1)[(t-1)(n-2t+1)+3][n^2-(3t-2)n+3t^2-4t]}{n^2}. \tag{4.7}$$

Note that $n^2-(3t-2)n+3t^2-4t \geq \frac{n^2}{4}$. This is because for any fixed $t \geq 3$, the function $g(n) = 3n^2-(12t-8)n+12t^2-16t$ is increasing in $[2t, \infty]$. Thus, $g(n) \geq g(2t) = 0$. Using this fact in inequality (4.7), we obtain that

$$s \geq t(t-1)[(t-1)(n-2t+1)+3]. \tag{4.8}$$

Let $f(n) = t(t-1)[(t-1)(n-2t+1)+3]-[(2t-1)n-t^2+t-2]$. When $t \geq 3$, $f(n)$ is increasing in $[2t, \infty]$. Thus,

$$\begin{aligned} f(n) \geq f(2t) &= t(t-1)(t+2)-(3t^2-t-2) \\ &> t(t-1)(t+2)-3t^2 \\ &= t(t^2-2t-2) > 0. \end{aligned}$$

Combining with inequality (4.8), we have $s > (2t-1)n-t^2+t-2$. This is a contradiction that finishes our proof. □

The following consequence of Theorem 5.4.5 extends Lemma 9.1 from [31].

Corollary 4.4.2. *For $t \in \{3,4\}$ and any integer $n \geq t$, any strongly regular graph associated with an $OA(t,n)$ satisfies Brouwer's Conjecture.*

Proof. For $t = 3$, we need to see what happens when $n \in \{3,4,5\}$. If $n = 3$, the corresponding graph is the complete three-partite graph $K_{3,3,3}$ which is OK by [31, Lemma 2.1]. If $n = 4$, the corresponding graph is a $(16, 9, 4, 6)$-SRG which is OK by [31, Lemma 2.1]. If $n = 5$, the corresponding graph is a $(25, 12, 5, 6)$-SRG which is OK by [31, Lemma 2.5].

For $t = 4$, we need to check what happens when $n \in \{4, 5, 6, 7\}$. If $n = 4$, the corresponding graph is the complete four-partite graph $K_{4,4,4,4}$ which is OK by [31, Lemma 2.1]. If $n = 5$, the corresponding graph is a $(25, 16, 9, 12)$-SRG is OK by [31, Lemma 2.1]. If $n = 6$, as shown by Tarry (see also Stinson [98]) an orthogonal array $OA(4, 6)$ does not exist. We remark here that McKay and Spence proved that there are exactly 32458 strongly regular graphs with parameters $(36, 20, 10, 12)$. If $n = 7$, the corresponding graph is a $(49, 24, 11, 12)$-SRG which is OK [31, Lemma 2.5]. $\square$

4.5 The Edge Version of Brouwer's Conjecture

In this section, we give a short proof of the edge version of Brouwer's Conjecture. We remark here that similar results were obtained by Hamidoune, Lladó, Serra and Tindell [61] for some families of vertex-transitive graphs.

Theorem 4.5.1. *Let Γ be a connected (v, k, λ, μ)-SRG. If A is a subset of vertices with $2 \le |A| \le v/2$, then*

$$e(A, A^c) \ge 2k - 2. \tag{4.9}$$

Equality happens if and only if A consists of two adjacent vertices or A induces a triangle in $K_{2,2,2}$ or A induces a triangle in the line graph of $K_{3,3}$.

Proof. If $k = 3$, then Γ is $K_{3,3}$ or the Petersen graph. If Γ is $K_{3,3}$, the proof is immediate. If Γ is the Petersen graph, and A is a subset of vertices with $3 \le |A| \le 4$, then the number of edges contained in A is at most $|A| - 1$ and therefore $e(A, A^c) \ge 3|A| - 2(|A| - 1) = |A| + 2 \ge 5$. If $|A| = 5$, then the number of edges inside A is at most $|A| = 5$ and therefore $e(A, A^c) \ge 3|A| - 2|A| = |A| = 5$. If $k = 4$, then Γ is $K_{4,4}, K_{2,2,2}$ or the line graph of $K_{3,3}$. If Γ is $K_{4,4}$, the proof is immediate. If Γ is $K_{2,2,2}$, and A is a subset of 3 vertices inducing a triangle, then $e(A, A^c) = 6$; in all the other cases, $e(A, A^c) > 6$. If Γ is the line graph of $K_{4,4}$ and A is a subset of vertices with $|A| = 3$, then $e(A, A^c) \geq 6$ with equality if and only if A induces a clique of order 3. If $|A| = 4$, then the number of edges contained in A is at most 4 and therefore, $e(A, A^c) \geq 4|A| - 8 = 8$.

Assume $k \geq 5$. If $3 \leq |A| \leq k-2$, then $e(A, A^c) \geq |A|(k+1-|A|) \geq 3(k-2) > 2k - 2$ as $k \geq 5$. If $k - 1 \leq |A| \leq v/2$, then $e(A, A^c) \geq \frac{(k-\theta_2)a(v-a)}{v} \geq \frac{(k-\theta_2)(k-1)}{2}$ (by Theorem 1.4.1). If Γ is a conference graph of parameters $(4t + 1, 2t, t - 1, t)$, then $k - \theta_2 = \frac{4t+1-\sqrt{4t+1}}{2} > 4$ for $t \geq 3$. These inequalities imply that $e(A, A^c) > 2(k - 1)$ which finishes the proof of this case. If Γ is not a conference graph and $k - \theta_2 > 4$, then $e(A, A^c) > 2k - 2$ and we are done again. The only case left is when Γ is not a conference graph and $k - \theta_2 \leq 4$. In this case, the eigenvalues of Γ are integers, $\theta_2 \geq k - 4$ and $\theta_v \leq -2$ as Γ is not a complete graph. Since $k - 1 \geq k - \mu = \theta_2(-\theta_v) \geq 2k - 8$, we get $5 \leq k \leq 7$. If $\theta_v = -2$, then by Theorem 3.3.1, Γ must have parameters $(16, 6, 2, 2)$ and eigenvalue $6, 2$ and -2. In this case, $e(A, A^c) \geq \frac{(k-\theta_2)a(v-a)}{v} = \frac{a(16-a)}{4} > 10$ for $k - 1 = 5 \leq a \leq 8 = v/2$. If $\theta_v \leq -3$, then we obtain $k - 1 \geq k - \mu = \theta_2(-\theta_v) \geq 3k - 12$ which implies $k = 5$. In this case, the graph Γ is the folded 5-cube which has parameters $(16, 5, 0, 2)$ and eigenvalues $5, 1$ and -3. As before, $e(A, A^c) \geq \frac{(k-\theta_2)a(v-a)}{v} = \frac{a(16-a)}{4} > 8$ for $3 \leq a \leq 8$. This finishes our proof. □

4.6 Remarks

Recall that, by Theorem 3.4.1, with finitely many exceptions, the strongly regular graphs with smallest eigenvalue $-m$ are of one of the following types:

(a) Complete multipartite graphs with classes of size m,

(b) Block graphs of Steiner m-systems with parameters

$$\left(\frac{n(n-1)}{m(m-1)}, \frac{m(n-m)}{m-1}, (m-1)^2 + \frac{n-1}{m-1} - 2, m^2\right),$$

(c) Latin square graphs with parameters

$$(n^2, m(n-1), n-2+(m-1)(m-2), m(m-1)).$$

For any fixed integer $m \geq 3$, the graphs in case (a) satisfy Brouwer's Conjecture by [31, Lemma 2.1]. By our results in Section 4.3 and Section 4.4, all but finitely many strongly regular graphs of type (b) or (c), satisfy Brouwer's Conjecture. This means that there are finitely many strongly regular graphs with smallest eigenvalue $-m$ that might not satisfy Brouwer's Conjecture. When $m = 2$, Cioabă, Kim and Koolen [31] proved that among the strongly regular graphs with smallest eigenvalue -2, the only counterexamples of Brouwer's Conjecture are the triangular graphs $T(m)$, where $m \geq 6$. By Theorem 4.3.1, Theorem 4.3.2 and Corollary 4.4.2, we know that when $m = 3, 4$, the strongly regular graphs of type (b) and (c) satisfy Brouwer's Conjecture.

We performed a computer search among feasible parameter sets of strongly regular graphs with smallest eigenvalue -3 or -4 for parameter sets (v, k, λ, μ) that do not satisfy the hypothesis of Proposition 2.4 in [31]. When $m = 3$, there are 321 such parameters that are the same as the parameters of block graphs of Steiner triple systems and 66 other parameter sets. The smallest strongly regular graph with other parameter sets is a $(36, 21, 12, 12)$-SRG. Therefore, the status of Brouwer's Conjecture is established for all strongly regular graphs with minimum eigenvalue -3 with the exception of 387 possible parameters. When $m = 4$, there are 1532 parameters that are the same as the parameters of block graphs of Steiner quadruple system graphs and 232 other parameters. The smallest strongly regular graph with other parameter sets is a $(45, 32, 22, 24)$-SRG. Thus, the status of Brouwer's Conjecture is established for all strongly regular graphs with minimum eigenvalue -4 with the exception of 1764 possible parameter sets.

Chapter 5

THE EXTENDABILITY OF MATCHINGS IN STRONGLY REGULAR GRAPHS

Most of the results of this chapter have appeared in Cioabă and Li [36].

5.1 Introduction

A set of edges M of a graph Γ is a matching if no two edges of M share a vertex. A matching M is perfect if every vertex is incident with exactly one edge of M. A matching is near perfect if all but one of the vertices of Γ are incident with edges of the matching. A graph Γ of even order v is called t-extendable if it contains at least one perfect matching, $t < v/2$ and any matching of size t is contained in some perfect matching. Graphs that are 1-extendable are also called matching covered (see Lovász and Plummer [80, page 113]). A graph Γ of odd order v is called t-near-extendable (or $t\frac{1}{2}$-extendable in the notation of Yu [106]) if it contains at least one near perfect matching, $t < (v-1)/2$ and for every vertex x, any matching of size t that does not cover x is contained in some near perfect matching that misses x. Graphs that are 0-near-extendable are also called factor-critical or hypomatchable (see Lovász and Plummer [80, page 89]). The extendability of a graph Γ of even order is defined as the maximum $t < v/2$ such that Γ is t-extendable; in this case, we say that the graph Γ precisely t-extendable. The extendability of a graph Γ of odd order is the largest $t < (v-1)/2$ such that Γ is t-near-extendable; in this case, we say that Γ is precisely t-near-extendable. In this chapter, the notion of (precisely) t-extendable will be used for graphs of even order and the notion of (precisely) t-near-extendable will refer to graphs of odd order.

Motivated among other things by work of Lovász [77] on canonical decomposition of graphs containing perfect matchings, the notion of extendability was introduced in 1980 by Plummer [90] for graphs of even order, was later extended to graphs of odd order by Yu [106] and has attracted a lot of attention (see the surveys [91, 92], the book [107] and the references therein). Zhang and Zhang [109] obtained an $O(mn)$ algorithm for determining the extendability of a bipartite graph Γ of order n and size m. At present time, the complexity of determining the extendability of a non-bipartite graph is unknown.

In this chapter, we study the extendability of primitive strongly regular graphs. Brouwer and Mesner (Theorem 3.3.2) proved that the vertex-connectivity of any connected strongly regular graph equals its valency. Plesník [89] (or [79, Chapter 7, Problem 30]) showed that if Γ is a k-regular and $(k-1)$-edge-connected graph with an even number of vertices, then the graph obtained by removing any $k-1$ edges of Γ, contains a perfect matching. It follows that every connected strongly regular graph is 1-extendable. Holton and Lou [65] showed that strongly regular graphs with certain connectivity properties are 2-extendable and conjectured that all but a few strongly regular graphs are 2-extendable. Lou and Zhu [81] proved this conjecture and showed that every connected strongly regular graph of valency $k \geq 3$ is 2-extendable with the exception (See Figure 5.2) of the complete 3-partite graph $K_{2,2,2}$ (the $(6,4,2,4)$-SRG) and the Petersen graph (the $(10,3,0,1)$-SRG). Another result worth mentioning is that any vertex-transitive graph is 1-extendable or 0-near-extendable (see [56, Theorem 3.5.1] or [80, Theorem 5.5.24]). For other results involving the extendability of vertex or edge-transitive graphs (with large cyclic connectivity) see Aldred, Holton and Lou [1]. Many strongly regular graphs have trivial automorphism groups and our techniques are different than the ones used for vertex-transitive graphs.

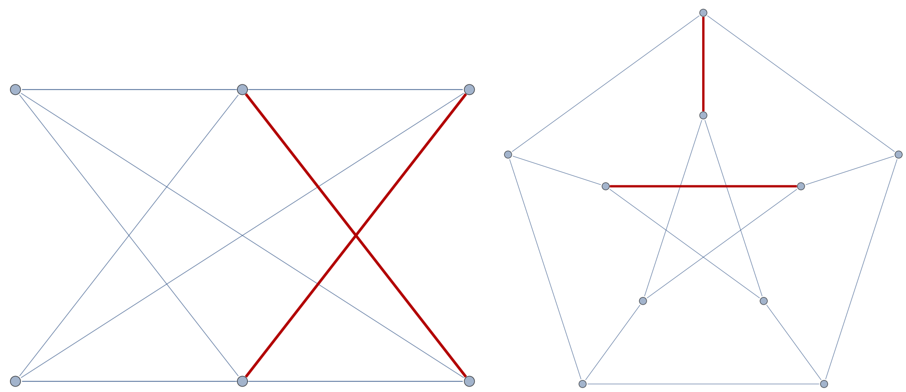

Figure 5.1: The only two strongly regular graphs which are not 2-extendable. The non-extendable matchings of size 2 are highlighted

In this chapter, we show that every connected (v, k, λ, μ)-SRG of valency $k \geq 5$ is 3-extendable with exception of the complete 4-partite graph $K_{2,2,2,2}$ (the $(8, 6, 4, 6)$-SRG), the complement of the Petersen graph (the $(10, 6, 3, 4)$-SRG) and the Shrikhande graph (one of the two $(16, 6, 2, 2)$-srgs). We also prove that any connected (v, k, λ, μ)-SRG with $\lambda \geq 1$ is $\lfloor k/3 \rfloor$-extendable when $\mu \leq k/2$ and $\lceil \frac{k+1}{4} \rceil$-extendable when $\mu > k/2$. This result is close to being best possible as we will prove that many connected strongly regular graphs with valency $k, \lambda \geq 1$ are not $\lceil k/2 \rceil$-extendable. On the other hand, we also determine the extendability of many families of strongly regular graphs including Latin square graphs, block graphs of Steiner systems, triangular graphs, lattice graphs and all known triangle-free strongly regular graphs. For each graph of valency k that we considered, the extendability is at least $\lceil k/2 \rceil - 1$. We conjecture that this is true for all connected strongly regular graphs of even order. We also obtain similar results for strongly regular graphs of odd order.

5.2 Main Tools

In this section, we introduce the notation used in this chapter and describe the main tools used in our proofs. Let $o(\Gamma)$ denote the number of components of odd order of a graph Γ. If S is a subset of vertices of Γ, then $\Gamma - S$ denotes the subgraph of Γ obtained by deleting the vertices in S. Let $N(T)$ denote the set of vertices outside T that are adjacent to at least one vertex of T. When $T = \{x\}$, let $N(x) = N(\{x\})$. If x is a vertex of a strongly regular graph Γ, let $N_2(x) = V(\Gamma) \setminus (\{x\} \cup N(x))$; the first subconstituent $\Gamma_1(x)$ of x is the subgraph of Γ induced by $N(x)$ and the second subconstituent $\Gamma_2(x)$ of x is the subgraph of Γ induced by $N_2(x)$.

Theorem 5.2.1 (Tutte [100]). *A graph Γ has a perfect matching if and only if $o(\Gamma - S) \leq |S|$ for every $S \subset V(\Gamma)$.*

Theorem 5.2.2 (Gallai [53]). *A graph Γ is factor-critical (or 0-near-extendable) if and only if Γ has an odd number of vertices and $o(\Gamma - S) \leq |S|$ for all $\emptyset \neq S \subset V(\Gamma)$.*

Using the above theorems, Yu [106] obtained the following characterizations of graphs that are not t-extendable (resp. not t-near-extendable).

Lemma 5.2.3 (Yu [106]). *Let $t \geq 1$ and Γ be a graph containing a perfect matching. The graph Γ is not t-extendable if and only if it contains a subset of vertices S such that S contains t independent edges and $o(\Gamma - S) \geq |S| - 2t + 2$.*

Lemma 5.2.4 (Yu [106]). *Let $t \geq 1$ and Γ be a factor-critical graph. The graph Γ is not t-near-extendable if and only if it contains a subset of vertices S such that S contains t independent edges, $|S| \geq 2t + 1$, and $o(\Gamma - S) \geq |S| - 2t + 1$.*

Lemma 5.2.5. *If Γ is a distance-regular graph of degree $k \geq 2$ and diameter $D \geq 3$, then, for any $x \in V(\Gamma)$, the subgraph induced by the vertices at distance 2 or more from x is connected.*

Proof. As $D \geq 3$, Γ contains an induced path P_4 with 4 vertices. By eigenvalue interlacing, the second largest eigenvalue $\theta_2(\Gamma)$ of Γ, is at least $\theta_2(P_4) = \frac{-1+\sqrt{5}}{2} > 0$. Cioabă and Koolen [32] proved that if the i-th entry of the standard sequence (See [17, Page 128]) corresponding to the second largest eigenvalue of a distance-regular graph Γ is positive, then for any vertex x, the subgraph of Γ induced by the vertices at distance at least i from x is connected. The second entry of the standard sequence corresponding to $\theta_2(\Gamma)$ is $\theta_2(\Gamma)/k > 0$ and this finishes our proof. □

Lemma 5.2.6. *Let Γ be a (v, k, λ, μ)-SRG with $\lambda \geq 1$. If T is an independent set, then $|N(T)| \geq 2|T|$.*

Proof. For any $x \in N(T)$, $N(x) \cap T$ is an independent set in the subgraph $\Gamma_1(x)$ which is induced by $N(x)$. The subgraph $\Gamma_1(x)$ is λ-regular. Consider the edges in $\Gamma_1(x)$. By counting the edges coming out of $N(x)\cap T$, we get that $|N(x)\cap T|\lambda \leq k\lambda/2$ and thus, $|N(x) \cap T| \leq k/2$. Therefore, $|T|k = e(T, N(T)) = \sum_{x\in N(T)} |N(x) \cap T| \leq |N(T)|k/2$ implying $|N(T)| \geq 2|T|$. □

Lemma 5.2.7. *Let Γ be a (v, k, λ, μ)-SRG. If T is an independent set, then*

$$|N(T)| \geq \frac{k^2|T|}{k + |T|\mu - \mu}. \tag{5.1}$$

Proof. For $x \in N(T)$, let $d_x = |T \cap N(x)|$ and $\bar{d} = \frac{\sum_{x\in N(T)} d_x}{|N(T)|}$. Counting the edges between T and $N(T)$, we have $|T|k = |N(T)|\bar{d}$. Counting the 3-subsets of the form $\{x, y, z\}$ such that $x, y \in T, z \in N(T), x \sim z, y \sim z$, we get that

$$\binom{|T|}{2}\mu = \sum_{x\in N(T)} \binom{d_x}{2} \geq |N(T)|\binom{\bar{d}}{2}. \tag{5.2}$$

Combining these equations, we obtain that $(|T|-1)\mu \geq k\left(\frac{k|T|}{|N(T)|}-1\right)$ which implies the desired inequality $|N(T)| \geq \frac{k^2|T|}{k+|T|\mu-\mu}$. $\square$

Note that the result of Lemma 5.2.7 is better than the one obtained by applying Inclusion-Exclusion Principle $|N(T)| = |\cup_{x\in T} N(x)| \geq \sum_{x\in T}|N(x)| - \sum_{x\neq y\in T}|N(x)\cap N(y)| = k|T| - \mu\binom{|T|}{2}$ when $|T| \geq 1+k/\mu$.

Lemma 5.2.8. *Let Γ be a primitive (v,k,λ,μ)-SRG with $\lambda \geq 1$. If T is an independent set, then*

$$|N(T)|-|T| \geq \begin{cases} k-1 & \text{if } 1 \leq |T| \leq \frac{(k-\mu)(k-1)}{\mu} \text{ or if } |T| \geq k-1. \\ \frac{(k-2)[k(k-1)-(k-3)\mu]}{(k-3)\mu+k} & \text{otherwise.} \end{cases} \tag{5.3}$$

Proof. Define $f(x) = \frac{k^2x}{\mu x+k-\mu} - x$ for $x \geq 1$. Note that $f(1) = f\left(\frac{(k-\mu)(k-1)}{\mu}\right) = k-1$ and that $f'(x) = \frac{k^2(k-\mu)}{(\mu x+k-\mu)^2} - 1$. Hence, $x_0 = \frac{k\sqrt{k-\mu}-(k-\mu)}{\mu}$ is the only critical point of $f(x)$. Since $k-\mu > 1$, we deduce that $1 < x_0 < \frac{(k-\mu)(k-1)}{\mu}$. Also, $f'(x) > 0$ for $x \in [1, x_0)$ and $f'(x) < 0$ for $x > x_0$. This implies that $|N(T)|-|T| \geq f(|T|) \geq f(1) = f\left(\frac{(k-\mu)(k-1)}{\mu}\right) = k-1$ whenever $1 \leq |T| \leq \frac{(k-\mu)(k-1)}{\mu}$. If $|T| \geq k-1$, Lemma 5.2.6 implies that $|N(T)|-|T| \geq |T| \geq k-1$. If $\frac{(k-\mu)(k-1)}{\mu} < |T| \leq k-2$, then the previous arguments and Lemma 5.2.7 imply that $|N(T)|-|T| \geq f(|T|) \geq f(k-2) = \frac{(k-2)[k(k-1)-(k-3)\mu]}{(k-3)\mu+k}$. $\square$

Note that if $\lambda \geq 1$ and $\mu \leq k/2$, then Lemma 5.2.8 implies that

$$|N(T)|-|T| \geq k-1 \tag{5.4}$$

for any independent set of vertices T.

The following lemma extends Theorem 5.1 of [33].

Lemma 5.2.9. *Let Γ be a primitive (v,k,λ,μ)-SRG. If A is a subset of vertices with $3 \leq |A| \leq v/2$ and A^c denotes its complement, then*

$$e(A, A^c) \geq 3k - 6. \tag{5.5}$$

Proof. If $k = 3$, then Γ is $K_{3,3}$ or the Petersen graph. If Γ is $K_{3,3}$, the proof is immediate. If Γ is the Petersen graph, and A is a subset of vertices with $3 \leq |A| \leq 4$, then the number of edges contained in A is at most $|A| - 1$ and therefore $e(A, A^c) \geq 3|A| - 2(|A| - 1) = |A| + 2 \geq 5$. If $|A| = 5$, then the number of edges inside A is at most $|A| = 5$ and therefore $e(A, A^c) \geq 3|A| - 2|A| = |A| = 5$. If $k = 4$, then Γ is $K_{4,4}, K_{2,2,2}$ or the Lattice graph $L_2(3)$ which is the unique $(9, 4, 1, 2)$-SRG. If Γ is $K_{4,4}$ or $K_{2,2,2}$, the proof is immediate. If Γ is the Lattice graph $L_2(3)$, and A is a subset of vertices with $|A| = 3$, then $e(A, A^c) \geq 6$ with equality if and only if A induces a clique of order 3. If $|A| = 4$, then the number of edges contained in A is at most 4 and therefore, $e(A, A^c) \geq 4|A| - 8 = 8$.

Assume $k \geq 5$. If $|A| \leq k - 2$, then $e(A, A^c) \geq |A|(k - |A| + 1) \geq 3(k - 2)$. Assume $|A| = k - 1$. If every vertex of A has at least 3 neighbors outside A, then $e(A, A^c) \geq 3(k - 1)$. Otherwise, there exists a vertex $x \in A$ that has exactly 2 neighbors outside A. Therefore, $e(N(x) \cap A^c, A) \geq 2 + 2(\lambda - 1) = 2\lambda$. Each vertex in $N(x)$ has $k - \lambda - 1$ neighbors outside $\{x\} \cup N(x)$. Thus, $e(N(x) \cap A, V(\Gamma) \setminus (N(x) \cup \{x\})) \geq (k - 2)(k - \lambda - 1)$. Hence, $e(A, A^c) \geq (k - 2)(k - \lambda - 1) + 2\lambda \geq 3(k - 2)$ since $\lambda \leq k - 2$.

If $k \leq |A| \leq v/2$, then $e(A, A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} \geq \frac{(k-\theta_2)k}{2}$ (by Theorem 1.4.1). If Γ is a conference graph of parameters $(4t + 1, 2t, t - 1, t)$, then $k - \theta_2 = \frac{4t+1-\sqrt{4t+1}}{2} > 6$ for $t \geq 4$ and consequently, $e(A, A^c) > 3k$. If $t = 3$, Γ has parameters $(13, 6, 2, 3)$ and therefore, $e(A, A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} > \frac{(13-\sqrt{13})|A|(13-|A|)}{2\cdot 13} > 12$. If Γ is not a conference graph and $k - \theta_2 \geq 6$, then $e(A, A^c) \geq 3k$ and we are done again. The only case left is when Γ is not a conference graph and $k-\theta_2 \leq 5$. In this case, the eigenvalues of Γ are integers, $\theta_2 \geq k-5$ and $\theta_v \leq -2$ as Γ is not a complete graph. Since $k-1 \geq k-\mu = \theta_2(-\theta_v) \geq 2k-10$, we get $5 \leq k \leq 9$. If $\theta_v = -2$, then by Seidel's characterization of strongly regular graphs with minimum eigenvalue -2 (see [19, Section 9.2] or [95]), there are three cases to consider. If Γ is a $(16,6,2,2)$-SRG, then its second largest eigenvalue is 2 and $e(A,A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} = \frac{|A|(16-|A|)}{4} \geq 15$ for $k=6 \leq |A| \leq 8 = v/2$. If Γ is a $(15,8,4,4)$-SRG, then since $k > v/2$, $|A| \leq k-1$ and $e(A,A^c) \geq 3k-6$ by a previous case. If Γ is a $(25,8,3,2)$-SRG, then its second largest eigenvalue is 3 and $e(A,A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} = \frac{|A|(25-|A|)}{5} \geq 24$ for $k=8 \leq |A| \leq 12 = \lfloor v/2 \rfloor$. If $\theta_v \leq -3$, then we obtain $k-1 \geq k-\mu = \theta_2(-\theta_v) \geq 3k-15$ which implies $5 \leq k \leq 7$. If $k=5$, then Γ is a $(16,5,0,2)$-SRG whose second largest eigenvalue is 1. Thus, $e(A,A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} = \frac{|A|(16-|A|)}{4} \geq 13$ for $k=5 \leq |A| \leq 8 = \lfloor v/2 \rfloor$. If $k=6$, then Γ is a $(15,6,1,3)$-SRG whose second largest eigenvalue is 1. Thus, $e(A,A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} = \frac{|A|(15-|A|)}{3} \geq 18$ for $k=6 \leq |A| \leq 7 = \lfloor v/2 \rfloor$. If $k=7$, then Γ is a $(50,7,0,1)$-SRG whose second largest eigenvalue is 2. Therefore, $e(A,A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} = \frac{|A|(50-|A|)}{10} \geq 28$ for $k=7 \leq |A| \leq 25 = v/2$. This finishes our proof. □

Lemma 5.2.10. *If Γ is a (v,k,λ,μ)-SRG of even order with independence number 2, then the extendability of Γ is $\lceil \frac{k}{2} \rceil - 1$.*

Proof. The fact that the extendability is at least $\lceil \frac{k}{2} \rceil - 1$ follows from Lemma 5.2.3. If Γ is imprimitive, then Γ must be $K_{2\times m}$ for some m and the conclusion will follow from Section 3.1. Assume that Γ is primitive and $\alpha(\Gamma)=2$. For any vertex $x \in V$, let $N_2(x) = V(\Gamma) \setminus (\{x\} \cup N(x))$. The second subconstituent $\Gamma_2(x)$ must be a complete graph with $k-\mu+1$ vertices. As the clique number is at most $\lambda+2$, we have

$\theta_2(-\theta_v) = k - \mu \leq \lambda + 1$. Since $\theta_v \leq -2$, we obtain that $\theta_2 \leq \frac{\lambda+1}{2} \leq \lambda - 1$ (when $\lambda \geq 3$). The first subconstituent $\Gamma_1(x)$ is λ-regular with second largest eigenvalue at most θ_2. By [30], $\Gamma_1(x)$ contains a matching of size $\lfloor k/2 \rfloor$. If k is even, then this matching cannot be extended to a maximum matching of Γ. If k is odd, one can add one disjoint edge to this matching such that the result matching of size $\lceil \frac{k}{2} \rceil$ cannot be extended to a maximum matching of Γ. It is easy to see that when λ is 1 or 2, $\Gamma_1(x)$ contains a perfect matching or an almost perfect matching. □

Lemma 5.2.11. *If Γ is a primitive (v, k, λ, μ)-SRG of even order with independence number $\alpha(\Gamma) \geq 3$, then the extendability of Γ is at least $\lceil \frac{k+3}{2} - \frac{\alpha(\Gamma)}{2} \rceil - 1 \geq \lceil \frac{k+3}{2} - \frac{v}{2(1+k/(-\theta_v))} \rceil - 1$.*

Proof. Let $t = \lceil \frac{k+3}{2} - \frac{\alpha(\Gamma)}{2} \rceil - 1$. Assume that Γ is not t-extendable. By Lemma 5.2.3, there is a vertex disconnecting set S containing t independent edges, such that $o(\Gamma - S) \geq |S| - 2t + 2 \geq k - 2t + 2 \geq 3$. Because Γ is primitive, Theorem 3.3.2 implies that $|S| \geq k + 1$. Thus, $\alpha(\Gamma) \geq o(\Gamma - S) \geq |S| - 2t + 2 \geq k - 2t + 3$, contradiction with $t = \lceil \frac{k+3}{2} - \frac{\alpha(\Gamma)}{2} \rceil - 1$. The second part follows from the Hoffman-ratio bound (Theorem 1.3.4) stating that $\alpha(\Gamma) \leq \frac{v}{1+k/(-\theta_v)}$. □

Lemma 5.2.12. *If Γ is a (v, k, λ, μ)-SRG with $\lambda = 0$, then $v > 3\alpha(\Gamma)$ except when Γ is the Petersen graph.*

Proof. The Hoffman-ratio bound (Theorem 1.3.4) states that $\alpha(\Gamma) \leq \frac{v}{1+k/(-\theta_v)}$. As $\theta_2(-\theta_v) = k - \mu < k$, when $\theta_2 \geq 2$, we have $k/(-\theta_v) > 2$, thus $\alpha(\Gamma) < v/3$. If $\theta_2 = 1$, then $\theta_v = \mu - k$ and $-\mu = \lambda - \mu = \theta_2 + \theta_v = 1 + \mu - k$. Thus, $k = 2\mu + 1$ and $v = 1 + k + k(k-1)/\mu = 3k + 1 = 6\mu + 4$. Let f be the multiplicity of θ_2 and g be the multiplicity of θ_v. We have $1 + f + g = v$ and $k + f + (\mu - k)g = 0$ and therefore, $v = 1 - k + (k - \mu + 1)g$. Hence, $g = \frac{8\mu+4}{\mu+2} = 8 - \frac{12}{\mu+2}$. As g is an integer,

$\mu \in \{1, 2, 4, 10\}$. When $\mu = 4$ or 10, we get $v > g(g + 3)/2$, which is impossible by Seidel's absolute bound (see [19, Section 9.1.8] or [45]). When $\mu = 1$, Γ must be the Petersen graph, where $\alpha(\Gamma) = 4$ and $v = 10 \leq 3\alpha(\Gamma) = 12$. Finally, when $\mu = 2$, Γ is the $(16, 5, 0, 2)$-SRG, where $16 = v > 3\alpha(\Gamma) = 15$. $\square$

Lemma 5.2.13. *Let Γ be a (v, k, λ, μ)-SRG with v even, $\lambda = 0$ and $k \geq 7$. If A is a subset of vertices such that $5 \leq |A| \leq v-k-1$ and $|A|$ is odd, then $e(A, A^c) \geq 5k-12$.*

Proof. Assume that $5 \leq |A| \leq 2k - 5$. As Γ is triangle-free, so is the subgraph induced by A. By Turán's Theorem, the number of edges inside A is at most $\frac{|A|^2-1}{4}$. So, $e(A) \leq \frac{|A|^2-1}{4}$ and $e(A, A^c) = k|A| - 2e(A) \geq k|A| - \frac{|A|^2-1}{2}$. The minimum is attained at $|A| = 5$ or $2k - 5$. In either case, we have $e(A, A^c) \geq 5k - 12$.

Assume that $2k - 3 \leq |A| \leq v/2$. We have $e(A, A^c) \geq \frac{(k-\theta_2)|A|(v-|A|)}{v} \geq \frac{(k-\theta_2)(2k-3)}{2}$ (by Theorem 1.4.1). Since $\theta_2(-\theta_v) = k - \mu \leq k - 1$ and $\theta_v \leq -3$ when $k \geq 7$, we have $\theta_2 \leq \frac{k-1}{3}$ and $k - \theta_2 \geq \frac{2k+1}{3}$. So, when $k \geq 7$, $k - \theta_2 \geq 5$ and $e(A, A^c) \geq \frac{5(2k-3)}{2} > 5k - 12$.

Assume that $v/2 < |A| \leq v-k-1$. It is equivalent to that $k+1 \leq |A^c| < v/2$. We can apply the same argument to A^c and show that $e(A, A^c) \geq 5k - 12$. $\square$

Lemma 5.2.14. *Let Γ be a primitive (v, k, λ, μ)-SRG and S be a disconnecting set of vertices. If $\Gamma - S$ contains at least two singleton components, then S contains at least $\mu(k - \mu)$ edges.*

Proof. Let x and y be two singleton components of $\Gamma - S$. Then $N(x) \cup N(y) \subseteq S$. If $z \in N(x) \setminus N(y)$, then z and y are non adjacent and they have exactly μ common. So, z is adjacent to at least μ vertices of S, and there are $|N(x) \setminus N(y)| = k - \mu$ such z. By the same argument, each vertex inside $N(y) \setminus N(x)$ is adjacent to at least μ vertices of S. Thus, $2e(S) \geq 2\mu(k - \mu)$. $\square$

5.3 The Extendability of Strongly Regular Graphs

5.3.1 Imprimitive strongly regular graphs

Recall that a strongly regular graph is imprimitive if it, or its complement, is disconnected. The only imprimitive strongly regular graphs are disjoint unions of cliques of the same order and their complements (complete multipartite regular graphs). A disjoint union mK_a of some number $m \geq 2$ of cliques K_a does not contain a perfect matching nor a near perfect matching if a is odd. If a is even, the extendability of this graph is $am/2 - 1$. The complete multipartite graph $K_{a\times m}$ (which is the complement of mK_a) has extendability $am/2 - 1$ if $m = 2$. When $m \geq 3$, if am is even, the extendability of $K_{a\times m}$ is $\frac{a(m-2)}{2} = \frac{k-a}{2}$ and if am is odd, the extendability of $K_{a\times m}$ is $\frac{a(m-2)-1}{2} = \lfloor \frac{k-a}{2} \rfloor$.

5.3.2 Lower bounds for the extendability of strongly regular graphs

In this section, we classify the primitive strongly regular graphs of even order that are not 3-extendable. We first provide results giving some general lower bounds for the extendability of a primitive strongly regular graphs.

Theorem 5.3.1. *If Γ is a (v, k, λ, μ)-SRG with $k/2 < \mu < k$ and $\alpha(\Gamma) \geq 3$, then the extendability of Γ is at least* $\max\left(\lceil \frac{k+3}{2} - \frac{3k-2\lambda-3}{2(2\theta_2+1)} \rceil - 1, \lceil \lambda/2 + 1 \rceil\right)$.

Proof. Since $\mu > k/2$, we get that $v = 1 + k + k(k - \lambda - 1)/\mu < 3k - 2\lambda - 1$. Also, $\mu > k/2$ implies that Γ has integer eigenvalues and $\theta_2(-\theta_v) = k - \mu < k/2$. By the Hoffman-ratio bound (Theorem 1.3.4), $\alpha(\Gamma) \leq \frac{v}{1+k/(-\theta_v)} < \frac{v}{1+2\theta_2}$. Combining these inequalities, we get that $\alpha(\Gamma) \leq \frac{3k-2\lambda-3}{1+2\theta_2}$. The second subconstituent $\Gamma_2(x)$ is connected (see [19, Proposition 9.3.1]) and $|N_2(x)| = k(k - \lambda - 1)/\mu < 2(k - \lambda - 1)$. Thus, $\alpha(\Gamma) \leq 1 + \frac{|N_2(x)|}{2}$ which implies $\alpha(\Gamma) \leq k - \lambda - 1$. By Lemma 5.2.11, the extendability of Γ is at least $\max\left(\lceil \frac{k+3}{2} - \frac{3k-2\lambda-3}{2(2\theta_2+1)} \rceil - 1, \lceil \lambda/2 + 1 \rceil\right)$. □

Note that the bounds in Theorem 5.3.1 are incomparable. When $\theta_2 = 1$, the first bound gives us $\lceil \lambda/3 + 1 \rceil$ and the second bound $\lceil \lambda/2 + 1 \rceil$ is better. On the other hand, when $\lambda = k/2$ and $\theta_2 \geq 2$, the first bound gives us $\lceil 3k/10 + 4/5 \rceil$ which is better than the second bound $\lceil k/4 + 1 \rceil$. There exist strongly regular graphs with $\lambda = k/2$ and $\theta_2 \geq 2$, for example, the $(36, 20, 10, 12)$-SRG.

Corollary 5.3.2. *Any primitive (v, k, λ, μ)-SRG with $\mu > k/2$ and $k \geq 8$ is 3-extendable.*

Proof. If $\theta_2 \geq 2$, then $\theta_v = \frac{\mu - k}{\theta_2} > -k/4$ as $\mu > k/2$. Thus, $\lambda = \mu + \theta_2 + \theta_v > k/2 + 2 - k/4 = k/4 + 2$. By Theorem 5.3.1, the extendability is at least $\lceil \frac{2k+2\lambda+8}{10} \rceil > \lceil k/4 + 1.2 \rceil \geq 4$ for $k \geq 8$. If $\theta_2 = 1$, we will show that $\lambda \geq \frac{k-3}{2}$. The second bound in Theorem 5.3.1 will then imply the extendability is at least $\lceil \frac{k+1}{4} \rceil \geq 3$. Since $\theta_2 = 1$, then $-\theta_v = k - \mu, 1 + \theta_v = \lambda - \mu$ and consequently, $\lambda = 2\mu + 1 - k$ and $v = 1 + k + \frac{k(k-\lambda-1)}{\mu} = 1 + k + \frac{2k(k-\mu-1)}{\mu}$. If $3k/4 \leq \mu < k$, then $\lambda \geq k/2 + 1 > \frac{k-3}{2}$. Otherwise, assume that $k/2 < \mu < 3k/4$, and let f be the multiplicity of θ_2 and g be the multiplicity of θ_v. We have $1 + f + g = v$ and $k + f + (\mu - k)g = 0$ and therefore, $1 - k + (k - \mu + 1)g = v = 1 + k + \frac{2k(k-\mu-1)}{\mu}$. Hence, $2(k-1)k = g(k - \mu + 1)\mu > g(1 + k/4)(3k/4)$ (as $(k - \mu + 1)\mu$ attains minimum at $\mu = 3k/4$) and $g < \frac{32(k-1)}{3(k+4)} < 11$. Thus, $g \leq 10$. By Seidel's absolute bound (see [19, Section 9.1.8] or [45]), $v < \frac{g(g+3)}{2} \leq 65$. We checked all the possible parameter sets of strongly regular graphs with v even and $\theta_2 = 1$ from the list [16]. All of them have the property that $\lambda \geq \frac{k-3}{2}$, and there is exactly one parameter set which attains the equality. It is the $(28, 15, 6, 10)$ and there are 4 such strongly regular graphs, the triangular graph $T(8)$ and the three Chang graphs (see Brouwer's list [16] or [19, page 123]). $\square$

Note that Chen [29] proved the following theorem (see also Aldred and Plummer [2] for extensions of Chen's result).

Theorem 5.3.3 (Chen [29]). *Let $t \geq 1$ and $n \geq 2$ be two integers. If Γ is a $(2t+n-2)$-connected $K_{1,n}$-free graph of even order, then Γ is t-extendable.*

When $\lambda \geq 1$, every (v, k, λ, μ)-SRG is k-connected and $K_{1,\lfloor k/2 \rfloor+1}$-free. If we let $t = \lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$ and $n = \lfloor k/2 \rfloor + 1$, then Chen's result implies that such strongly regular graph is $\lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$-extendable. This is the same as our result when $\theta_2 = 1$, $\mu > k/2$ and $\lambda \leq \frac{k-3}{2}$. For other cases, our lower bound is better than Chen's result. Note that Chen's bound can be improved if one has a better bound than $k/2$ for the independence number of the first subconstituents of the strongly regular graph.

Theorem 5.3.4. *Let Γ be a primitive (v, k, λ, μ)-SRG with $\lambda \geq 1$. If $\mu \leq k/2$, then Γ is n-extendable, where $n = \lceil \frac{k^2-k-3}{3k-7} \rceil - 1$.*

Proof. If Γ is not n-extendable, by Lemma 5.2.3, there is a vertex set S with s vertices such that S contains n independent edges, and $\Gamma - S$ has at least $s - 2n + 2$ odd components. Let $O_1, O_2, \ldots, O_r$ be all the odd components of $\Gamma - S$, with $r \geq s - 2n + 2$. Let $a \geq 0$ denote the number singleton components among $O_1, \ldots, O_r$. Counting the number of edges between S and $O_1 \cup \cdots \cup O_r$ and using Lemma 5.2.9, we get the following

$$ks - 2n \geq e(S, O_1 \cup \cdots \cup O_r) \geq ak + (r-a)(3k-6) \geq ak + (s-2n+2-a)(3k-6). \quad (5.6)$$

This inequality is equivalent to

$$n \geq \frac{(k-3)(s-a) + 3k - 6}{3k-7} \quad (5.7)$$

and since $s - a \geq k - 1$ (see the remark following Lemma 5.2.8), we obtain that

$$n \geq \frac{(k-3)(k-1) + 3k - 6}{3k-7} = \frac{k^2 - k - 3}{3k-7}. \quad (5.8)$$

This is a contradiction with $n = \lceil \frac{k^2-k-3}{3k-7} \rceil - 1 < \frac{k^2-k-3}{3k-7}$. $\square$

Corollary 5.3.5. *Any primitive (v, k, λ, μ)-SRG with $\lambda \geq 1$ and $k \geq 8$ is 3-extendable.*

Proof. Note that $\lceil \frac{k^2-k-3}{3k-7} \rceil - 1 = \lfloor k/3 \rfloor$ if $k \equiv 0, 1 \pmod 3$ and $\lfloor k/3 \rfloor + 1$, otherwise. Corollary 5.3.2 and Theorem 5.3.4 imply that any primitive (v, k, λ, μ)-SRG with $\lambda \geq 1$ and $k \geq 8$ is 3-extendable. □

Theorem 5.3.6. *Any primitive (v, k, λ, μ)-SRG with $\lambda = 0$ and $k \geq 8$ is 3-extendable.*

Proof. We show that Γ is 3-extendable by contradiction. Assume that Γ is not 3-extendable. Lemma 5.2.3 implies that Γ has a vertex subset S, such that S contains 3 independent edges, and $o(\Gamma - S) \geq |S| - 4$. Let S be such disconnecting set with maximum size. We first claim that any non-singleton component of $\Gamma - S$ cannot induce a bipartite graph. If that was the case, the respective component would have two partitions X and Y. Assume that $|X| > |Y|$, then define $S' = S \cup Y$. Then $|S'| > |S|$ and $o(\Gamma - S') \geq |S'| - 4$, contradicting the maximality of $|S|$. Note that $\Gamma - S$ cannot contain exactly 3 vertices, because Γ is triangle free and any component with 3 vertices must be a path, which is bipartite. By similar argument, $\Gamma - S$ contains no even components. If it contains a even component, we can put one vertex of this even component into S, which make $|S|$ larger and S still satisfy $o(\Gamma - S) \geq |S| - 4$. But it contradicts to the maximality of $|S|$. Let $O_1, O_2, \ldots, O_r$ be all the odd components of $\Gamma - S$. If $\Gamma - S$ has only singleton components, then $\alpha(\Gamma) \geq o(\Gamma - S) \geq |S| - 4 \geq v - \alpha(\Gamma) - 4$. Thus, $3\alpha(\Gamma) < v \leq 2\alpha(\Gamma) + 4$ and $k \leq \alpha(\Gamma) < 4$, which contradicts to $k \geq 8$. If $\Gamma - S$ has at most one singleton component, as $o(\Gamma - S) \geq |S| - 4 \geq 3$, $\Gamma - S$ has at least two non-singleton components, thus $|S| \geq k + 1$. By using Lemma 5.2.13 and counting the edges between S and $O_1 \cup \cdots \cup O_r$, we will get the following,

$$k|S| - 6 \geq e(S, O_1 \cup \cdots \cup O_r) \geq k + (|S| - 5)(5k - 12)$$

which implies that

$$12k - 33 \geq (2k-6)|S| \geq (2k-6)(k+1).$$

Thus, $2k^2 - 16k + 27 \leq 0$, contradiction with $k \geq 8$.

If $\Gamma - S$ has at least one non-singleton components and at least two singleton components, by using Lemma 5.2.13 and Lemma 5.2.14 and counting the edges between S and $O_1 \cup \cdots \cup O_r$, we will get that

$$k|S|-2(k-1) \geq k|S|-2\mu(k-\mu) \geq k|S|-2e(S) \geq e(S, O_1 \cup \cdots \cup O_r) \geq 5k-12+(|S|-5)k$$

which will yield another contradiction with $k \geq 8$. $\square$

Note that this theorem covers all known primitive triangle-free strongly regular graph with even order with precisely three exceptions. These are the Petersen graph, which is the unique $(10,3,0,1)$-SRG (and has extendability 1), the folded 5-cube, which is the unique $(16,5,0,2)$-SRG (and has extendability 3; see Theorem 5.4.9) and the Hoffman-Singleton graph, which is the unique $(50,7,0,1)$-SRG (and has extendability 5; see Theorem 5.4.10).

Corollary 5.3.7. *Let Γ be a primitive (v,k,λ,μ)-SRG with v even and $k \geq 5$. Then Γ is 3-extendable unless Γ is the complete 4-partite graph $K_{2,2,2,2}$ (the $(8,6,4,6)$-SRG), the complement of the Petersen graph (the $(10,6,3,4)$-SRG) or the Shrikhande graph (one of the two $(16,6,2,2)$-SRGs).*

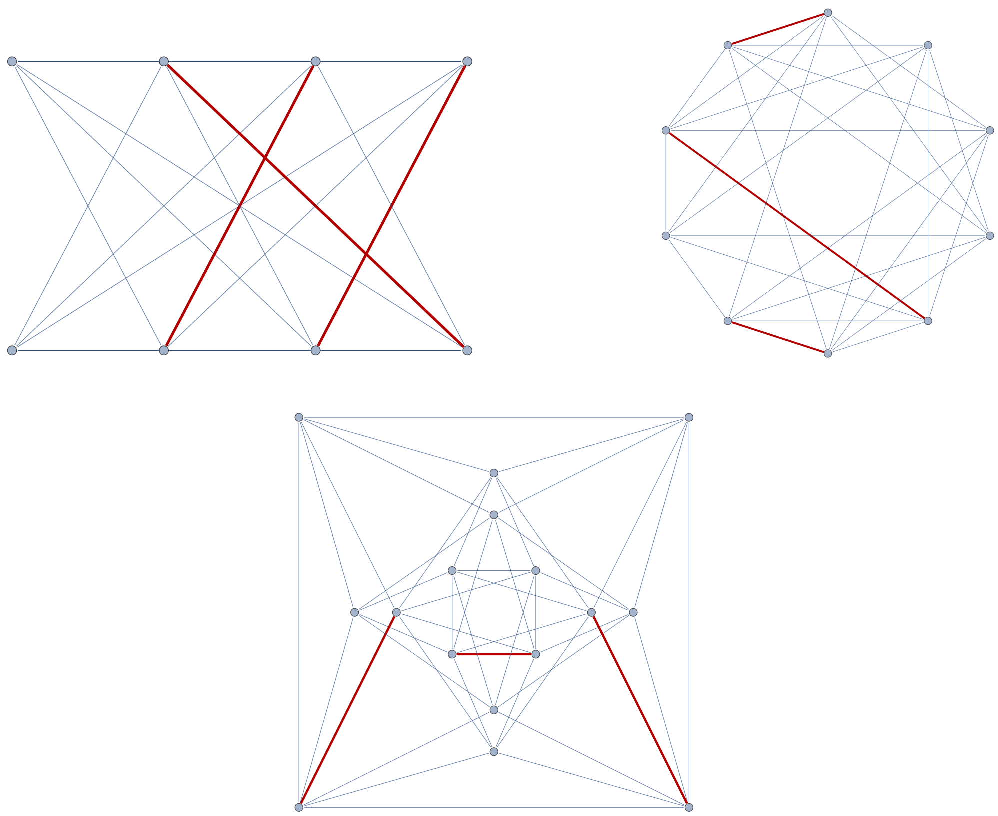

Figure 5.2: The three strongly regular graphs with $k \geq 6$ which are not 3-extendable. The non-extendable matchings of size 3 are highlighted. Deleting the highlighted matching in each graph will isolate a vertex, thus the remaining graph does not have a perfect matching.

Proof. If $k \geq 8$, then Γ is 3-extendable by Corollary 5.3.5 and Theorem 5.3.6. There are two primitive parameter sets with v even, $\lambda \geq 1$ and $5 \leq k \leq 7$: $(10, 6, 3, 4)$ and $(16, 6, 2, 2)$. There is a unique $(10, 6, 3, 4)$-SRG, the complement of Petersen graph or the triangular graph $T(5)$. Proposition 5.4.2 will show that the extendability of this graph is 2. There are two non-isomorphic strongly regular graphs with parameter set

$(16, 6, 2, 2)$. One is the Shrikhande graph (see [19, page 123] for a description) and the other is the line graph of $K_{4,4}$. In the Shrikhande graph, the first subconstituent of a fixed vertex is isomorphic to the cycle C_6 and thus contains a matching of size 3. This matching is not contained in any perfect matching. Thus, the Shrikhande graph is not 3-extendable; by Lou and Zhu [81], the extendability of the Shrikhande graph is 2. Proposition 5.4.7 will show that the extendability of the line graph of $K_{4,4}$ is 3. To finish the proof, the only strongly regular graph with $5 \leq k \leq 7$ and $\lambda = 0$ is the folded 5-cube whose extendability is 3 (see Theorem 5.4.9). □

The previous argument can be generalized to classify primitive 4-extendable strongly regular graphs. By a more extensive case analysis which we omit here, we can show that a primitive strongly regular graph Γ with even number of vertices and $\lambda \geq 1$ is 4-extendable if and only if $k \geq 9$. Similarly, we can also classify all the primitive 1-near-extendable and the 2-near-extendable strongly regular graphs. When $\lambda \geq 1$, every strongly regular graph of odd order is 1-near-extendable and there is exactly one primitive strongly regular graph with $\lambda \geq 1$ which is not 2-near-extendable, namely the Paley graph on 9 vertices (the unique $(9, 4, 1, 2)$-SRG).

5.4 The Extendability of Some Specific Strongly Regular Graphs

In this section, we determine the extendability of several families of strongly regular graphs. In the first three subsections, we show that there are many strongly regular graphs with extendability equal or slightly larger than $\lceil k/2 \rceil - 1$. In the last subsection, we show that the extendability of any known triangle-free strongly regular graph of even order and valency k equals $k - 2$.

The reason that the graphs considered in the next three subsections (except for the graphs in Theorem 5.4.7) are not $\lceil k/2 \rceil$-extendable (when v is even) or not $k/2$-near-extendable (when v is odd) is the following. Consider the first subconstituent

$\Gamma_1(x)$ of any fixed vertex x; this is the subgraph induced by $N(x)$. If v is even, we will show that $\Gamma_1(x)$ has a matching of size $k/2$ if k is even and of size $(k-1)/2$ if k is odd. When k is odd, there is one vertex y not covered by the matching of size $(k-1)/2$ and we choose a vertex z not adjacent to x such that z is adjacent with y. In each case, we construct a matching of size $\lceil \frac{k}{2} \rceil$ that cannot be contained in a perfect since its removal leaves x isolated. If v is odd, then k is even. We will show that $\Gamma_1(x)$ has a matching of size $k/2$. Choose a vertex $y \in N_2(x)$. The matching of size $k/2$ in $\Gamma_1(x)$ does not cover y. Thus, we construct a matching that cannot be contained in a near perfect matching that misses y since the removal of $N(x) \cup \{y\}$ leaves x isolated. We will also use the following lemma.

Lemma 5.4.1. *Let Γ be a graph of order am whose vertex set can be partitioned into m subsets, $A_1, A_2, \ldots, A_m$ with equal size a, such that for $1 \leq i \leq m$, A_i induce a clique, and the graph obtained by vertex contracting each A_i contains a perfect matching (when m is even) or a near perfect matching (when m is odd). Then Γ contains a perfect matching if am is even, and Γ contains a near perfect matching if am is odd.*

Proof. If a is even, the lemma is obvious. If a is odd and m is even, we can find a matching $u_1u_2, \cdots, u_{m-1}u_m$ such that $u_i \in A_i$ for $1 \leq i \leq m$. Now, each subgraph induced by $A_i \backslash u_i$ contain a perfect matching. Thus Γ contains a perfect matching. If a is odd and m is odd, we can find a matching $u_1u_2, \cdots, u_{m-2}u_{m-1}$ such that $u_i \in A_i$ for $1 \leq i \leq m-1$. Now, each subgraph induced by $A_i \backslash u_i$ contain a perfect matching for $1 \leq i \leq m-1$ and A_m contains a near perfect matching. Thus Γ contains a near perfect matching. □

5.4.1 Triangular graphs

Recall that the triangular graph $T(m)$ is the line graph of the complete graph K_m; its vertices are the 2-subsets of $[m] := \{1, \ldots, m\}$ and $\{u, v\} \sim \{x, y\}$ if and only if $|\{u, v\} \cap \{x, y\}| = 1$. The triangular graph $T(m)$ is an $(\binom{m}{2}, 2(m-2), m-2, 4)$-SRG.

Theorem 5.4.2. *If $m \geq 4$, the extendability of $T(m)$ is $k/2 - 1 = m - 3$.*

Proof. We consider first the case when $\binom{m}{2}$ is even. The subgraph induced by $N(\{1, 2\})$ contains a perfect matching; take $\{(\{1, i\}, \{2, i\}) | 3 \leq i \leq m\}$ for example. By the observation at the beginning of Section 5.4, this shows that $T(m)$ is not $(m-2)$-extendable.

Assume that $T(m)$ is not $(m-3)$-extendable. By Lemma 5.2.3, there is a subset of vertices S such that S contains $m-3$ independent edges and $r = o(\Gamma - S) \geq |S| - 2(m-3) + 2$. Let $O_1, O_2, \ldots, O_r$ be the odd components of $\Gamma - S$. Denote by P_i the union of the 2-subsets corresponding to the vertices of O_i, for $1 \leq i \leq r$. If $r \leq 3$, then by Theorem 3.3.2, $|S| \geq 2(m-2)$ and therefore, $3 \geq o(\Gamma - S) \geq |S| - 2(m-3) + 2 \geq 4$, contradiction. If $r \geq 4$, then since $P_1, P_2, \ldots, P_r$ are disjoint subsets of $[m]$, and $|P_i| \geq 2$, we have $m \geq 8$. There exists two odd components, says O_1, O_2, such that $3 \leq |P_1 \cup P_2| \leq m - 3$. We have $\{\{u, v\} \mid u \in P_1 \cup P_2, v \in [m] - (P_1 \cup P_2)\} \subset N(O_1 \cup O_2) \subset S$. Thus $|S| \geq 3(m-3)$. On the other hand, as $2r \leq |P_1| + |P_2| + \ldots + |P_r| \leq m$, we have $r \leq m/2$. So, $m/2 \geq o(\Gamma - S) \geq |S| - 2(m-3) + 2 \geq 3(m-3) - 2(m-3) + 2 = m - 1$, contradiction with $m \geq 8$.

If $\binom{m}{2}$ is odd, by the same argument as above, it is easy to see that $T(m)$ is not $(m-2)$-near-extendable. Assume that $T(m)$ is not $(m-3)$-near-extendable. By Lemma 5.2.4, there is a subset of vertices S such that S contains $m-3$ independent edges, $|S| \geq 2(m-3) + 1$ and $r = o(\Gamma - S) \geq |S| - 2(m-3) + 1$. If $r = 2$, then by

Theorem 3.3.2, $|S| \geq 2(m-2)$ and therefore, $2 = o(\Gamma - S) \geq |S| - 2(m-3) + 1 \geq 3$, contradiction. If $r = 3$, S is not the neighborhood of some vertex. By Theorem 3.3.2, $|S| \geq 2(m-2)+1$, and therefore, $3 = o(\Gamma - S) \geq |S| - 2(m-3) + 1 \geq 4$, contradiction. The rest of the proof is the same as in the case $\binom{m}{2}$ even. $\square$

5.4.2 Block graphs of Steiner systems

A 2-$(n, K, 1)$-design or a Steiner K-system is a point-block incidence structure on n points such that each block has K points and any two distinct points are contained in exactly one block. The block graph of such a Steiner system has as vertices the blocks and two distinct blocks are adjacent if they intersect. The block graph of a Steiner K-system is a $\left(\frac{n(n-1)}{K(K-1)}, \frac{K(n-K)}{K-1}, (K-1)^2 + \frac{n-1}{K-1} - 2, K^2\right)$-SRG.

Theorem 5.4.3. *Let Γ be the block graph of a Steiner K-system on n points such that $\frac{n(n-1)}{K(K-1)}$ is even. If $K \in \{3,4\}$ and $n > K^2$ or $K \geq 5$ and $n > 4K^2 + 5K + 24 + \frac{96}{K-4}$, the extendability of Γ is $\lceil k/2 \rceil - 1$, where k is the valency of Γ.*

Proof. Let Γ be the block graph of Steiner K-system, and B denote the block sets of the 2-$(n, K, 1)$ design. Consider the neighborhood $N(\{1, \ldots, K\})$ of the vertex $\{1, 2, \ldots, K\}$. There is a partition of $N(\{1, \ldots, K\})$ into cliques, which is $A_i = \{b \in B \mid b \cap \{1, 2, \ldots, K\} = \{i\}\}$ for $1 \leq i \leq K$. For any A_i and A_j, there exist $b_i \in A_i$ and $b_j \in A_j$ such that $n \in b_i$ and $n \in b_j$. So, b_i and b_j are adjacent. The graph obtained by contracting each A_i is a complete graph. By Lemma 5.4.1, the first subconstituent $\Gamma_1(x)$ contains a perfect matching or a near perfect matching. There are $\lceil k/2 \rceil$ independent edges incident with all $N(x)$ and not incident with x. This implies Γ is not $\lceil k/2 \rceil$-extendable.

Assume that Γ is not $(\lceil k/2 \rceil - 1)$-extendable. By Lemma 5.2.3, there is a subset of vertices S such that S contains $\lceil k/2 \rceil - 1$ independent edges and $r =$

$o(\Gamma - S) \geq |S| - 2(\lceil k/2 \rceil - 1) + 2$. Let $O_1, O_2, \ldots, O_r$ be all the odd components of $\Gamma - S$, and P_i denote the union of the blocks corresponding to the vertices of O_i, where $1 \leq i \leq r$. Since $|P_i| \geq K$ and $P_i \cap P_j = \emptyset$ for $i \neq j$, we have that $n \geq |P_1| + |P_2| + \ldots + |P_r| \geq Kr$.

If $r \leq 2$, then as $|S| \geq k$ by Theorem 3.3.2, we get that $2 \geq o(\Gamma - S) \geq |S| - 2(\lceil k/2 \rceil - 1) + 2 \geq k - 2\lceil k/2 \rceil + 4 \geq 3$, contradiction. Otherwise, if $r \geq 3$ and there exists two singleton components among $O_1, \ldots, O_r$, then $|S| \geq 2k - \mu$. This implies that $n/K \geq r \geq |S| - 2(\lceil k/2 \rceil - 1) + 2 \geq 2k - \mu - 2\lceil k/2 \rceil + 4 \geq k - \mu + 3 = \frac{K(n-K)}{K-1} - K^2 + 3$, contradiction with $K \in \{3, 4\}$ and $n > K^2$, or $K \geq 5$ and $n > 4K^2 + 5K + 24 + \frac{96}{K-4}$. Otherwise, if there is at most one singleton component, then there are at least two non-singleton odd components, says O_1, O_2. The neighborhood of O_1 is a non-local disconnecting set whose removal disconnects the graph into non-singleton components. By the results in [33, Section 3], $|S| \geq |N(O_1)| \geq 2k - \lambda - 2$. As before, $n/K \geq r \geq |S| - 2(\lceil k/2 \rceil - 1) + 2 \geq 2k - \lambda - 2 - 2\lceil k/2 \rceil + 4 \geq k - \lambda + 1 = \frac{K(n-K)}{K-1} - (K-1)^2 - \frac{n-1}{K-1} + 3 = n - K^2 + K + 1$, contradiction. $\square$

Note that when $K \in \{3, 4\}$ and $n \leq K^2$, the block graph of Steiner K-system is either a complete graph or a complete multipartite graph. If $v = \frac{n(n-1)}{K(K-1)}$ is odd, then k is even. The proof of the next result is similar to the previous one and will be omitted.

Theorem 5.4.4. *Let Γ be the block graph of a Steiner K-system on n points such that $\frac{n(n-1)}{K(K-1)}$ is odd. If $K \in \{3, 4\}$ and $n > K^2$ or $K \geq 5$ and $n > 4K^2 + 5K + 24 + \frac{96}{K-4}$, the extendability of Γ is $k/2 - 1$, where k is the valency of Γ.*

5.4.3 Latin square graphs

Recall that an orthogonal array $OA(t, n)$ with parameters t and n is a $t \times n^2$ matrix with entries from the set $[n] = \{1, \ldots, n\}$ such that the n^2 ordered pairs defined by any two distinct rows of the matrix are all distinct. Given an orthogonal array $OA(t, n)$, one can define a graph Γ as follows. The vertices of Γ are the n^2 columns of the orthogonal array and two distinct columns are adjacent if they have the same entry in one coordinate position. The graph Γ is an $(n^2, t(n-1), n-2+(t-1)(t-2), t(t-1))$-SRG. Any strongly regular graph with such parameters is called a Latin square graph. When $t = 2$ and $n \neq 4$, such a graph must be the line graph of $K_{n,n}$ which is also the graph associated with an orthogonal array $OA(2, n)$ (see [19, page 123]).

Theorem 5.4.5. *Let $n \geq 2t \geq 6$ be two integers with n even. If Γ is a Latin square graph corresponding to an $OA(t, n)$, then the extendability of Γ is $\lceil k/2 \rceil - 1$.*

Proof. Let C denote the column set of the orthogonal array $OA(t, n)$ corresponding to Γ. Consider the neighborhood $N(c_1)$ of a column $c_1 = (c_1(1), \ldots, c_1(t))^T$ of C. There is a partition of $N(c_1)$ into cliques, which is $A_i = \{c \in C \mid c(i) = c_1(i)\}$ for $1 \leq i \leq t$. Let $l \in [n]$ such that $l \neq c_1(3)$. There exist $c \in A_1$ such that $c(3) = l$ and there is $c' \in A_2$ such that $c'(3) = l$. Thus c and c' are adjacent. The graph obtained by contracting each A_i is a complete graph. By Lemma 5.4.1 and the same argument in Theorem 5.4.3, we deduce that Γ is not $\lceil k/2 \rceil$-extendable.

Assume that Γ is not $(\lceil k/2 \rceil - 1)$-extendable. By Lemma 5.2.3, there is a subset of vertices S such that S contains $\lceil k/2 \rceil - 1$ independent edges and $r = o(\Gamma - S) \geq |S| - 2(\lceil k/2 \rceil - 1) + 2$. Let $O_1, O_2, \ldots, O_r$ be all the odd components of $\Gamma - S$.

If $r \leq 2$, then as $|S| \geq k$ by Theorem 3.3.2, we have $2 \geq r \geq |S| - 2(\lceil k/2 \rceil - 1) + 2 \geq k - 2\lceil k/2 \rceil + 4 \geq 3$, contradiction. Otherwise, if $r \geq 3$ and there exists two singleton components among $O_1, \ldots, O_r$, then $|S| \geq 2k - \mu$. Because $\alpha(\Gamma) \leq n$, we deduce that $n \geq r \geq |S| - 2(\lceil k/2 \rceil - 1) + 2 \geq 2k - \mu - 2\lceil k/2 \rceil + 4 \geq k - \mu + 3 = t(n-1) - t(t-1) + 3 = t(n-t) + 3 \geq 2(n-2) + 3$. Thus, $n \leq 1$, contradiction. If $r \geq 3$ and there is at most one singleton component, then there are at least two non-singleton odd components, says O_1, O_2. The neighborhood $N(O_1)$ of O_1 is a non-local disconnecting set whose removal disconnects the graph into non-singleton components. By the result in [33], $|N(O_1)| \geq 2k - \lambda - 2$. Thus $|S| \geq 2k - \lambda - 2$. As before, $n \geq o(\Gamma - S) \geq k - \lambda + 1 = t(n-1) - n + 2 - (t-1)(t-2) + 1 = (t-1)(n-t+1) + 2 \geq n + 1$, contradiction. $\square$

The proof of our next result is similar and will be omitted.

Theorem 5.4.6. *Let $n \geq 2t \geq 6$ be two integers with n odd. If Γ is a Latin square graph corresponding to an $OA(t, n)$, then the extendability of Γ is $k/2 - 1$, where k is the valency of Γ.*

The line graph of $K_{n,n}$ is a $(n^2, 2(n-1), n-2, 2)$-SRG. It can be regarded as a strongly regular graph corresponding to an $OA(2, n)$.

Theorem 5.4.7. *Let Γ be the line graph of $K_{n,n}$ with $n \geq 4$ and n even. The extendability of Γ is $k/2 = n - 1$.*

Proof. If n is even, then the first subconstituent $\Gamma_1(x)$ of some vertex x of Γ is the disjoint union of two cliques K_1, K_2 of odd order. Pick two vertices y, z in the second subconstituent of x. If $S = N(x) \cup \{y, z\}$, then S contains a matching of size n that is not contained in any perfect matching. Therefore, Γ is not n-extendable.

Assume that Γ is not $k/2$-extendable. By Lemma 5.2.3, there is a subset of vertices S such that S contains $k/2$ independent edges (therefore, S is not the neighborhood of some vertex) and $r = o(\Gamma - S) \geq |S| - 2(k/2) + 2$. Let $O_1, O_2, \ldots, O_r$ be all the odd components of $\Gamma - S$. If $r \leq 2$, then as S is not the neighborhood of some vertex, by Theorem 3.3.2 $|S| \geq k+1$ and therefore, $2 \geq r \geq |S| - k + 2 \geq 3$, contradiction. If $r \geq 3$, the proof is the same as Theorem 5.4.5 except for the case when there is at most one singleton component among $O_1, \ldots, O_r$. In this case, we need to show that $o(\Gamma - S) \leq n - 1$. This inequality can be proved by contradiction. We know that $o(\Gamma - S) \leq \alpha(\Gamma) = n$. If $o(\Gamma - S) = n$, then we can pick a vertex x_i from each component O_i and $I = \{x_1, \ldots, x_n\}$ will form an independent set of size n. I can be considered as a perfect matching in $K_{n,n}$. Any edge in $K_{n,n}$ outside this perfect matching will intersect with two edges in this perfect matching. Hence, any vertex in the line graph of $K_{n,n}$ but not in I will adjacent with two vertices in I. Assume that O_1 is not singleton, then there is a vertex $y \in O_1$, where $y \neq x_1$, and y is adjacent to x_i for some $i \neq 1$. It contradicts to that O_1 and O_i are two distinct components in $\Gamma - S$. □

Theorem 5.4.8. *Let Γ be the line graph of $K_{n,n}$ with $n \geq 3$ and n odd. The extendability of Γ is $k/2 - 1 = n - 2$.*

Proof. If n is odd, then the first subconstituent $\Gamma_1(x)$ of some vertex x of Γ is the disjoint union of two cliques K_1, K_2 of even order. Thus, $\Gamma_1(x)$ contains a matching of size $n-1$. Therefore, Γ is not $(n-1)$-near-extendable. The proof of fact that Γ is $(n-2)$-near-extendable is simpler and will be omitted. □

5.4.4 The extendability of the known triangle-free strongly regular graphs

We determine the extendability of the known primitive triangle-free strongly regular graphs. There are seven known examples of such graphs and they have parameter sets: $(5, 2, 0, 1)$, $(10, 3, 0, 1)$, $(16, 5, 0, 2)$, $(50, 7, 0, 1)$, $(56, 10, 0, 2)$, $(77, 16, 0, 4)$, $(100, 22, 0, 6)$. Lou and Zhu [81] proved that the extendability of the $(10, 3, 0, 1)$-SRG (the Petersen graph) is 1.

The $(16, 5, 0, 2)$-SRG is called the folded 5-cube as it can be obtained from the 5-dimensional cube on 32 vertices by identifying antipodal vertices. The $(16, 5, 0, 2)$-SRG is known as the complement of the Clebsch graph (see [19, page 117]) or as the Clebsch graph (see [76, Example 21.4, page 263]).

Theorem 5.4.9. *The folded* 5*-cube is precisely* 3*-extendable.*

Proof. Let Γ be the $(16, 5, 0, 2)$-SRG. We first show that Γ is not 4-extendable. Let x be a vertex of Γ. It is known that the second subconstituent $\Gamma_2(x)$ of x is isomorphic to the Petersen graph. Consider four independent edges of $\Gamma_2(x)$. We claim that these four edges are not contained in a perfect matching of Γ. Let S be the complement of the neighborhood of x in Γ. Then S contains four independent edges and $5 = o(\Gamma - S) \geq |S| - 2 \cdot 4 + 2 = 5$. Lemma 5.2.3 implies that Γ is not 4-extendable.

We show that Γ is 3-extendable by contradiction. Assume that Γ is not 3-extendable. Lemma 5.2.3 implies that Γ has a vertex subset S, such that S contains 3 independent edges, and $o(\Gamma - S) \geq |S| - 4$. Let S be such disconnecting set with maximum size. By the same argument in the proof of Theorem 5.3.6, any non-singleton component of $\Gamma - S$ cannot contain exactly 3 vertices. If $\Gamma - S$ has no singleton components, then $\Gamma - S$ has at most two odd components (since $\alpha(\Gamma) = 5$ and each non-singleton component has two non-adjacent vertices). Thus, $|S| \leq o(\Gamma - S) + 4 \leq 6$. Lemma 4.1.2 implies that $|S| \geq 25/2$, a contradiction. If $\Gamma - S$

has one or two singleton components, then $\Gamma - S$ has at most three odd components. Thus, $|S| \leq o(\Gamma - S) + 4 \leq 7$. However, S contain the neighborhood of a vertex, which is an independent set of size 5. Since S also contains three independent edges, $|S| \geq 8$, contradiction. The remaining case is when $\Gamma - S$ has at least three singleton components, say x, y, z. As $o(\Gamma - S) \leq 5$, $|S| \leq o(\Gamma - S) + 4 \leq 9$. However, $|N(\{x, y, z\})| = 10$. This is because y, z are contained in $\Gamma_2(x)$ which is isomorphic to the Petersen graph. Because y and z are not adjacent, they have one common neighbor in $\Gamma_2(x)$. Hence, there are five vertices adjacent to y or z in $\Gamma_2(x)$. Thus, $|S| \geq |N(\{x, y, z\})| = 10$, contradiction. □

The $(50, 7, 0, 1)$-SRG is called the Hoffman-Singleton graph and its independence number is 15 (see [17, Section 13.1] or [19, page 117]).

Theorem 5.4.10. *The Hoffman-Singleton graph is precisely* 5*-extendable.*

Proof. Let Γ be the $(50, 7, 0, 1)$-srg. We first show that Γ is not 6-extendable. Let x and y be two non-adjacent vertices of Γ and let $S = N(x) \cup N(y)$. Then $|S| = 13$ and S contains 6 independent edges. Since the second subconstituent $\Gamma_2(x)$ of x is a distance-regular graph with intersection array $\{6, 5, 1; 1, 1, 6\}$, Lemma 5.2.5 implies that the subgraph of $\Gamma_2(x)$ obtained by removing the neighbors of y in $\Gamma_2(x)$, has exactly two odd components. Hence, $3 = o(\Gamma - S) \geq 3 = |S| - 2 \cdot 6 + 2$. By Lemma 5.2.3, Γ is not 6-extendable.

We show that Γ is 5-extendable by contradiction. Assume that Γ is not 5-extendable. By Lemma 5.2.3, there exists a subset of vertices S such that S contains five independent edges and $o(\Gamma - S) \geq |S| - 8$. Consider such a set S of maximum size. By the same argument as in the proof of Theorem 5.3.6, the maximality of $|S|$ implies that any non-singleton odd component of $\Gamma - S$ cannot be bipartite. We use this observation to prove that any non-singleton odd component must has at least 7 vertices. Assume that C is a non-singleton component with 5 vertices (Otherwise, C induces a bipartite graph). Then C must induce a cycle on 5 vertices. Thus, $o(\Gamma - S) \leq \alpha(\Gamma) - 1 = 14$ and $|S| \leq o(\Gamma - S) + 8 \leq 22$. However, $|N(C)| = 25$ because each vertex in C has 5 neighbors in S, and any two vertices in C have no common neighbors in S. Thus, $22 \geq |S| \geq |N(C)| \geq 25$, a contradiction.

Thus, any odd non-singleton component of $\Gamma - S$ has at least 7 vertices. If $\Gamma - S$ has at least 2 non-singleton odd components, then $o(\Gamma - S) \leq 11$, and $|S| \leq o(\Gamma - S) + 8 \leq 19$. Lemma 4.1.2 implies that $|S| \geq \frac{4\times 7\times(50-19-7)}{25} > 26$, a contradiction. If $\Gamma - S$ has exactly one non-singleton odd component, then $o(\Gamma - S) \leq 13$ and $|S| \leq o(\Gamma - S) + 8 \leq 21$. If $\Gamma - S$ contains at least 7 singleton components, then Lemma 4.1.2 implies that $|S| \geq \frac{4\times 7\times(50-21-7)}{25} > 24$, a contradiction. If $\Gamma - S$ contains between 3 and 6 singleton components, then $|S| \leq o(\Gamma - S) + 8 \leq 15$. However, for any three independent vertices x, y, z, $|N(x) \cup N(y) \cup N(z)| \geq 3\cdot 7 - \binom{3}{2}\cdot 1 = 18$, a contradiction. If $\Gamma - S$ has one or two singleton components, then $|S| \leq o(\Gamma - S) + 8 \leq 11$. Since S contains the neighborhood $N(x)$ of a vertex x and S contains 5 independent edges, S must contain another 5 vertices outside of $N(x)$. Thus, $|S| \geq 12$, a contradiction. If $\Gamma - S$ only has singleton odd components, then $\Gamma - S$ has no even components. Otherwise, we can put one vertex of the even components into S. In this way, $|S|$ will increase by one, and $o(\Gamma - S)$ will increase at least by one, contradicting the maximality of $|S|$. Thus, $|S| = 50 - o(\Gamma - S) \geq 35$ contradicting the inequality $|S| \leq o(\Gamma - S) + 10 \leq 25$. $\square$

The $(56, 10, 0, 2)$-SRG is known as the Gewirtz graph or the Sims-Gewirtz graph and its independence number is 16 (see [17, page 372] or [19, page 117]).

Theorem 5.4.11. *The Gewirtz graph is precisely* 8*-extendable.*

Proof. Let Γ be the Gewirtz graph. We first show that Γ is not 9-extendable. Let x and y be two non-adjacent vertices of Γ. Because every vertex in $N(x) \setminus N(y)$ has exactly 2 neighbors in $N(y) \setminus N(x)$ and every vertex in $N(y) \setminus N(x)$ has exactly 2 neighbors in $N(x) \setminus N(y)$, we can find 8 independent edges with one endpoint in $N(x) \setminus N(y)$ and the other endpoint in $N(y) \setminus N(x)$. Let $z \in N(x) \cap N(y)$ and let w be a neighbor of z that is not x nor y. Let $S = N(x) \cup N(y) \cup \{w\}$. It follows that S contains 9 independent edges, $|S| = 19$ and $o(\Gamma - S) \geq 3$. Thus, $o(\Gamma - S) \geq 3 = |S| - 16$ and by Lemma 5.2.3, Γ is not 9-extendable.

Assume Γ is not 8-extendable. Lemma 5.2.3 implies the existence of subset of vertices S such that S contains 8 independent edges, and $o(\Gamma - S) \geq |S| - 14$. Take such a set S of maximum size. By a similar argument as before, any non-singleton odd component of $\Gamma - S$ must have at least 7 vertices.

Assume that C is a non-singleton component with 5 vertices. If C has no odd cycles, then C is a bipartite graph and we can add two vertices of C to S. Then $|S|$ will increase by 2 and $o(\Gamma - S)$ will increase by 2, contradicting the maximality of S. If C induces a pentagon, then $o(\Gamma - S) \leq \alpha(\Gamma) - 1 = 15$ and, $|S| \leq o(\Gamma - S) + 14 \leq 29$. However, $|N(C)| = 8 \times 5 - 5 = 35$, a contradiction. Thus, any non-singleton component should have size at least 7.

If $\Gamma - S$ has at least 2 non-singleton odd components, then $o(\Gamma - S) \leq 12$, and $|S| \leq o(\Gamma - S) + 14 \leq 26$. Lemma 4.1.2 implies that $|S| \geq \frac{2\times 7\times(56-26-7)}{9} = \frac{322}{9} = 35.77 > 35$, a contradiction. If $\Gamma - S$ has exactly one non-singleton odd component, then $o(\Gamma - S) \leq 14$, and $|S| \leq o(\Gamma - S) + 14 \leq 28$. Let t be the number of singleton components. If $t \geq 7$, then Lemma 4.1.2 implies that $|S| \geq \frac{2\times 7\times(56-28-7)}{9} = \frac{98}{3} = 32.66 > 32$, a contradiction. If $3 \leq t \leq 6$, then $|S| - 14 \leq o(\Gamma - S) = 7$. Thus, $|S| \leq 21$. However, for any three independent vertices x, y and z, $|S| \geq |N(x) \cup N(y) \cup N(z)| \geq 10 \times 3 - 6 = 24$, a contradiction. If $1 \leq t \leq 2$, then

$|S| - 14 \leq o(\Gamma - S) \leq 3$. Thus, $|S| \leq 17$. Since S contains the neighborhood of a vertex, which is an independent set, in order for S to contain 8 independent edges, S must contain another 8 vertices. Thus, $|S| \geq 18$, a contradiction. If $\Gamma - S$ only has singleton components. Then $\Gamma - S$ has no even components. Otherwise, we can put one vertex of an even components into S. In this way, $|S|$ will increase by one, and $o(\Gamma - S)$ will increase at least by one, contradicting the maximality of $|S|$. Thus, $|S| = 56 - o(\Gamma - S) \geq 40$, contradicting $|S| \leq o(\Gamma - S) + 14 \leq 30$. $\square$

The $(100, 22, 0, 6)$-SRG is called Higman-Sims graph and its independence number is 22 (see [19, Section 3.5 and Section 9.1.7]).

Theorem 5.4.12. *The Higman-Sims graph is precisely* 20*-extendable.*

Proof. Let Γ denote the Higman-Sims graph. Let x and y be two non-adjacent vertices. Because every vertex in $N(x) \setminus N(y)$ has exactly 6 neighbors in $N(y) \setminus N(x)$ and every vertex in $N(y) \setminus N(x)$ has exactly 6 neighbors in $N(x) \setminus N(y)$, we can find 16 independent edges with one endpoint in $N(x) \setminus N(y)$ and the other endpoint in $N(y) \setminus N(x)$. Every vertex from $N(x) \cap N(y)$ has exactly 21 neighbors in the second subconstituent of x. Each vertex in the second subconstituent of x has 16 neighbors in $N(x) \cap N(y)$. By Hall's Marriage Theorem, we can find five independent edges $w_i u_i$ such that $u_i \in N_2(x)$ and $w_i \in N(x) \cap N(y)$ for $1 \leq i \leq 5$. Let $S = N(x) \cup N(y) \cup \{w_1, w_2, w_3.w_4, w_5\}$. It follows that S contains 21 independent edges, $|S| = 43$ and $o(\Gamma - S) \geq 3 = |S| - 21 \times 2 + 2$. Lemma 5.2.3 implies that Γ is not 21-extendable.

If Γ were not 20-extendable, by Lemma 5.2.3, there is a subset of vertices S such that S contains 20 independent edges and $o(\Gamma - S) \geq |S| - 38$. As before, we may assume that S is such a disconnecting set with maximum size. Then any non-singleton odd component of $\Gamma - S$ has at least 5 vertices. Furthermore, we can prove that any non-singleton odd component has at least 7 vertices. Assume that C is a non-singleton component with 5 vertices. If C has no odd cycle, then C is bipartite graph and we can add two vertices from the same color class of C into S. Then $|S|$ will increase by 2 and $o(\Gamma - S)$ will increase by 2, contradicting the maximality of $|S|$. If C induces a pentagon, then $|S| \geq |N(C)| = 20 \times 5 - 5 \times 5 = 75$. However, since $o(\Gamma - S) \leq \alpha(\Gamma) - 1 = 21$, we get that $|S| \leq o(\Gamma - S) + 38 \leq 59$, a contradiction. Note that any non-singleton odd component of $\Gamma - S$ contains an independent set of order 3.

If $\Gamma - S$ has at least two non-singleton odd components, then $o(\Gamma - S) \leq 18$, and $|S| \leq o(\Gamma - S) + 38 \leq 56$. However, by Lemma 4.1.2, $|S| \geq \frac{6\times7\times(100-56-7)}{25} > 62$, contradiction. If $\Gamma - S$ has exactly one non-singleton odd component, then $o(\Gamma - S) \leq 20$, and $|S| \leq o(\Gamma - S) + 38 \leq 58$. Let t be the number of singleton components. If $t \geq 7$, then Lemma 4.1.2 implies that $|S| \geq \frac{6\times7\times(100-58-7)}{25} > 58$, a contradiction. If $3 \leq t \leq 6$, then $|S| \leq o(\Gamma - S) + 38 = 45$. However, for any three independent vertices x, y and z from $\Gamma - S$, $|S| \geq |N(x) \cup N(y) \cup N(z)| \geq 22\times3 - 3\times6 = 48$, a contradiction. If $1 \leq t \leq 2$, then $|S| \leq o(\Gamma - S) + 38 \leq 41$. Since S contains the neighborhood of a vertex, which is an independent set, in order for S to contain 20 independent edges, S must contain another 20 vertices. Thus, $|S| \geq 42$, a contradiction.

If $\Gamma - S$ only has singleton odd components, then $\Gamma - S$ has no even components. Otherwise, we can put one vertex of an even components into S. In this way, $|S|$ will increase by one, and $o(\Gamma - S)$ will increase at least by one, contradicting the maximality of $|S|$. Thus, $|S| = 100 - o(\Gamma - S) \geq 78$ which contradicts with $|S| \leq o(\Gamma - S) + 38 \leq 60$. $\square$

We can also compute the extendability of the known triangle-free strongly regular graphs with odd number of vertices. The $(5, 2, 0, 1)$-SRG is precisely 0-near-extendable. The only other known triangle-free strongly regular graph of odd order is the M_{22} graph which is $(77, 16, 0, 4)$-SRG with independence number 21 (see [19, page 118]).

Theorem 5.4.13. *The M_{22} graph is precisely* 13*-near-extendable.*

Proof. Let Γ denote the M_{22} graph. Let x and y be two non-adjacent vertices. Because every vertex in $N(x) \setminus N(y)$ has exactly 4 neighbors in $N(y) \setminus N(x)$ and every vertex in $N(y) \setminus N(x)$ has exactly 4 neighbors in $N(x) \setminus N(y)$, we can find 12 independent edges with one endpoint in $N(x) \setminus N(y)$ and the other endpoint in $N(y) \setminus N(x)$. Every vertex from $N(x) \cap N(y)$ has exactly 14 neighbors in the second subconstituent of x. We can find two independent edges $w_i u_i$ such that u_i is in the second subconstituent of x and $w_i \in N(x) \cap N(y)$ for $1 \leq i \leq 2$. Let $S = N(x) \cup N(y) \cup \{w_1, w_2\}$. It follows that S contains 14 independent edges, $|S| = 30 \geq 29$ and $o(\Gamma - S) \geq 3 = |S| - 14 \times 2 + 1$. Lemma 5.2.4 implies that Γ is not 14-near-extendable.

If Γ were not 13-near-extendable, by Lemma 5.2.4, there is a subset of vertices S such that $|S| \geq 27$ and S contains 13 independent edges and $o(\Gamma - S) \geq |S| - 25$. As before, we may assume that S is such a disconnecting set with maximum size. Then any non-singleton odd component of $\Gamma - S$ has at least 5 vertices. Furthermore, we can prove that any non-singleton odd component has at least 7 vertices. Assume that C is a non-singleton component with 5 vertices. If C has no odd cycle, then C is bipartite graph and we can add two vertices from the same color class of C into S. Then $|S|$ will increase by 2 and $o(\Gamma - S)$ will increase by 2, contradicting the maximality of $|S|$. If C induces a pentagon, then $|S| \geq |N(C)| = 14 \times 5 - 3 \times 5 = 55$. However, since $o(\Gamma - S) \leq \alpha(\Gamma) - 1 \leq 20$, we get that $|S| \leq o(\Gamma - S) + 25 \leq 45$, a contradiction. Note that any non-singleton odd component of $\Gamma - S$ contains an independent set of order 3.

If $\Gamma - S$ has at least two non-singleton odd components, then $o(\Gamma - S) \leq 17$, and $|S| \leq o(\Gamma - S) + 25 \leq 42$. However, by Lemma 4.1.2, $|S| \geq \frac{7\times(77-42-7)}{4} = 49$, contradiction. If $\Gamma - S$ has exactly one non-singleton odd component, then $o(\Gamma - S) \leq 19$, and $|S| \leq o(\Gamma - S) + 25 \leq 44$. Let t be the number of singleton components. If $t \geq 7$, then Lemma 4.1.2 implies that $|S| \geq \frac{7\times(77-42-7)}{4} = 49$, a contradiction. If $3 \leq t \leq 6$, then $|S| \leq o(\Gamma - S) + 25 \leq 32$. However, for any three independent vertices x, y and z from $\Gamma - S$, $|S| \geq |N(x) \cup N(y) \cup N(z)| \geq 16\times 3 - 3\times 4 = 36$, a contradiction. If $1 \leq t \leq 2$, then $|S| \leq o(\Gamma - S) + 25 \leq 27$. Since S contains the neighborhood of a vertex, which is an independent set, in order for S to contain 13 independent edges, S must contain another 13 vertices. Thus, $|S| \geq 29$, a contradiction. If $t = 0$, then $o(\Gamma - S) = 1$. It is also impossible since by our assumption $o(\Gamma - S) \geq |S| - 25 \geq 2$.

If $\Gamma - S$ only has singleton odd components, then $\Gamma - S$ has no even components. Otherwise, we can put one vertex of an even components into S. In this way, $|S|$ will increase by one, and $o(\Gamma - S)$ will increase at least by one, contradicting the maximality of $|S|$. Thus, $|S| = 77 - o(\Gamma - S) \geq 56$ which contradicts with $|S| \leq o(\Gamma - S) + 25 \leq 46$. □

5.5 Remarks

The extendability of a strongly regular graph is not determined by its parameters. The Shrikhande graph and the line graph of $K_{4,4}$ both have parameter set $(16, 6, 2, 2)$. The extendability of the Shrikhande graph is 2 and the extendability of $L(K_{4,4})$ is 3. However, we find it remarkable that the extendability of every known primitive triangle-free strongly regular graph of valency k and even order, equals $k - 2$.

We make the following conjecture regarding the extendability of strongly regular graphs of valency k.

Conjecture 5.5.1. *If Γ is a primitive strongly regular graph of valency k, then its extendability is at least $\lceil k/2 \rceil - 1$.*

Note that this conjecture is not true for imprimitive strongly regular graph. For example, the extendability of $K_{a\times 3}$ is $a/2 = k/4$. The conjecture above would be essentially best possible since there are many strongly regular graphs of valency k that are not $\lceil k/2 \rceil$-extendable. If Γ is a (v, k, λ, μ)-SRG with $\lambda > \theta_2$, then the first subconstituent $\Gamma_1(x)$ of any vertex x is connected by eigenvalue interlacing. If Γ is not a conference graph, then $\lambda - \theta_2 \geq 1$ as θ_2 is an integer. The first subconstituent $\Gamma_1(x)$ is λ-regular with second largest eigenvalue at most θ_2. By [30], $\Gamma_1(x)$ contains a matching of size $\lfloor k/2 \rfloor$. If k is even, then this matching cannot be extended to a maximum matching of Γ. If k is odd, one can add one disjoint edge to this matching such that the result matching of size $\lceil \frac{k}{2} \rceil$ cannot be extended to a maximum matching of Γ. If Γ is a conference graph with parameters $(4t+1, 2t, t-1, t)$-SRG, $\lambda - \theta_2 > 1$ when $t \geq 4$. If $t = 2$ or $t = 3$, then the first subconstituent contains a matching of size t that cannot be extended to a maximum matching of Γ. We also remark that there are strongly regular graphs Γ such that the first subconstituent $\Gamma_1(x)$ does not contain a matching of size $\lfloor k/2 \rfloor$ for any vertex x. For example, if $\Gamma_1(x)$ is a disjoint union of cliques $K_{\lambda+1}$ and λ is even, then $\Gamma_1(x)$ will not contain a matching of size $\lfloor k/2 \rfloor$.

It would be nice to use the extendability properties of strongly regular graphs to study the edge-chromatic number of such graphs of even order. Results from [18, 30] imply that any k-regular graph with second largest eigenvalue θ_2 contains at least $(k-\theta_2)/2$ edge disjoint perfect matchings. It would be interesting to improve this bound for strongly regular graphs.

Counting perfect matchings in regular graphs is an important problem in discrete mathematics (see [51, 80]) and a well-known conjecture (see [80, Conjecture 8.18]) states that for any $k \geq 3$, there exists positive constants $c_1(k)$ and $c_2(k)$ such that any k-regular 1-extendable graph of order v contains at least $c_2(k)c_1(k)^v$ perfect matchings (also $c_1(k) \to \infty$ as $k \to \infty$). Seymour (see [48]) showed that k-regular $(k-1)$-edge-connected graphs of order v contain at least $2^{(1-1/k)(1-2/k)v/3656}$ perfect matchings. It would be nice to improve these estimates for strongly regular graphs.

Determining or obtaining bounds for the extendability properties of distance-regular graphs is also an interesting problem that we will discuss in the next chapter.

Chapter 6

MAX-CUT AND EXTENDABILITY OF MATCHINGS IN DISTANCE-REGULAR GRAPHS

A connected graph Γ with diameter D is called *distance-regular* if there are constants c_i, a_i, b_i, which are called intersection numbers, such that for all $i = 0, 1, \ldots, D$, and all vertices x and y at distance $i = d(x, y)$, among the neighbors of y, there are exactly c_i at distance $i - 1$ from x, exactly a_i at distance i, and exactly b_i at distance $i + 1$. It follows that Γ is a regular graph with valency $k = b_0$, and that $c_i + a_i + b_i = k$ for all $i = 0, 1, \ldots, D$. By these equations, the intersection numbers a_i can be expressed in terms of the others, and it is standard to put others in the *intersection array*

$$\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}.$$

Note that b_D and c_0 are not included in the array because $b_D = c_0 = 0$, whereas $c_1 = 1$ is included. Also the number of vertices can be obtained from the intersection array. Denote the set of vertices at distance i from a given vertex $z \in V$ by $N_i(z)$, for $i = 0, 1, \ldots, D$. Every vertex has a constant number of vertices k_i at given distance i, that is, $k_i = |N_i(z)|$ for all $z \in V$. Indeed, this follows by induction and counting the number of edges between $N_i(z)$ and $N_{i+1}(z)$ in two ways. In particular, it follows that $k_0 = 1$ and $k_{i+1} = b_i k_i / c_{i+1}$ for all $i = 0, 1, \ldots, D - 1$. The number of vertices now follows as $v = k_0 + k_1 + \cdots + k_D$. In combinatorial arguments such as the above, it helps to draw picture; in particular, of the so-called *distance-distribution diagram*, as depicted in Figure 6.1.

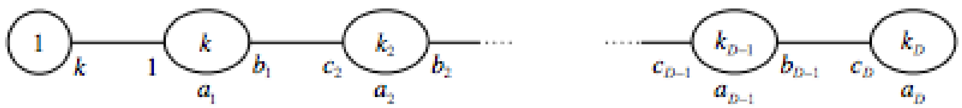

Figure 6.1: Distance-distribution diagram

Distance-regular graphs are generalizations of strongly regular graphs, where the connected strongly regular graphs are distance-regular graphs with diameter $D = 2$. For example, the Petersen graph is distance-regular graph with intersection array: $\{3, 2; 1, 1\}$.

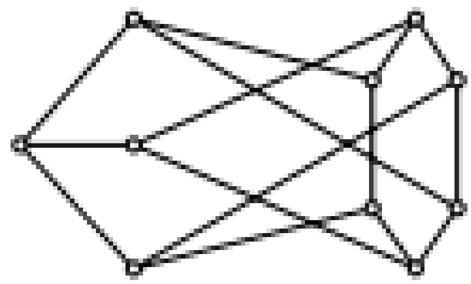

Figure 6.2: Petersen graph with distance partition

Distance-regular graphs have links to coding theory, design theory, finite group theory, representation theory, finite geometry, association schemes, and orthogonal polynomials. See Brouwer, Cohen and Neumaier [17] and a recent survey by van Dam, Koolen and Tanaka [40]. Distance-regular graphs are important also because many combinatorial problems can be tested on them before we consider general graphs. In this chapter, we consider two combinatorial problems in distance-regular graphs: the max-cut problem and the extendability of matchings problem. Most of the results of this chapter have appeared in Cioabă, Koolen and Li [34].

6.1 Introduction

Recall that If $A \subset V$ and $A^c = V \setminus A$, $e(A, A^c)$ denotes the number of edges between A and A^c. The max-cut of Γ is defined as $\text{mc}(\Gamma) := \max_{A \subset V} e(A, A^c)$ and measures how close is Γ from being a bipartite graph. Given a graph Γ, determining $\text{mc}(\Gamma)$ is a well-known NP-hard problem (see [54, Problem ND16, page 210] or

[69]) and designing efficient algorithms to approximate $\mathrm{mc}(\Gamma)$ has attracted a lot of attention [4, 42, 43, 44, 57, 86, 99].

In Section 6.2, we obtain a simple upper bound for the max-cut of certain regular graphs in terms of their odd girth (the shortest length of an odd cycle). In Section 2, we prove that if Γ is a non-bipartite distance-regular graph with odd girth g, then $\mathrm{mc}(\Gamma) \leq e(1-\frac{1}{g})$, where $e := |E|$. As a consequence of this result, we show that if Γ is a non-bipartite distance-regular graph with odd girth g and independence number $\alpha(\Gamma)$, then $\alpha(\Gamma) \leq \frac{v}{2}(1-\frac{1}{g})$. We show that these bounds are incomparable with some spectral bounds of Mohar and Poljak [86] for the max-cut and of Hoffman [19, Theorem 3.5.2] for the independence number.

In Section 6.3, we generalize results in Chapter 5 and study the extendability of distance-regular graphs with diameter $D \geq 3$. Brouwer and Haemers [18] proved that distance-regular graphs are k-edge-connected. Plesník ([89] or [79, Chapter 7]) showed that if Γ is a k-regular $(k-1)$-edge-connected graph with an even number of vertices, then the graph obtained by removing any $k-1$ edges of Γ contains a perfect matching. Thus, every distance-regular graph is 1-extendable. We improve this result and we show that all distance-regular graphs with diameter $D \geq 3$ are 2-extendable. Let λ be the number of common neighbors of two adjacent vertices and μ be the number of common neighbors of two vertices in distance 2. We also show that distance-regular graphs of valency $k \geq 3$ with $\lambda \geq 1$ are $\lfloor \frac{k+1-\frac{k}{\lambda+1}}{2} \rfloor$-extendable (when $\mu = 1$), $\lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$-extendable (when $\mu = 2$), $\lceil \frac{k-3}{3} \rceil$-extendable (when $3 \leq \mu \leq k/2$), $\lfloor k/3 \rfloor$-extendable (when $\mu > k/2$) and $\lfloor \frac{k+1}{2} \rfloor$-extendable (when the graph is bipartite).

6.2 Max-cut of Distance Regular Graphs

The following theorem is the main result of this section.

Theorem 6.2.1. *Assume that Γ is a non-bipartite graph with odd girth g. If every edge of Γ is contained in a constant number of cycles of length g, then*

$$\mathrm{mc}(\Gamma) \le e\left(1 - \frac{1}{g}\right). \tag{6.1}$$

Proof. Let γ be the number of cycles of length g containing an edge e_0. Let $\mathcal{C}$ be the set of cycles of length g. Double count $\{(e_0, C) \mid e_0 \in E(\Gamma),\ C \in \mathcal{C}$ and e_0 is an edge of $C\}$. We have $e\gamma = |\mathcal{C}|g$. Thus, $|\mathcal{C}| = \frac{e\gamma}{g}$. Let A be any subset of V and T be all the edges in the graphs with both end points in A or both end points in A^c. If we delete T, we will obtain a bipartite graph. Assume that we delete the edges in T one by one. Every time we delete an edge in T, we destroy at most γ cycles in $\mathcal{C}$. And deleting T will destroy all the cycles of length g. We must have $|T|\gamma \ge |\mathcal{C}|$. As a result, $|T| \ge \frac{e}{g}$ and $e(A, A^c) = e - |T| \le e(1 - \frac{1}{g})$. As A is arbitrary, the theorem follows. $\square$

Our theorem can be applied to the family of m-walk regular graphs with $m \ge 1$. This family of graphs contains the distance-regular graphs. A connected graph Γ is m-walk-regular if the number of walks of length l between any pair of vertices only depends on the distance between them, provided that this distance does not exceed m. The family of m-walk-regular graphs was first introduced by Dalfó, Fiol, and Garriga [39, 50].

Note that the upper bound of Theorem 6.2.1 is tight as shown for example by the blow up of an odd cycle C_g. Consider the odd cycle C_g. If we replace each vertex i of C_g by a coclique A_i ($1 \le i \le g$) of size m and add all the possible edges between A_i and A_j whenever i and j are adjacent in C_g, then we will obtain a graph with gm vertices and gm^2 edges. The odd girth of this graph is g, each edge of the graph is contained in the same number of cycles of length g and there is a cut of size $e\left(1 - \frac{1}{g}\right) = (g-1)m^2$.

Mohar and Poljak [86] obtained an upper bound for the max-cut in terms of the Laplacian eigenvalues (see also [4, 42, 43, 44]). Translated to regular graphs, their result implies the following inequality:

$$\mathrm{mc}(\Gamma) \leq \frac{e}{2}\left(1 - \frac{\lambda_v}{k}\right). \tag{6.2}$$

Note that the inequalities (6.1) and (6.2) are incomparable. This fact can be seen by considering the complete graph and the odd cycle, but we give other examples of distance-regular graphs later in this section.

The Hamming graph $H(D, q)$ is the graph whose vertices are all the words of length D over an alphabet of size q with two words being adjacent if and only their Hamming distance is 1. The graph $H(D, q)$ is distance-regular of diameter D, has eigenvalues $(q-1)D - qi$ for $0 \leq i \leq D$ and is bipartite when $q = 2$ [19, page 174]. When $q \geq 3$, inequality (6.1) always gives an upper bound $\frac{2e}{3}$. The upper bound from inequality (6.2) is $\frac{e}{2}(1 + \frac{1}{q-1})$. When $q = 3$, (6.1) is better. When $q \geq 5$, inequality (6.2) is better. When $q = 4$, both inequalities give the same upper bound.

The Johnson graph $J(n, m)$ is the graph whose vertices are the m-subsets of a set of size n with two m-subsets being adjacent if and only if they have $m-1$ elements in common. The graph $J(n, m)$ is distance-regular with diameter $D = \min(m, n-m)$, eigenvalues $(m-i)(n-m-i) - i$, where $0 \leq i \leq D$ [19, page 175]. Inequality (6.1) always gives an upper bound $\frac{2e}{3}$. Inequality (6.2) is $\frac{e}{2}(1 + \frac{D}{m(n-m)})$. When $\max(m, n-m) \geq 4$, (6.2) is better. In other cases ($m \in \{2, 3\}, n-m \in \{1, 2, 3\}$), (6.1) is better.

In the following examples, we compare (6.1) and (6.2) for other distance-regular graph with larger odd girth.

1. The Dodecahedral graph [17, page 417] is a 3-regular graph of order 20 and

size 30. It has $\lambda_v = -\sqrt{5}$ and $g = 5$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq 24$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq 26$.

2. The Coxeter graph [17, page 419] is a 3-regular graph of order 28 and size 42. It has $\lambda_v = -\sqrt{2} - 1 \approx -2.414$ and $g = 7$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq 36$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq 37$.

3. The Biggs-Smith graph [17, page 414] is a 3-regular graph of order 102 and size 153. It has $\lambda_v \approx -2.532$ and $g = 9$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq 136$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq 141$.

4. The Wells graph [17, page 421] is a 5-regular graph of order 32 and size 80. It has $\lambda_v = -3$ and $g = 5$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq 64$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq 64$.

5. The Hoffman-Singleton graph [17, page 391] is a 7-regular graph of order 50 and size 175. It has $\theta_v = -3$ and $g = 5$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq 140$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq 125$.

6. The Ivanov-Ivanov-Faradjev graph [17, page 414] is a 7-regular graph of order 990 and size 3465. It has $\theta_v = -4$ and $g = 5$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq 2772$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq 2722$.

7. The Odd graph O_{m+1} [17, page 259-260] is the graph whose vertices are the m-subsets of a set with $2m+1$ elements, where two m-subsets are adjacent if and only if they are disjoint. Note that O_3 is Petersen graph. The graph O_{m+1} is a distance-regular graph of valency $m+1$, order $v = \binom{2m+1}{m}$ and size $e = \frac{m+1}{2}\binom{2m+1}{m}$. It has $\theta_v = -m$ and $g = 2m+1$. Inequality (6.1) gives $\mathrm{mc}(\Gamma) \leq e(1 - \frac{1}{2m+1})$ and inequality (6.2) gives $\mathrm{mc}(\Gamma) \leq e(1 - \frac{1}{2m+2})$.

Theorem 6.2.1 can be used to obtain an upper bound for the independence number of certain regular graphs.

Corollary 6.2.2. *Let Γ be a non-bipartite regular graph with valency k and odd girth g. If every edge of Γ is contained in the same number of cycles of length g, then*

$$\alpha(\Gamma) \leq \frac{v}{2}\left(1 - \frac{1}{g}\right). \tag{6.3}$$

Proof. Let A be an independent set of size $\alpha(\Gamma)$. Then $k\alpha(\Gamma) = e(A, A^c) \leq \frac{vk}{2}(1 - \frac{1}{g})$ which implies the conclusion of the theorem. □

The Cvetković inertia bound (see [19, Theorem 3.5.1] or [56, Lemma 9.6.3]) states that if Γ is a graph with n vertices, n_+ positive eigenvalues of the adjacency matrix and n_- negative eigenvalues of the adjacency matrix, then

$$\alpha(\Gamma) \leq \min(n - n_-, n - n_+). \tag{6.4}$$

The Hoffman-ratio bound (Theorem 1.3.4) states that if Γ is a k-regular graph with v vertices, then

$$\alpha(\Gamma) \leq \frac{v}{1 + k/(-\theta_v)}. \tag{6.5}$$

In the table below, we compare the bounds (6.3), (6.4) and (6.5) for some of the previous examples. When the bounds obtained are not integers, we round them below. The exact values of the independence numbers below were computed using Sage.

Graph	α	(6.3)	(6.4)	(6.5)
Dodecahedral	8	8	8	11
Coxeter	12	12	13	12
Biggs-Smith	43	45	58	46
Wells	10	12	13	12
Hoffman-Singleton	15	20	21	15

For the Hamming graph $H(D, q)$ with $D = 2$ and $q \geq 3$, (6.3) is better than (6.5). For the Hamming graph $H(D, q)$ with $D \geq 3$ and $q \geq 3$, (6.5) is better. For the Odd graph O_{m+1}, the inequalities (6.3) and (6.5) give the same bound equals the independence number of O_{m+1}.

6.3 Extendability of Matchings in Distance-regular Graphs

In this section, we will focus on the extendability of distance-regular graphs. A set of edges M of a graph Γ is a matching if no two edges of M share a vertex. A matching M is perfect if every vertex is incident with exactly one edge of M. A graph Γ of even order v is called t-extendable if it contains at least one perfect matching, $t < v/2$ and any matching of size t is contained in some perfect matching. In Subsection 6.3.1, we describe the main tools which will be used in our proofs. In Subsection 6.3.2, we give various lower bounds for the extendability of distance-regular graphs. In Subsection 6.3.3, we show that all distance-regular graphs with diameter $D \geq 3$ are 2-extendable.

6.3.1 Main tools

Theorem 6.3.1 (Brouwer and Haemers [18]). *Let Γ be a distance-regular graph of valency k. Then Γ is k-edge-connected. Moreover, if $k > 2$, then the only disconnecting set of k edges are the set of k edges on a single vertex.*

Theorem 6.3.2 (Brouwer and Koolen [21]). *Let Γ be a distance-regular graph of valency k. Then Γ is k-connected. Moreover, if $k > 2$, then the only disconnecting sets of k vertices are the set of the neighbors of some vertex.*

Lemma 6.3.3. *Let Γ be a distance-regular graph with $k \geq 4$. If $A \subset V$ with $3 \leq |A| \leq k - 1$, then $e(A, A^c) \geq 3k - 6$.*

Proof. If $|A| \leq k-2$, then each vertex in A has at least $k-(|A|-1)$ neighbors in A^c and consequently $e(A, A^c) \geq |A|(k-|A|+1) \geq 3(k-2)$. Let $A \subset V$ with $|A| = k-1$. If $|N_1(x) \cap A| \leq k-3$ for any $x \in A$, then $e(A, A^c) \geq 3(k-1)$. Otherwise, let $x \in A$ such that $|N_1(x) \cap A| = k-2$. Denote $N_1(x) \cap A^c = \{y, z\}$. At least $\lambda - 1$ of the λ common neighbors of x and y are contained in A. Therefore, y has at least λ neighbors in A. A similar statement holds for z. Thus, $e(A, N_1(x) \cap A^c) \geq 2\lambda = 2(k-b_1-1)$. Also, $e(N_1(x) \cap A, N_2(x)) \geq (k-2)b_1$ so $e(A, A^c) \geq (k-2)b_1 + 2(k-b_1-1) = 3k-6+(k-4)(b_1-1) \geq 3k-6$. □

We give the following characterization of bipartite non t-extendable graphs that might be of independent interest.

Lemma 6.3.4. *Let Γ be a bipartite graph with color classes X and Y, where $|X| = |Y| = m$. The graph Γ is not t-extendable if and only if Γ has an independent set I of size at least $m-t+1$, such that $I \not\subset X$ and $I \not\subset Y$.*

Proof. Assume that Γ is not t-extendable. Lemma 5.2.3 implies that there is a vertex disconnecting set S such that the subgraph induced by S contains at least t independent edges and $o(\Gamma - S) \geq |S| - 2t + 2$. Let S be such a disconnecting set of maximum size. Our key observation is that $\Gamma - S$ does not have non-singleton odd components. Indeed, note that any non-singleton odd component of $\Gamma - S$ induces a bipartite graph with color classes A and B. Since $|A| + |B|$ is odd, we get that $|A| \neq |B|$ and assume that $|A| > |B|$. If $S' = S \cup B$, then S' is a vertex disconnecting set with $|S'| > |S|$ and $o(\Gamma - S') \geq |S'| - 2t + 2$, contradicting to the maximality of $|S|$. By a similar argument, $\Gamma - S$ contains no even components. Let $I = V(\Gamma) \setminus S$. Then I is an independent set of size at least $m-t+1$ since $|I| + |S| = 2m$ and $|I| \geq |S| - 2t + 2$. Assume that $I \subset X$. Then S induces a bipartite graph with one partite set of size at most $t-1$. This makes it impossible for the subgraph induced by S to contain t independent edges. The converse implication is immediate. $\square$

Note that the study of such independent sets in regular bipartite graphs has been done by other authors in different contexts (see [41] for example).

Lemma 6.3.5 (Brouwer and Haemers [18])**.** *Let Γ be a distance-regular graph and T be a disconnecting set of edges of Γ, and let A be the vertex set of a component of Γ minus T. Fix a vertex $a \in A$ and let t_i be the number of edges in T that join $N_{i-1}(a)$ and $N_i(a)$. Then $|A \cap N_i(a)| \geq (1 - \sum_{j=1}^{i} \frac{t_j}{c_j k_j})k_i$, so that*

$$|A| \geq v - \sum_i \frac{t_i}{c_i k_i}(k_i + \cdots + k_D).$$

Further more, when T is a disconnecting set of edges none of which is incident with a,

$$|A| > v\left(1 - \frac{|T|}{\mu k_2}\right).$$

Lemma 6.3.6. *Let Γ be a distance-regular graph with $\lambda \geq 1$. If A is an independent set of Γ, then $|N(T)| \geq 2|T|$.*

Proof. The same as Lemma 5.2.6. $\square$

Lemma 6.3.7. *Let Γ be a distance-regular graph with valency $k \geq 3$, $\lambda \geq 1$ and $\mu \leq k/2$. If A is an independent set of Γ, then $|N(A)| \geq k + |A| - 1$.*

Proof. The same as Lemma 5.2.8. $\square$

A distance-regular graph with intersection array $\{k, \mu, 1; 1, \mu, k\}$ is called a *Taylor graph.*

Lemma 6.3.8. *Let Γ be a non-bipartite distance-regular graph with $D \geq 3$. If $k < 2\mu$, then Γ is a Taylor graph.*

Proof. First we show that $D \geq 4$ implies $k \geq 2\mu$; this is known (see [17, Theorem 1.9.3]), but we include a short proof here for the sake of completeness. If $D \geq 4$, then let x and y be two vertices of Γ at distance 4. If $z \in N_2(x) \cap N_2(y)$, then z has μ neighbors in $\Gamma(x) \cap \Gamma(z)$ and μ neighbors in $\Gamma(y) \cap \Gamma(z) \subset N_3(x)$. Therefore, $k \geq 2\mu$, contradicting to the condition that $k < 2\mu$.

Thus, $D = 3$. Let $x \in V(\Gamma)$. We just need to prove that $|N_3(x)| = 1$. This implies $k_2 = c_3 = k$, and that Γ has intersection array $\{k, \mu, 1; 1, \mu, k\}$. Suppose that $|N_3(x)| > 1$. Note that $a_2 > 0$ or $a_3 > 0$. Otherwise, Γ being not bipartite and $a_2 = a_3 = 0$ imply that $a_1 > 0$. Let $z \in N_3(x)$. There is at least one triangle containing z and any such triangle is contained in $N_{\geq 2}(x)$ contradicting $a_2 = a_3 = 0$. Also, if every vertex in $N_2(x)$ is adjacent to every vertex in $N_3(x)$, then $k_2 = c_3$. Since $k_2 \geq k \geq c_3$, this implies $\mu = c_2 = k$, impossible. Hence, there exists $y \in N_2(x), z \in N_3(x)$, such that $y \not\sim z$. Without loss of generality, we may assume that y and z are at distance 2. By Lemma 5.2.5, $N_{\geq 2}(x)$ is connected. As y has μ neighbors in $\Gamma(x) \cap \Gamma(y)$ and μ neighbors in $\Gamma(y) \cap \Gamma(z) \subset N_2(x) \cup N_3(x)$, we have $k \geq 2\mu$, contradiction. □

6.3.2 Lower bounds for the extendability of distance-regular graphs

In this subsection, we give some sufficient conditions, in term of k, λ and μ, for a distance-regular graph to be t-extendable, where $t \geq 1$.

Theorem 6.3.9. *If Γ is a distance-regular graph with even order and $\lambda \geq 1$, then Γ is $\lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$-extendable.*

Proof. The graph Γ is $K_{1,\lfloor k/2 \rfloor + 1}$-free because $\lambda \geq 1$. Let $t = \lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$ and $n = \lfloor k/2 \rfloor + 1$. Then $k \geq 2t + n - 2$. The result follows from Lemma 6.3.2 and Theorem 5.3.3. □

We improve the previous result when $\mu = 1$.

Theorem 6.3.10. *If Γ be a distance-regular graph with even order, $\lambda \geq 1$ and $\mu = 1$, then Γ is $\left\lfloor \frac{k+1-\frac{k}{\lambda+1}}{2} \right\rfloor$-extendable.*

Proof. The condition $\mu = 1$ implies that $\Gamma_1(x)$ is a disjoint union of cliques on $\lambda + 1$ vertices. Hence, $\lambda + 1$ divides k and Γ is $K_{1,\frac{k}{\lambda+1}+1}$-free. Let $t = \left\lfloor \frac{k+1-\frac{k}{\lambda+1}}{2} \right\rfloor$ and $n = \frac{k}{\lambda+1} + 1$. Then $2t + n - 2 \leq k$. The conclusion follows from Lemma 6.3.2 and Theorem 5.3.3. □

The following theorem is an improvement of Theorem 6.3.9 when $3 \leq \mu \leq k/2$.

Theorem 6.3.11. *Let Γ be a distance-regular graph with even order, and $D \geq 3$. If $\lambda \geq 1$ and $3 \leq \mu \leq k/2$, then Γ is t-extendable, where $t = \left\lceil \frac{(k-3)(k-1)}{3k-6} \right\rceil$.*

Proof. If Γ is not t-extendable, by Lemma 5.2.3, there is a vertex set S with s vertices such that the subgraph induced by S contains t independent edges, and $o(\Gamma - S) \geq s - 2t + 2$. Let S be a disconnecting set with minimum cardinality and $o(\Gamma - S) \geq s - 2t + 2$. Note that such S may not contain t independent edges. Let $O_1, O_2, \ldots, O_r$ be all the odd components of $\Gamma - S$, with $r \geq s - 2t + 2$. Let $a \geq 0$ denote the number singleton components among $O_1, \ldots, O_r$.

We claim that $e(A, S) \geq 3k - 6$ for any non-singleton component A of $\Gamma - S$.

Let A be a non-singleton odd component of $\Gamma - S$ and $B = (A \cup N(A))^c$. If $|A| \leq k - 1$, the claim follows from Lemma 6.3.3. Assume that $|A| \geq k$. Let $S' := \{s \in N(A) \mid N(s) \subseteq A \cup N(A)\}$. Then $|S'| \leq 1$. Otherwise, assume that $x \neq y \in S'$. Define $S_0 = S \setminus \{x, y\}$ and $A_0 = A \cup \{x, y\}$. Then S_0 is a disconnecting set with $o(\Gamma - S') = o(\Gamma - S) \geq |S| - 2t + 2 > |S'| - 2t + 2$, contradicting the minimality of $|S|$.

If we let $A' := \{a \in A \mid d(a, b) = 2 \text{ for some } b \in B\}$, then $e(A, S) \geq \mu|A'|$. If $|A'| \geq k - 2$, we get $e(A, S) \geq \mu|A'| \geq 3(k - 2)$ and we are done. Otherwise, if

$|A'| < k-2$, then the set $A' \cup S'$ is a disconnecting set with less than $k-1$ vertices, contradicting Lemma 6.3.2. This finishes our proof of the claim.

Counting the number of edges between S and $O_1 \cup \cdots \cup O_r$, we obtain the following

$$ks \geq e(S, O_1 \cup \cdots \cup O_r) \geq ak + (r-a)(3k-6) \geq ak + (s-2t+2-a)(3k-6). \quad (6.6)$$

This inequality is equivalent to

$$t \geq \frac{(k-3)(s-a) + 3k - 6}{3k-6} \quad (6.7)$$

and since $s - a \geq k-1$ (Lemma 6.3.7), we obtain that

$$t \geq \frac{(k-3)(k-1)}{3k-6} + 1. \quad (6.8)$$

This is a contradiction with $t = \left\lceil \frac{(k-3)(k-1)}{3k-6} \right\rceil$. □

Theorem 6.3.12. *Let Γ be a non-bipartite distance-regular graph with $D \geq 3$ and $\mu > k/2$, then Γ is t-extendable, where $t = \lfloor k/3 \rfloor$ when $\lambda \geq 1$ and $t = k-1$ when $\lambda = 0$.*

Proof. Lemma 6.3.8 implies that Γ is a Taylor graph with intersection array $\{k, \mu, 1; 1, \mu, k\}$. If $\lambda = 0$, then $\mu = k-1$ and Γ is obtained by deleting a perfect matching from $K_{(k+1)\times(k+1)}$ (see [17, Corollary 1.5.4]). It is straightforward to show that Γ is $(k-1)$-extendable.

Assume that $\lambda \geq 1$. It is known that for any $x \in V(\Gamma)$, $\Gamma_1(x)$ is a strongly regular graph with parameters $\left(k, \lambda, \frac{3\lambda-k-1}{2}, \frac{\lambda}{2}\right)$ (see [17, Section 1.5]). If $\frac{3\lambda-k-1}{2} \geq 1$, then Lemma 6.3.6 implies that $\alpha(\Gamma_1(x)) \leq k/3$. If Γ is not t-extendable, then there is a vertex disconnecting set S containing t independent edges, such that $\Gamma - S$ has at least $s - 2t + 2 \geq k - 2t + 2 \geq 3$ odd components. Picking one vertex from each odd component yields an independent set I in Γ. If two vertices of this independent set were at distance 3, then the neighborhood of these two vertices will be formed by the remaining $2k$ vertices of the graph and therefore, $\Gamma - S$ would have only two odd components, contradiction. Thus, assume that any two vertices of this independent set are at distance 2 to each other. Pick a vertex x in this independent set. Any subset of $k - 2t + 1$ vertices of $I \setminus \{x\}$ will be an independent set in $\Gamma_1(y)$, where y is the antipodal vertex to x. Thus, $k - 2t + 1 \leq k/3$, contradiction with $t = \lfloor k/3 \rfloor$. If $\frac{3\lambda - k - 1}{2} = 0$, then $\Gamma_1(x)$ has parameters $(3\lambda - 1, \lambda, 0, \lambda/2)$. If $\lambda = 2$, $\Gamma_1(x)$ is C_5 which implies that $k = 5$ and $\mu = 2$, contradiction with $k/2 < \mu$. If $\lambda \geq 4$, then $\Gamma_1(x)$ must have integer eigenvalues implying that $x^2 + \frac{\lambda}{2}x - \frac{\lambda}{2} = 0$ has integer roots. However, $(\lambda/2)^2 + 2\lambda$ is not a perfect square, contradiction. $\square$

In the end of this subsection, we will show that bipartite distance-regular graphs have high extendability.

Theorem 6.3.13. *If Γ is a bipartite distance-regular graph with valency k, then Γ is t-extendable, where $t = \lfloor \frac{k+1}{2} \rfloor$.*

Proof. Let X and Y be the color classes of Γ, where $|X| = |Y| = m$. Assume that Γ is not t-extendable. By Lemma 6.3.4, Γ has an independent set I of size at least $m-t+1$, such that $I \not\subset X$ and $I \not\subset Y$. Let $A = I \cap X$, $B = I \cap Y$, $C = X \setminus A$, $D = Y \setminus B$. If $|A| = a$, then $|B| \geq m - a - t + 1$, $|C| = m - a$ and $|D| \leq a + t - 1$. As there are ak edges between A and D, and $(a + t - 1)k \geq |D|k = e(D, X) = e(A, D) + e(C, D)$, there are at most $(t - 1)k$ edges between C and D. This implies that Γ has an edge cut of size at most $(t - 1)k$, which disconnects Γ into two vertex sets $B \cup C$ and

$A \cup D$. Without loss of generality, assume that $|A \cup D| \leq m$. By the second part of Lemma 6.3.5, we have

$$|A \cup D| > v\left(1 - \frac{e(A \cup D, B \cup C)}{\mu k_2}\right) \geq 2m\left(1 - \frac{(t-1)k}{(k-1)k}\right) \geq 2m(1 - 1/2) = m,$$

contradiction with $|A \cup D| \leq m$. $\square$

6.3.3 The 2-extendability of distance-regular graphs of valency $k \geq 3$

Lou and Zhu [81] proved that any strongly regular graph of even order is 2-extendable with the exception of the complete tripartite graph $K_{2,2,2}$ and the Petersen graph. Cioabă and Li [36] showed that any strongly regular graph of even order and valency $k \geq 5$ is 3-extendable with the exception of the complete 4-partite graph $K_{2,2,2,2}$, the complement of the Petersen graph and the Shrikhande graph (see [18, page 123] for a description of this graph).

In this subsection, we prove that any distance-regular graph of diameter $D \geq 3$ is 2-extendable. By Theorem 6.3.9, any distance-regular graph with $\lambda \geq 1$ and $k \geq 5$ is 2-extendable. Note also that any distance-regular graph of even order having valency $k \leq 4$ and diameter $D \geq 3$ must have $\lambda = 0$ (see [13, 20]). Theorem 6.3.13 implies that any bipartite distance-regular graph of valency $k \geq 3$ is 2-extendable. Thus, we only need to settle the case of non-bipartite distance-regular graphs with $\lambda = 0$. We will need the following lemma.

Lemma 6.3.14. *If Γ is a non-bipartite distance-regular graph with valency $k \geq 5$ and $\lambda = 0$, then $\alpha(\Gamma) < v/2 - 1$.*

Proof. If g is the odd girth of Γ, then $v > 2g$ and Corollary 6.2.2 implies that $\alpha(\Gamma) \leq \frac{v}{2}(1 - \frac{1}{g}) < v/2 - 1$. $\square$

Theorem 6.3.15. *If Γ is a non-bipartite distance-regular graph with even order, valency $k \geq 3$ and $\lambda = 0$, then Γ is 2-extendable.*

Proof. We prove this result by contradiction and the outline of our proof is the following. We assume that Γ is not 2-extendable. Lemma 5.2.3 implies that there is a vertex disconnecting set S, such that the graph induced by S contains at least 2 independent edges and $o(\Gamma - S) \geq |S| - 2$. Without loss of generality, we may assume that S is such a disconnecting set with the maximum size. We then prove that $\Gamma - S$ does not have non-singleton components which implies that $V(\Gamma) - S$ is an independent set of size at least $v/2 - 1$, contradiction to Lemma 6.3.14.

Assume $k \geq 5$ first.

Note that any odd non-singleton component of $\Gamma - S$ is not bipartite. Otherwise, assume there is a bipartite odd component of $\Gamma - S$ with color classes X and Y such that $|X| > |Y|$. Let $S' = S \cup Y$. Then $|S'| > |S|$ and $o(\Gamma - S') \geq |S'| - 2$, contradiction with $|S|$ being maximum. Also, $\Gamma - S$ has no even components. Otherwise, we can add one vertex of one such even component to S and creating a larger disconnecting set and an extra odd component, contradicting again the maximality of $|S|$. It is easy to see that $\Gamma - S$ does not have any components with 3 vertices, because Γ is triangle free and any component with 3 vertices must be a path, hence bipartite.

Assume that A is an odd non-singleton component of $\Gamma - S$. If we can show that $e(A, S) \geq 3k - 3$, then we obtain a contradiction by counting the edges between S and S^c:

$$k|S| - 4 \geq e(S, S^c) \geq 3k - 3 + k(|S| - 3) = k|S| - 3, \tag{6.9}$$

finishing our proof.

We now prove $e(A, S) \geq 3k - 3$ whenever A is a non-singleton odd component of $\Gamma - S$.

If $5 \leq |A| \leq 2k - 3$, then as A has no triangle, Turán's theorem implies that A

contains at most $\frac{|A|^2-1}{4}$ edges. Thus, $e(A,S) \geq k|A| - 2e(A) \geq k|A| - \frac{|A|^2-1}{2} \geq 3k-4$. The last equality is attained when A induces a bipartite graph $K_{k-1,k-2}$. This is impossible as the graph induced by A is not bipartite. Hence, $e(A,S) \geq 3k-3$.

Let A be an odd component of $\Gamma - S$ such that $|A| \geq 2k-1$. If every vertex of A sends at least one edge to S, then we have two subcases: $\mu \geq 2$ and $\mu = 1$.

If $\mu \geq 2$, then we can define $S' := \{s \in N(A) \mid N(s) \subseteq A \cup N(A)\}$. If $|S'| \geq 3$, then $e(A,S)+2e(S) \geq 3k+1$. This is because $e(A,S)+2e(S) = \sum_{x\in S} |N(x) \cap (A\cup S)|$. As the graph induced by S contains at least 2 independent edges, the previous sum contains at least 4 positive terms, and at least 3 of such terms are equal to k. On the other hand, $e(A,S)+(|S|-3)k \leq e(S,S^c) = |S|k - 2e(S)$. Thus, $e(A,S)+2e(S) \leq 3k$, contradiction. If $|S'| \leq 2$, then let $B = (A \cup N(A))^c$ and $A' = \{a \in A \mid \exists b \in B$ such that $d(a,b) = 2\}$. Because $A' \cup S'$ is a disconnecting set, Lemma 3.3.2 implies that $|A' \cup S'| \geq k$ and therefore, $|A'| \geq k-2$. As each vertex in A' sends at least μ edges to S and $\mu \geq 2$, we get that $e(A,S) \geq 2k-1+(k-2)(\mu-1) \geq 3k-3$.

If $\mu = 1$, then A contains no triangles or four-cycles. If $|A| \geq 3k-3$, then $e(A,S) \geq 3k-3$, as every vertex of A sends at least one edge to S. If $|A| \leq 3k-4$, then $e(A) \leq \frac{|A|\sqrt{|A|-1}}{2}$ since A contains no triangles or four-cycles (see [55, Theorem 2.2] or [76, Theorem 4.2]). Since also $2k-1 \leq |A| \leq 3k-4$, we get that $e(A,S) = k|A| - 2e(A) \geq |A|(k - \sqrt{|A|-1}) \geq (2k-1)(k-\sqrt{3k-5}) \geq 3k-3$.

The only case remaining is when $|A| \geq 2k-1$ and A has a vertex x having no neighbors in S (such a vertex is called a deep point in [20]). Note that A^c always has a deep point because every vertex in $V(\Gamma) \setminus (A \cup S)$ is a deep point of A^c. We have two cases:

1. When $k \geq 6$, we will show that $e(A,S) \geq 3k-3$. Otherwise, by Lemma 6.3.5,

$$|A| > v\left(1 - \frac{3k-4}{\mu k_2}\right) = v\left(1 - \frac{3k-4}{k(k-1)}\right) \geq v/2. \tag{6.10}$$

The last inequality is true since $k \geq 6$. As A^c always has a deep point, by Lemma 6.3.5 again, we get that $|A^c| > v/2$, contradiction.

2. When $k = 5$, we do not have inequality (6.10) so we need a different proof. If $\mu \geq 3$, by Lemma 6.3.8, Γ must be a Taylor graph. As $\lambda = 0$, by Theorem 6.3.12, Γ is 4-extendable. So, we must have $1 \leq \mu \leq 2$.

   We first show that A is the only non-singleton component of $\Gamma - S$. Assume that there are at least two non-singleton components in $\Gamma - S$. Let B be another non-singleton component of $\Gamma - S$. Then B has a deep point, by previous arguments. If $e(A, S) \geq 2k-1$ and $e(B, S) \geq 2k-1$, then $k|S| - 4 \geq e(S, S^c) \geq 2(2k-1) + (|S|-4)k = k|S| - 2$, contradiction. Without loss of generality, assume that $e(A, S) \leq 2k-2$. By Lemma 6.3.5, $|A| > v\left(1 - \frac{2k-2}{\mu k_2}\right) = v\left(1 - \frac{2}{k}\right) = \frac{3v}{5}$. On the other hand, Lemma 6.3.5 also implies that $|A^c| > \frac{3v}{5}$, contradiction.

   Thus, A is the only non-singleton components in $\Gamma - S$. Recall that $|A| \geq 2k-1$ and A has a deep point x. If $e(A, S) \leq 3k-5$, by Lemma 6.3.5, $|A| > v\left(1 - \frac{3k-5}{\mu k_2}\right) = v\left(1 - \frac{10}{20}\right) \geq v/2$. Lemma 6.3.5 also implies that $|A^c| > v/2$, contradiction. If $e(A, S) = 3k - 4 = 11$, by counting the edges between S and S^c, we know that S contains exactly two independent edges. Also, $o(\Gamma - S) = |S| - 2$. Let T be the set of singleton components of $\Gamma - S$. We have $|T| = |S| - 3$. By Theorem 6.3.2, $|S| \geq k + 1 = 6$ and $|T| \geq 3$.

   Now, we have two subcases:

   (i) Assume that $\mu = 2$. Let $W = \{a \in A \mid \exists s \in S, a \sim s\}$. Note that $W \subset A$ and W is a disconnecting set of Γ. By Theorem 6.3.2, $|W| \geq 5$ and the only disconnecting sets of 5 vertices are the neighbors of some vertex. If $|W| = 5$, then we have $W = N(x)$ for some vertex x. By Lemma

5.2.5, the subgraph induced by the vertices at distance 2 or more from x is connected. In other words, W disconnects Γ into two components, x and $V \setminus (W \cup \{x\})$. Since $|A^c| > 1$, we must have $A^c = V \setminus (W \cup \{x\})$ and $A \setminus W = \{x\}$. Hence, $|A| = 6$, contradicting to that $|A|$ is odd. So, $|W| \geq 6$.

We claim that for any $x \in W$, there exists $t \in T$ such $d(x,t) = 2$. As $\mu = 2$, each vertex in W has at least 2 neighbors in S and $e(A, S) \geq 12$, which is also a contradiction.

Assume that the claim above is not true. Then there is $s \in S$ such that $N(s) \subset A \cup S$. Since the graph induced by S contains exactly two independent edges, s has at most one neighbor in S and at least four neighbors in A. If we let $A' = A \cup \{s\}$ and $S' = S \setminus \{s\}$, then $e(A', S') \leq 8$. By Lemma 6.3.5, $|A'| > v\left(1 - \frac{8}{\mu k_2}\right) = v\left(1 - \frac{8}{20}\right) = \frac{3v}{5}$. On the other hand, Lemma 6.3.5 also implies that $|(A')^c| > \frac{3v}{5}$, contradiction.

(ii) Assume that $\mu = 1$. We will first prove that $a_2 \leq 1$. If for every $s \in S$, $|N(s) \cap T| \leq 2$, by counting the edges between S and T, we have $5|T| = e(S,T) \leq 2|S|$. On the other hand, $|T| = |S| - 3 \geq \frac{5}{2}|T| - 3$, thus $|T| \leq 2$, contradicting to $|T| \geq 3$. Hence, there exists $s \in S$ such that $|N(s) \cap T| \geq 3$. Let $x, y, z \in N(s) \cap T$. As $\mu = 1$, $N(x) \cap N(y) = N(y) \cap N(z) = N(x) \cap N(z) = \{s\}$. Let $U = (N(x) \cup N(y) \cup N(z)) \setminus \{s\}$. It is easy to check that $U \subset N_2(s)$, $|U| = 12$, $|N_2(s)| = 20$, and $\Gamma_2(s)$ is a_2-regular. Since there are at most two edges inside U, $12a_2 - 4 \leq e(U, N_2(s) \setminus U) \leq 8a_2$ and thus $a_2 \leq 1$.

Note that $\mu = 1$ and $a_2 \leq 1$ imply that $b_2 \geq 3$. If there exists $r \in S$, such that $N(r) \subset T$, then $d(r, A) \geq 3$. By Lemma 6.3.5, $|A^c| > v\left(1 - \frac{3k-4}{k_2 b_2}\right) \geq$

$v\left(1-\frac{11}{60}\right) = \frac{49v}{60}$. On the other hand, Lemma 6.3.5 also implies that $|A| > \left(1-\frac{3k-4}{\mu k_2}\right) = v\left(1-\frac{3k-4}{k(k-1)}\right) = \frac{9v}{20}$, contradiction. Thus, for all $r \in S$, we have $N(r) \not\subset T$. Consider the edges between T and S. We have $5|T| = e(T,S) \leq 4|S|$ and therefore, $|T| \geq |S|-3 \geq 5|T|/4-3$. Thus, $|T| \leq 12$, $|S| \leq 15$, $27 \geq |A^c| > 9v/20$ and $v < 60$. Note that there is no distance-regular graph with $v < 60$, $k = 5$, $\lambda = 0$, $\mu = 1$ and $a_2 \leq 1$, see the table [13].

This finishes the proof of the case $k \geq 5$.

When $k = 4$, all the distance-regular graphs with even order are bipartite [20] so we are done by Theorem 6.3.13.

When $k = 3$, there are 3 non-bipartite triangle-free distance-regular graphs with even order (see [9] or [17, Chapter 7]): the Coxeter graph (intersection array $\{3,2,2,1;1,1,1,2\}$), the Dodecahedral graph (intersection array $\{3,2,1,1,1;1,1,1,2,3\}$) and the Biggs-Smith graph (intersection array $\{3,2,2,2,1,1,1,;1,1,1,1,1,1,3\}$). We will show that each one of them is 2-extendable.

Let Γ be the Coxeter graph. Then Γ has 28 vertices, girth 7 and independence number 12 (see [8] for example). If Γ is not 2-extendable, there is a disconnecting set S of maximum size, such that the graph induced by S contains 2 independent edges and $o(\Gamma - S) \geq |S|-2$. As $|S| \geq 4$, we have $o(\Gamma - S) \geq 2$. Assume that $\Gamma - S$ contains a non-singleton component A. As $|A^c| \geq |S|+1 \geq 5$, we have $3 \leq |A| \leq v - |A^c| \leq 23$. If $3 \leq |A| \leq 5$, the graph induced by A is bipartite as the girth of Γ is 7. As in the case $k \geq 5$, we can construct a larger disconnecting set contradicting the maximality of S. If $|A^c| = 5$, then A^c induces a bipartite graph and $e(A,A^c) = 3|A^c| - 2e(A^c) \geq 7$. If $7 \leq |A| \leq 21$ and $e(A,A^c) \geq \frac{|A|(28-|A|)}{28} \geq \frac{7\times 21}{28} > 5$ (Theorem 1.4.1). In the above two cases, we have $e(A,S) \geq 3k-3 = 6$. Similar to

(6.9), this will lead to a contradiction. Thus, $\Gamma - S$ has only singleton components. Thus, $\alpha(\Gamma) \geq o(\Gamma - S) \geq \max(28 - |S|, |S| - 2) \geq 13$, contradiction with $\alpha(\Gamma) = 12$.

Let Γ be the Dodecahedral graph. Then Γ has 20 vertices, girth 5 and independence number 8 (see [56, pp.116] for example). If Γ is not 2-extendable, there is a disconnecting set S of maximum size, such that the graph induced by S contains 2 independent edges and $o(\Gamma - S) \geq |S| - 2$. As $|S| \geq 4$, we have $o(\Gamma - S) \geq 2$. Assume that $\Gamma - S$ contains a non-singleton component A. As $|A^c| \geq |S| + 1 \geq 5$, we have $3 \leq |A| \leq 15$. We will prove that $|A| \neq 3, 5, 7, 9$ and $|A^c| \neq 5, 7, 9$. By maximality of $|S|$, the graph induced by A is not bipartite. So, $|A| \neq 3$. If $|A| = 7$, then the graph induced by A contains at most one cycle. Thus, $e(A) \leq 7$ and $e(A, A^c) = 3|A| - 2e(A) \geq 7$. If $|A| = 9$, then the graph induced by A contains at most two cycles. Thus, $e(A) \leq 10$ and $e(A, A^c) = 3|A| - 2e(A) \geq 7$. In either case, we will obtain a contradiction by inequality (6.9). Using the same argument, we can show that $|A^c| \neq 7, 9$. If $|A^c| = 5$, then A^c induces either a bipartite graph or a pentagon. If A^c induces a bipartite graph, then $e(A^c) \leq 4$ and $e(A, A^c) = 3|A^c| - 2e(A^c) \geq 7$, contradiction by inequality (6.9). If A^c induces a pentagon, then every vertex in A^c is connected to A, and $\Gamma - S$ has only one odd component A, which is because $S \subset A^c$. The last case is $|A| = 5$. Since $\Gamma - S$ has no bipartite component, A must induce a pentagon. Consider the edges between S and S^c, we have $3|S| - 4 \geq e(S, S^c) \geq 5 + 3(o(\Gamma - S) - 1)$. Thus, $|S| - 2 \geq o(\Gamma - S)$. Combining with $o(\Gamma - S) \geq |S| - 2$, we have $o(\Gamma - S) = |S| - 2$ and equality implies that S contains exactly 2 edges and $\Gamma - S$ contains exactly one non-singleton component. Since $|S| + |A| + o(\Gamma - S) - 1 = 20$, we have $|S| = 9$ and $o(\Gamma - S) = 7$. Assume that $x, y \in A$ such that x and y are not adjacent in A. Let $U = S \cup \{x\} \cup \{y\}$. Then the graph induced by U contains exactly 4 edges and the graph induced by U^c contains exactly one edge. Hence, $e(U, U^c) = 25$. However, by Theorem 6.2.1,

$\mathrm{mc}(\Gamma) \leq e\left(1 - \frac{1}{g}\right) = 24$, contradiction. Thus, $\Gamma - S$ has only singleton components. Therefore, $\alpha(\Gamma) \geq o(\Gamma - S) \geq \max(20 - |S|, |S| - 2) \geq 9$, contradiction with $\alpha(\Gamma) = 8$.

Let Γ be the Biggs-Smith graph. Then Γ has girth 9 and 102 vertices. If Γ is not 2-extendable, there is a disconnecting set S of maximum size, such that the graph induced by S contains 2 independent edges and $o(\Gamma - S) \geq |S| - 2$. Assume that $\Gamma - S$ contains a non-singleton component A. By similar argument as the previous cases, we can assume that $5 \leq |A| \leq 97$. When $5 \leq |A| \leq 7$, $e(A) = |A| - 1$ and $e(A, A^c) = 3|A| - 2e(A) = |A| + 2 \geq 7$. When $9 \leq |A| \leq 15$, $e(A) \leq |A|$ and $e(A, A^c) = 3|A| - 2e(A) \geq |A| \geq 9$. When $17 \leq |A| \leq 51$, $e(A, A^c) \geq \frac{(3-2.56155)|A|(102-|A|)}{102} \geq 6.21134 > 6$ (by Theorem 1.4.1). If $e(A, S) \geq 3k - 3 = 6$, we will obtain a contradiction by inequality (6.9). Using the same argument, we can obtain a contradiction when $5 \leq |A^c| \leq 51$. Thus, all the components of $\Gamma - S$ are singletons. Therefore, $\alpha(\Gamma) \geq o(\Gamma - S) \geq \max(102 - |S|, |S| - 2) \geq 50$, contradiction with $\alpha(\Gamma) = 43$ (see the table on page 5). $\square$

6.4 Remarks

Note that some of the bounds in this chapter can be improved if we have a good lower bound for $e(A, A^c)$ with $k \leq |A| \leq v - k$. We make the following conjecture at the end of this chapter.

Conjecture 6.4.1. *If Γ is a distance-regular graph of valency k, even order v and diameter $D \geq 3$, then the extendability of Γ is at least $\lceil k/2 \rceil - 1$.*

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