A MATHEMATICAL THEORY OF THE VERTICAL DISTRIBUTION OF TEMPERATURE AND SALINITY IN WATER UNDER THE ACTION OF RADIATION, CONDUCTION, EVAPORATION, AND MIXING DUE TO THE RESULTING CONVECTION

Derivation of a General Theory, and Illustrative Numerical Applications to a Tank, a Lake, and a Region of the North Pacific Ocean

BY

GEORGE FRANCIS McEWEN

BULLETIN OF THE SCRIPPS INSTITUTION OF OCEANOGRAPHY
LA JOLLA, CALIFORNIA
TECHNICAL SERIES, Volume 2, No. 6, pp. 197–306, 11 figures in text

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY, CALIFORNIA
1929

The SCRIPPS INSTITUTION OF OCEANOGRAPHY, LA JOLLA, CALIFORNIA, announces the beginning of a new series of its Bulletin—the **Technical Series**. The following numbers have been issued. Other numbers are in preparation.

Earlier papers from the Institution have appeared as **Bulletins of the Scripps Institution for Biological Research**, and as contributions to the series, **University of California Publications in Zoology**.

BULLETIN OF THE SCRIPPS INSTITUTION OF OCEANOGRAPHY, TECHNICAL SERIES.—Edited by T. WAYLAND VAUGHAN, F. B. SUMNER, and G. F. MCEWEN.

VOLUME 1. 1927–

1. Effect of Tidal Changes on Physical, Chemical, and Biological Conditions in the Sea Water of the San Diego Region, by Erik G. Moberg and Winfred Emory Allen. March, 1927 .... .25

2. Quantitative Studies on Inshore Marine Diatoms and Dinoflagellates of Southern California in 1921, by Winfred Emory Allen.

3. *Idem*, 1922.
Nos. 2 and 3 in one cover. March, 1927 .... .25

4. Surface Catches of Marine Diatoms and Dinoflagellates made by U.S.S. "Pioneer" in Alaskan Waters in 1923, by Winfred Emory Allen. March, 1927 .... .25

5. Studies on Marine Diatoms and Dinoflagellates Caught with the Kofoid Bucket in 1923, by Henry P. Dorman. March, 1927 .... .25

6. Notes on Cestodes and Trematodes of Marine Fishes of Southern California, by George F. Sleggs. March, 1927 .... .25

7. Quantitative Studies on Marine Diatoms and Dinoflagellates at Four Inshore Stations on the Coast of California in 1923, by Henry P. Dorman.

8. Pelagic Fish Eggs off La Jolla, California, by P. S. Barnhart.
Nos. 7 and 8 in one cover. March, 1927 .... .25

9. Marine Phytoplankton in the Region of La Jolla, California, during the Summer of 1924, by George F. Sleggs. March, 1927 .... .30

10. Recent Foraminifera from Off the West Coast of America, by Joseph A. Cushman. September, 1927 .... .90

11. Surface Catches of Marine Diatoms and Dinoflagellates off the Coast of Oregon by U.S.S. "Guide" in 1924, by Ralph Lewis. November, 1927 .... .25

12. Surface Catches of Marine Diatoms and Dinoflagellates from Pacific High Seas in 1925 and 1926, by Winfred Emory Allen and Ralph Lewis. November, 1927 .... .25

13. Catches of Marine Diatoms and Dinoflagellates Taken by Boat in Southern California Waters in 1926, by Winfred Emory Allen. January, 1928 .... .55

14. The Periodic Occurrence of Copepoda in the Marine Plankton of Two Successive Years at La Jolla, California, by C. O. Esterly. March, 1928 .... 1.25

15. Quantitative Studies on Inshore Marine Diatoms and Dinoflagellates Collected in Southern California in 1924, by Winfred Emory Allen. November, 1928 .... .25

A MATHEMATICAL THEORY OF THE VERTICAL DISTRIBUTION OF TEMPERATURE AND SALINITY IN WATER UNDER THE ACTION OF RADIATION, CONDUCTION, EVAPORATION, AND MIXING DUE TO THE RESULTING CONVECTION

DERIVATION OF A GENERAL THEORY, AND ILLUSTRATIVE NUMERICAL APPLICATIONS TO A TANK, A LAKE, AND A REGION OF THE NORTH PACIFIC OCEAN

BY

GEORGE FRANCIS McEWEN

Bulletin of the Scripps Institution of Oceanography
of the University of California
La Jolla, California
Technical Series, Volume 2, No. 6, pp. 197–306, 11 figures in text
Issued September 27, 1929

University of California Press
Berkeley, California

Cambridge University Press
London, England

CONTENTS

A MATHEMATICAL THEORY OF THE VERTICAL DISTRIBUTION OF TEMPERATURE AND SALINITY IN WATER UNDER THE ACTION OF RADIATION, CONDUCTION, EVAPORATION, AND MIXING DUE TO THE RESULTING CONVECTION

DERIVATION OF A GENERAL THEORY, AND ILLUSTRATIVE NUMERICAL APPLICATIONS TO A TANK, A LAKE, AND A REGION OF THE NORTH PACIFIC OCEAN

BY

GEORGE FRANCIS McEWEN

INTRODUCTION: QUALITATIVE STATEMENT OF THE PROBLEM

A well recognized and important result of oceanic circulation is its effect upon the "normal" distribution of temperature. The term "normal" is used in this paper to denote that distribution of temperature, salinity, or any other physical or chemical property of water which would prevail in the absence of a general drift or flow of the water as a whole, either vertical or horizontal. Investigations of regions like that off the coast of California where conditions are usually far from normal (McEwen, 1912, 1914, 1915) raise the questions: How can the normal distribution of temperature be determined from observations made on the actual one, disturbed by both horizontal and vertical drift? What is the rate of drift? At what rate does solar radiation penetrate the surface? At what rate is heat lost from the surface? Such considerations led to a general investigation of the relation of temperature to the ever-present factors, radiation, evaporation, and the resulting alternating or mixing motion of small water masses, and the effect of a given drift, horizontal or vertical, upon the normal distribution of any property of the water. In this paper are presented the derivation of a basic theory, and certain numerical applications selected to illustrate this theory. Investigations of fresh-water lakes and reservoirs have proved invaluable in attempting to deal with the more complicated phenomena of the ocean, which formed the incentive for developing this theory.

A general qualitative statement of the problem was presented at the second annual meeting of the American Geophysical Union (McEwen, 1921), also a more detailed qualitative explanation was presented at the fourth annual meeting (McEwen, 1924). The following brief general statement will serve to introduce the detailed mathematical treatment presented in this paper.

The distribution of heat, chemical properties, and substances dissolved in the water of reservoirs, lakes, and oceans depends upon external agencies, such as radiation and evaporation, and upon the internal processes of conduction and diffusion. But the phenomena of conduction and diffusion taking place in large bodies of water are of a type very different from those revealed by controlled laboratory experiments. The well-known laws of conduction and diffusion deduced from laboratory experiments cannot be carried over unaltered into the "field" where the corresponding phenomena are of a large-scale type peculiar to nature (McEwen, 1920). In order to deal mathematically with the problem, a set of assumptions underlying thermal phenomena of exposed bodies of water was formulated with the help of field observations, and corresponding mathematical formulae were derived involving the external agencies and internal processes. Briefly the assumptions are: The solar radiation that penetrates the surface decreases in geometrical proportion to the depth and its direct heating effect is limited to a region extending only a few meters below the surface. Heat is transferred from one level to another within the body of water by means of eddy motion or turbulence in a manner agreeing formally with the law of heat conduction in solids. Evaporation, back radiation, and conduction through the air constitute the cooling agency, which is confined to a very thin surface film and causes a surface loss of heat at a rate uniformly distributed over the area considered. The increase of the specific gravity of any small portion of water at the surface, due to this cooling agency, causes it to descend. The amount of increase of specific gravity required to cause the descent of any small portion of the surface film varies throughout the surface. Accordingly if the temperature of small portions of this thin surface film could be measured, large and irregular temperature variations in a horizontal direction would be indicated. The number of small water masses of a given specific gravity descending at any time is less, the greater the specific gravity. The downward velocity of each portion is assumed to vary directly as the difference between its specific gravity and the general average or observed value at that level. There is a continual disturbance of equilibrium resulting in a succession of overturning motions. Each such motion follows a sufficient increase of stress in the system to overcome its rigidity. Only those particles whose specific gravity exceeds the average value at any level will descend below that level.

The settling of particles must result in a compensating upward displacement of lighter, warmer water. Since the average or observed specific gravity increases with the distance below the surface, the descending particles of varying specific gravity will stop at different levels. They will be distributed vertically according to the combined effect of the average or observed distribution of specific gravity in a vertical direction and the frequency distribution of values of the specific gravity of the particles at the surface before descending.

The mechanism of convective or mixing circulation corresponding to these assumptions also affords a means of dealing with the distribution of other properties of the water, physical and chemical. This accords with the generally accepted idea that heat conduction, diffusion, and viscosity "so called," in large bodies of water, are really due to convection or the interchange of water particles or small portions of water having different properties (Thorade, 1923). The special theory developed in this paper, of the settling of relatively cool and heavy masses of surface water, applies to a limited part of the body of water extending from the surface down to a depth probably not exceeding about one hundred meters even in very deep water. Below this limited upper part the classical equations of motion, conduction, and diffusion may be adequate if "virtual" values of the constants found from applying the formulae to field observations are substituted for the laboratory values. These virtual values are of a much higher order of magnitude (Ekman, 1906; Taylor, 1915; McEwen, 1919, 1927; Jeffreys, 1920), than the laboratory values.

FUNDAMENTAL ASSUMPTIONS AND BASIC DEDUCTIONS FROM THEM

Precise Formulation of Fundamental Assumptions

In order to develop mathematical laws of the phenomena of the distribution of any property of water, the foregoing general ideas have been used as a basis for the following precisely formulated assumptions:

1. Heat is supplied to the water at each level by the absorption of radiant energy at the rate of (R) units per unit volume of water. That is, $(R)\times(A)\times(\Delta y)$ equals the time rate at which heat is supplied to the element where (A) is the horizontal cross-section area and the small quantity (Δy) is the thickness of the element. The radiant energy (R) depends upon the time (hour of the day or month of the year) and decreases as the depth increases. At depths exceeding a few meters it may be neglected.

2. At the surface small volume elements are cooled by evaporation, back radiation, and conduction through the air, at a rate assumed to be uniform throughout the whole surface area considered, but the actual reduction of the temperature and change of the salinity of any one element, and the corresponding increase in its specific gravity necessary to cause its descent, vary from one element to another.

That is, different elements are cooled for different lengths of time, and accordingly to different temperatures, before descending. Therefore, the greater the reduction of temperature, that is, the colder the elements, the longer will be the time required to produce the change, the less frequent will be their descent, and the greater will be the velocity of descent.

3. Each element descends to a depth where the average specific gravity (the value that is computed from the observed temperature and salinity) is slightly less than that of the descending element. That is, equilibrium is approached but may not be completely attained. Accordingly, all elements having specific gravities greater than the mean at a given level descend through the plane at that level, and therefore displace an equal volume of lighter water upward through the same plane. Therefore the amount of this upward flow is greatest at the top, and decreases as the depth increases.

4. The velocity of descent of each particle at any time is proportional to the difference between its specific gravity and the average specific gravity of the water at that level.

5. The observed temperature at any depth is the mean of the temperatures of all the elements both ascending and descending through the plane at that level. Any measured property of the water, the salinity, the acidity, etc., at a given depth, is likewise the average of the values for all the elements at that depth.

6. The usual Fourier expression $\mu^2 \frac{\partial^2 \theta}{\partial y^2}$ for the time rate of change of temperature due to heat conduction is approximately applicable to exposed bodies of water. The coefficient of turbulence μ^2 corresponds to the coefficient of heat conduction in solids, but a "virtual value" must be used to agree with "field" conditions. Some modification of this simple assumption regarding heat conduction may be necessary in certain cases, but would not alter the theory of the downward diffusion of surface loss of heat, presented in this paper.

7. Supplementary assumptions are introduced as needed in developing and applying the theory.

DEFINITION OF SYMBOLS

The following list of symbols introduced for reference in deriving fundamental equations is supplemented farther on as needed.

y = distance below the surface.

Δy = thickness of an element.

x = horizontal distance from vertical plane through the limited horizontal line L_1.

t = time.

L_1 = the length and L_2 the breadth of a rectangular volume of water extending from the surface to the bottom (fig. 1).

$u = f_1(y, t)$, the average temperature of the relatively warm and light ascending water masses.

u_o = surface value of u.

$u_o > u$.

$v = f_2(x, y, t)$, temperature of the relatively cold and heavy descending elements.

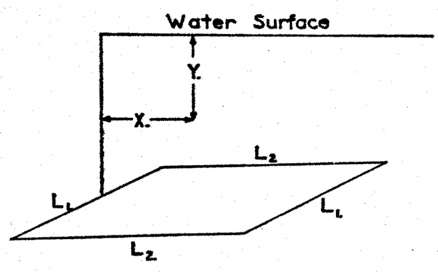

Fig. 1

θ = mean of the values of u and v at any level, equals the observed temperature at that level.

g = the cross-section area of a single element, equals $L_1(\Delta x)$.

ϕ = temperature reduction at the surface due to evaporation and back radiation, equals departure from (θ_o).

σ = specific gravity, at atmospheric pressure. It is therefore a function of temperature and salinity only.

ψ = the increase in specific gravity due to surface cooling, equals departure from (σ_o).

$F(\psi)$ = a function of (ψ) to which the frequency of this departure is proportional.

$F(R, T)$ = rate of observed temperature change due to radiation (R) and turbulence (T).

Table 1 shows limits of temperature and specific gravity at the surface and their corresponding frequencies.

Mathematical Formulation and Deduction of Basic Equations

In dealing with the system of fluid elements just described the attempt is made so to explain their invisible, or unobservable, relations as to account for and describe the observed phenomena. Of course consideration of the *precise* behavior of all the elements of such a system is impracticable if not impossible. However, in accordance with established concepts of *statistical mechanics*, definite results of value can be obtained by attempting to deduce the *gross* behavior of the system which is a consequence of the complex interaction of its elements and the external agencies.

TABLE 1

Limits of temperature and specific gravity at the surface, and their corresponding frequencies

Temperature limits	Specific gravity limits	Frequencies
$[\theta_o-(\phi_o+a_o)]$ to $[\theta_o-(\phi_o-a_o)]$	$[\sigma_o+(\psi_o-\delta_o)]$ to $[\sigma_o+(\psi_o+\delta_o)]$	$F(\psi_o)2\delta_o$
$[\theta_o-(\phi_1+a_1)]$ to $[\theta_o-(\phi_1-a_1)]$	$[\sigma_o+(\psi_1-\delta_1)]$ to $[\sigma_o+(\psi_1+\delta_1)]$	$F(\psi_1)2\delta_1$
$[\theta_o-(\phi_i+a_i)]$ to $[\theta_o-(\phi_i-a_i)]$	$[\sigma_o+(\psi_i-\delta_i)]$ to $[\sigma_o+(\psi_i+\delta_i)]$	$F(\psi_i)2\delta_i$
$[\theta_o-(\phi_n+a_n)]$ to $[\theta_o-(\phi_n-a_n)]$	$[\sigma_o+(\psi_n-\delta_n)]$ to $[\sigma_o+(\psi_n+\delta_n)]$	$F(\psi_n)2\delta_n$
$[\theta_o-(\phi_{n+r}+a_{n+r})]$ to $[\theta_o-(\phi_{n+r}-a_{n+r})]$	$[\sigma_o+(\psi_{n+r}-\delta_{n+r})]$ to $[\sigma_o+(\psi_{n+r}+\delta_{n+r})]$	$F(\psi_{n+r})\ (2\delta_{n+r})$

ϕ_o+a_o = upper limit of temperature reduction of one group of elements,

ϕ_1-a_1 = lower limit of temperature reduction of next group, etc., where $\phi_1>\phi_o$

Therefore, $\phi_o+a_o=\phi_1-a_1$, $\phi_1+a_1=\phi_2-\phi_2$ etc., $\psi_o+\delta_o=\psi_1-\delta_1$, $\psi_1+\delta_1=\psi_2-\delta_2$ etc., $\phi_1-\phi_o=a_o+a_1$, $\psi_2-\psi_1=\delta_1+\delta_2$, $\phi_3-\phi_2=a_2+a_3$ etc.,
and $\psi_1-\psi_o=\delta_o+\delta_1$, $\psi_2-\psi_1=\delta_1+\delta_2$, $\psi_3-\psi_2=\delta_2+\delta_3$ etc.

Also, $\theta_o-\phi_n$ = the bottom temperature $=\theta_n$, $\theta_o-\phi_{n+r}$ = the minimum temperature of the descending elements,

$\sigma_o+\psi_n$ = the specific gravity at the bottom $=\sigma_n$,

and $\sigma_o+\psi_{n+r}$ = the maximum specific gravity of the descending elements.

Assume that a water layer at any level $y=m\Delta y$, whose mean or observed specific gravity is σ_m will exchange $F(\psi_m)\ 2\delta_m$ elements having the specific gravity σ_m for the same number of descending elements having the specific gravity $(\sigma_o+\psi_m+\delta_m)=(\sigma_m+\delta_m)>\sigma_m$. In the development of this theory it was found convenient to give δ_m the form $\delta_m=C^2\left(\frac{\partial\sigma_m}{\partial y}\right)$ which will therefore be used from now on. Elements having a specific gravity less than $(\sigma_m-\delta_m)=\left[\sigma_m-C^2\left(\frac{\partial\sigma_m}{\partial y}\right)\right]$ do not reach the level $m\Delta y$, and those whose specific gravity exceeds $(\sigma_m+\delta_m)=\left[\sigma_m+C^2\left(\frac{\partial\sigma_m}{\partial y}\right)\right]$ sink below this level. θ_m is the mean temperature at the level $y=m\Delta y$.

The rate of descent at the level y or mth layer of elements having the specific gravity σ_i is $W_{m,\,i}=G^2(\sigma_i-\sigma_m)=f_3(x,\ y,\ t)$ where $\sigma_i>\sigma_m$.

Let $\mathbf{W}$ be the average upward velocity of the lighter elements at the mth layer or level y, and assume the heating effect due to absorbed radiation and turbulence to be confined to the rising portion of the water having the temperature u. To correct for the error made by this simplifying assumption, divide the terms involving these effects by the ratio of the cross-section occupied by rising elements to the total cross-section. Denote the ratio between these cross-sections by $\rho<1$, then

$$\frac{\partial u}{\partial t}=\mathbf{W}\frac{\partial u}{\partial y}+\frac{F(R,\,T)}{\rho} \tag{1}$$

The downward flow of the (i) group of elements at the level $y=m\Delta y$ would be

$$\left\{\left[2\,C^2\,F(\psi_i)\frac{\partial\sigma_i}{\partial y}\right]g\right\}W_{m,\,i}$$

if these elements filled the columns containing them and all were sinking with the velocity $W_{m,\,i}$. The derivative, $\frac{\partial\sigma_i}{\partial y}=$ gradient of specific gravity at the level where $\sigma=\sigma_i$.

But in each group of the columns having the cross-section area

$$\left\{\left[2\,C^2\,F(\psi_i)\frac{\partial\sigma_i}{\partial y}\right]g\right\}$$

there will be at each level an element of specific gravity (σ_i) next to one or more elements having a lower specific gravity corresponding to the temperature (u), followed by another element having the specific gravity σ_i, etc. As will be shown, the ratio of the downward-moving elements to the total number in a column is $\rho_i=\frac{\Delta y}{(W_{m,\,i})\Delta t_i}$, where Δt_i is the time required to reduce the temperature by the amount ϕ_i. Since this is also the ratio of downward-moving elements to the total number of elements in a cross-section of these columns, the downward flow of the (i) group is

$$\left[2\,C^2\,F(\psi_i)\frac{\partial\sigma_i}{\partial y}\right]\frac{g(W_{m,\,i})\Delta y}{(W_{m,\,i})\Delta t}$$

If (K), the rate per unit area at which heat is removed from the surface by evaporation, back radiation, and conduction through the air, is regarded as constant over the area (L_1L_2), it follows that

$$\frac{\phi_i\Delta y}{\Delta t_i}=K \text{ or } \frac{\Delta y}{\Delta t_i}=\frac{K}{\varphi_i} \tag{2}$$

Therefore the downward flow of the (i) group is

$$\frac{2\,gC^2K}{\phi_i}\,F(\psi_i)\left(\frac{\partial\sigma_i}{\partial y}\right)$$

which does not contain the velocity term $W_{m,\,i}$.

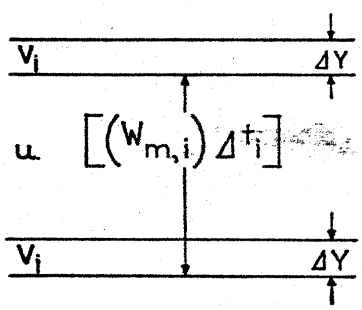

Fig. 2

In any arbitrary time interval (t_1), not too large because the velocity $W_{m,\,i}$ is regarded as constant for that time, $\frac{t_1}{\Delta t_i}$ groups of elements will be cooled by the amount ϕ_i and start to descend. The distance through which the first group descends is therefore $(W_{m,\,i})t_1$. At a time Δt_i later, the second group, cooled by the same amount, starts to descend, at a time $2\Delta t_i$ the third group starts, etc. Therefore during this time interval t_1, the second group moves through the distance $(W_{m,\,i})\ (t_1-\Delta t_i)$, the third through the distance $(W_{m,\,i})\ (t_1-2\Delta t_i)$, etc. Therefore the vertical distance between any two successive groups is $(W_{m,\,i})\ \Delta t_i$. In a column of height $(W_{m,\,i})\ \Delta t_i$ (fig. 2), the heavy descending elements of temperature v_i, and specific gravity σ_i, fill the space of height Δy. The remaining space $(W_{m,\,i}\ \Delta t_i-\Delta y)$ is filled by lighter rising elements of temperature (u). A column is assumed to have a cross-section area (g) equal to the average area of a descending element having any temperature departure φ_i. Therefore the ratio of the downward-moving elements to the total number in any thin horizontal section of the column is

$$\frac{\Delta y}{(W_{m,\,i})\Delta t_i}=\rho_i$$

which was used on page 205. As Δt_i increases, the specific gravity increases, and consequently the velocity $W_{m,\,i}$ and the distance $(W_{m,\,i})\Delta t_i$ increases.

Figure 3 has been drawn to aid in visualizing the distribution and relative numbers of the descending and ascending elements. The relative density of the descending elements is greater in proportion to the shading. The distance between the vertical lines $= \Delta x$. The distance (L_1) of figure 1 is measured in a direction perpendicular to the plane of

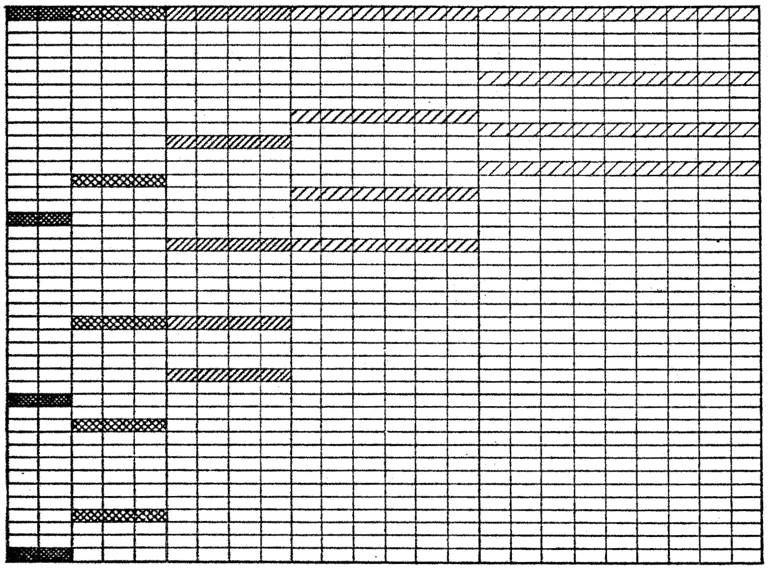

Fig. 3.—General qualitative illustration of the distribution of ascending and descending water. Cold and heavy water masses indicated by the intensity of shading, ascending masses, unshaded.

the paper in figure 3, and the cross-section area of an element is g or $L_1\Delta x$. All elements cooled by the same amount at the surface are collected together in corresponding (i) groups, and in figure 3 each such group is represented as descending as a whole. However, no such regularity is believed to exist actually. There may be a lag between the descent of different elements of each (i) group, and the columns may be interchanged in any manner.

The subscript $(_i)$ signifies a group of descending elements whose temperature is reduced by the amount ϕ_i, and whose specific gravity is increased by the amount ψ_i. The subscript $(_m)$ signifies an intermediate level $m\Delta y$ or y, such that the difference between the mean or observed temperature at that level and the surface temperature is ϕ_m, and the difference between the mean or observed specific gravity at that level and the surface specific gravity is ψ_m.

Since $\sigma_i=\sigma_o+\psi_i$ the expression for downward flow of the (i) group at the level (y) becomes

$$\frac{2gC^2K}{\phi_i}F(\psi_i)\frac{\partial\psi_i}{\partial y}$$

The upward flow in through the same section of the plane is

$$\left[2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}\right]g(\mathbf{W}_{m,\,i})\frac{W_{m,\,i}\,\Delta t_i-\Delta y}{W_{m,\,i}\,\Delta t_i}$$

Continuity of mass can be satisfied by equating the upward and downward flow in each group of $\left[2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}\right]$ columns thus

$$\frac{2gC^2K}{\phi_i}F(\psi_i)\frac{\partial\psi_i}{\partial y}=2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}g(\mathbf{W}_{m,\,i})\frac{(W_{m,\,i})\Delta t_i-\Delta y}{W_{m,\,i}\,\Delta t_i}$$

This reduces to

$$\mathbf{W}_{m,\,i}=\left[\frac{(W_{m,\,i})\,\Delta t_i}{(W_{m,\,i})\,\Delta t_i-\Delta y}\right]\frac{K}{\phi_i}=\frac{\Delta y}{(W_{m,\,i})\Delta t_i-\Delta y}W_{m,\,i}$$

In this equation $(W_{m,\,i})$ equals the downward velocity of the (v_i) elements through a plane at the depth (y), and $(\mathbf{W}_{m,\,i})$ equals the upward velocity of the (u) elements of the same column through the same plane. In a column of height $W_{m,\,i}\Delta t_i$, the number of (u) elements is $\frac{(W_{m,\,i})\,\Delta t_i-\Delta y}{\Delta y}$ and the number of (v_i) elements is $\frac{\Delta y}{\Delta y}=1$.

The number of such columns,

$$\left[2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}\right]$$

equals the number of (v_i) elements in the same portion of all the (i) columns. The number of (u) elements in all of the columns of height $(W_{m,\,i})\Delta t_i$ corresponding to a given value of (i) is

$$2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}\left(\frac{W_{m,\,i}\,\Delta t_i-\Delta y}{\Delta y}\right)$$

In order to concentrate attention on the problem of weighted averages, suppose the frequency to be the same for each column. That is, for the moment suppose

$$\left[2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}\right]$$

to be independent of (i) or the amount of temperature reduction, and denote this constant frequency by (N).

Then $\frac{W_{m,\,i}\Delta t_i-\Delta y}{\Delta y}N$ equals the number of (u) elements in the (i) group at the depth (y).

$\frac{\Delta y}{\Delta y}N$ equals the number of (v_i) elements in the (i) group at the depth (y).

Consider a layer of small thickness (λ), then at the surface where $y=0$, the weighted average of the (u_o) and (v_i) temperatures is

$$\frac{N\sum\limits_{i=0}^{i=n+r}\frac{[(W_{o,\,i})\Delta t_i-\Delta y]\lambda}{\Delta y(W_{o,\,i})\Delta t_i}u_o+N\sum\limits_{i=0}^{i=n+r}\frac{\Delta y\lambda}{(W_{o,\,i})(\Delta t_i)\Delta y}v_i}{N\sum\limits_{i=0}^{i=n+r}\frac{\lambda(W_{o,\,i})\,(\Delta t_i)}{(W_{o,\,i})\,(\Delta t_i)\Delta y}}=\theta_o \qquad (3)$$

where (N) equals the number of columns for a given value of (i), and is for the moment assumed to be independent of the value of (i). Accordingly since the total number of parts of a column between horizontal planes, separated by the distance $W_{o,\,i}\,\Delta t_i$ within a depth interval λ is $\frac{\lambda}{W_{o,\,i}\Delta t_i}$ there would be $(n+r)$ N columns of $\left(\frac{\lambda}{\Delta y}\right)$ elements in each part at all depths. At the surface there are $(n+r)$ values of (v_i), and (N) columns for each of these. But at the depth $y=m\Delta y$ there are only $[(n+r)-m]$ different values of (v_i). In the remaining (mN) columns all of the elements have the temperature (u), which is independent of the position in a given horizontal plane. Therefore at the depth (y) the mean or observed temperature would have the value

$$\frac{N\sum\limits_{i=m}^{i=n+r}\frac{W_{m,\,i}\Delta t_i-\Delta y}{\Delta y}\,\frac{\lambda}{W_{m,\,i}\Delta t_i}u+N\sum\limits_{i=m}^{i=n+r}\frac{\lambda v_i}{W_{m,\,i}\Delta t_i}+N\sum\limits_{i=0}^{i=m}\frac{\lambda}{\Delta y}u}{N\sum\limits_{i=0}^{i=n+r}\frac{\lambda}{\Delta y}}=\theta_m \qquad (4)$$

If (u) and (v_i) are omitted from the numerator of equation (4) the result equals the denominator, as it should.

Since the frequency $2C^2F(\psi_i)\frac{\partial\psi_i}{\partial y}$ does not have the constant value (N) but varies with respect to (i) we must write $\theta_m=(\theta-\phi_m)=$

$$\frac{\sum\limits_{i=m}^{i=n+r}\frac{W_{m,\,i}\,\Delta t_i-\Delta y}{\Delta y(W_{m,\,i})\Delta t_i}F(\psi_i)\frac{\partial\psi_i}{\partial y}u_m+\sum\limits_{i=m}^{i=n+r}\frac{F(\psi_i)\frac{\partial\psi_i}{\partial y}}{W_{m,\,i}\Delta t_i}v_i+\sum\limits_{i=0}^{i=m}\frac{F(\psi_i)}{\Delta y}\,\frac{\partial\psi_i}{\partial y}u_m}{\sum\limits_{i=0}^{i=n+r}\frac{F(\psi_i)}{\Delta y}\,\frac{\partial\psi_i}{\partial y}} \qquad (5)$$

Equation (5) can readily be transformed into the following, making use of equation (2).

$$\theta_m=\frac{\begin{array}{l}\sum\limits_{i=m}^{i=n+r}u_m\frac{(W_{m,\,i})\Delta t_i-\Delta y}{\Delta y(W_{m,\,i})\Delta t_i}F(\sigma_i-\sigma_o)\,\frac{\partial\sigma_i}{\partial y}\\ +\sum\limits_{i=m}^{i=n+r}\frac{v_iF(\sigma_i-\sigma_o)}{(W_{m,\,i})\Delta t_i}\,\frac{\partial\sigma_i}{\partial y}+\sum\limits_{i=0}^{i=m}u_m\frac{F(\sigma_i-\sigma_o)}{\Delta y}\,\frac{\partial\sigma_i}{\partial y}\end{array}}{\sum\limits_{i=0}^{i=n+r}\frac{F(\sigma_i-\sigma_o)}{\Delta y}\,\frac{\partial\sigma_i}{\partial y}} \qquad (6)$$

$$\theta_m=\frac{\sum_{i=m}^{i=n+r}\left[u_m\frac{(W_{m,\,i})\Delta t_i-\Delta y}{(W_{m,\,i})\Delta t_i}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}+\frac{v_iF(\sigma_i-\sigma_o)\Delta y}{(W_{m,\,i})\Delta t_i}\frac{\partial\sigma_i}{\partial y}\right]+\sum_{i=0}^{i=m}u_mF(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}}{\sum_{i=0}^{i=n+r}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}} \tag{7}$$

$$\theta_m=\frac{\sum_{i=m}^{i=n+r}\left(\left[1-\frac{K}{(W_{m,\,i})\,(\theta_o-v_i)}\right]u_m+\left[\frac{K}{(W_{m,\,i})\,(\theta_o-v_i)}\right]v_i\right)F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}+\sum_{i=0}^{i=m}u_m\,F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}}{\sum_{i=0}^{i=n+r}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}} \tag{8}$$

Summing the expression on page 208 the downward flow is

$$\sum_{i=m}^{i=n+r}\frac{2gC^2K}{\phi_i}F(\psi_i)\frac{\partial\psi_i}{\partial y}=2gC^2\sum_{i=m}^{i=n+r}F(\sigma_i-\sigma_o)\frac{\Delta y}{\Delta t}\frac{\partial\sigma_i}{\partial y}$$

The upward flow is

$$\mathbf{W}_m 2C^2g\sum_{i=m}^{i=n+r}\frac{[(W_{m,\,i})\,(\Delta t_i)-\Delta y]}{(W_{m,\,i})\Delta t_i}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}+\sum_{i=0}^{i=m}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y} \tag{9}$$

Therefore, after equating the upward and downward flow we obtain

$$\mathbf{W}_m=\frac{\sum_{i=m}^{i=n+r}F(\sigma_i-\sigma_o)\frac{\Delta y}{\Delta t_i}\frac{\partial\sigma_i}{\partial y}}{\sum_{i=m}^{i=n+r}\left[1-\frac{\Delta y}{(W_{m,\,i})\Delta t_i}\right]F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}+\sum_{i=0}^{i=m}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}} \tag{10}$$

or

$$\mathbf{W}_m=\frac{K\sum_{i=m}^{i=n+r}\frac{F(\sigma_i-\sigma_o)}{(\theta_o-v_i)}\frac{\partial\sigma_i}{\partial y}}{\sum_{i=m}^{i=n+r}\left[1-\frac{K}{(W_{m,\,i})\,(\theta_o-v_i)}\right]F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}+\sum_{i=0}^{i=m}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}} \tag{11}$$

If in the numerator of equation (8) the coefficients of (u_m) are retained but (u_m) itself is omitted and the (v_i) term is omitted, the result must be the ratio referred to on page 205, since (ρ) equals the ratio of the number of ascending elements at a given level to the total number of elements. The result is

$$\rho=\frac{\sum\limits_{i=m}^{i=n+r}\left[1-\frac{K}{(W_{m,\,i})\,(\theta_o-v_i)}\right]F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}+\sum\limits_{i=0}^{i=m}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}}{\sum\limits_{i=0}^{i=n+r}F(\sigma_i-\sigma_o)\frac{\partial\sigma_i}{\partial y}} \tag{12}$$

Multiplying numerator and denominator of equations (8), (10), and (12) by (dy) and replacing the sums by integrals results in the following three basic equations

$$\theta_m=\frac{u_m\int\limits_{\sigma_m}^{\sigma_{n+r}}\left[1-\frac{K}{W(\theta_o-v)}\right]F(\sigma-\sigma_o)d\sigma+K\int\limits_{\sigma_m}^{\sigma_{n+r}}\frac{v}{(W(\theta_o-v)}F(\sigma-\sigma_o)d\sigma+u_m\int\limits_{\sigma_o}^{\sigma_m}F(\sigma-\sigma_o)d\sigma}{\int\limits_{\sigma_o}^{\sigma_{n+r}}F(\sigma-\sigma_o)d\sigma} \tag{13}$$

$$\mathbf{W}_m=\frac{K\int\limits_{\sigma_m}^{\sigma_{n+r}}\frac{F(\sigma-\sigma_o)}{(\theta_o-v)}d\sigma}{\int\limits_{\sigma_m}^{\sigma_{n+r}}\left[1-\frac{K}{W(\theta_o-v)}\right]F(\sigma-\sigma_o)d\sigma+\int\limits_{\sigma_o}^{\sigma_m}F(\sigma-\sigma_o)d\sigma} \tag{14}$$

$$\rho_m=\frac{\int\limits_{\sigma_m}^{\sigma_{n+r}}\left[1-\frac{K}{W(\theta_o-v)}\right]F(\sigma-\sigma_o)d\sigma+\int\limits_{\sigma_o}^{\sigma_m}F(\sigma-\sigma_o)d\sigma}{\int\limits_{\sigma_o}^{\sigma_{n+r}}F(\sigma-\sigma_o)d\sigma} \tag{15}$$

where (W) is the velocity of descent through the layer at depth (y) of relatively cold and heavy water masses of temperature (v), and specific gravity (σ).

SIMPLIFIED EXPRESSION OF BASIC EQUATIONS IN TERMS OF CERTAIN INTEGRALS

Transformation of Basic Equations

For greater convenience, and clearness, the following notation is introduced:

$$z=h(\sigma-\sigma_o),\ F(\sigma-\sigma_o)=F_1[h(\sigma-\sigma_o)]=F_1(z),\ B=\frac{\theta_o-\theta}{\sigma-\sigma_o}.$$

In these expressions (θ), (σ), and (z) correspond to the depth $y=m\Delta y$, (θ'), (σ'), and $(z')=(x)$ will be used for other depths, and are regarded as variables when integrating, therefore $W=G^2(\sigma_i-\sigma_m)=G^2(\sigma'-\sigma)$.

There will be a depth such that the observed temperature equals the reduced temperature (v) of any surface element. In fresh water the temperature reduction corresponding to the increment $(\sigma-\sigma_o)$ of specific gravity of a surface element equals the difference between the surface temperature and the temperature at the depth where the specific gravity equals (σ). Therefore $\frac{1}{B}=\frac{\sigma-\sigma_o}{\theta_o-\theta}$ can be computed directly from the observed temperatures. But in sea water, after computing the densities (σ) and the differences $(\sigma-\sigma_o)$ from the given temperatures and salinities, the ratio $\frac{\sigma-\sigma_o}{\theta_o-\theta}$ cannot be used unchanged for $\left(\frac{1}{B}\right)$ as in the case of fresh water. This is because the lowering of the temperature of an element at the surface below θ_o is accompanied by a corresponding increase in salinity as a result of evaporation. These corresponding changes in temperature and salinity will not in general agree with the changes observed in a vertical direction at the depth of the given density difference. As shown on page 221 the relation between the corresponding differences in temperature and salinity at the surface is

$$(S-S_o)=\lambda\frac{S_o}{L}(\theta_o-v) \tag{77}$$

where (λ) is somewhat less than 1. Assuming the values 1, 33.75, and 600 for (λ), (S_o), and (L), respectively, the relation is

$$(S-S_o)=.056(\theta_o-v).$$

An explanation of methods of computing $\left(\frac{1}{B}\right)$ for sea water accompanies table 10, pages 301 to 305.

Introducing the value of

$$W=G^2(\sigma_i-\sigma_m)=G^2(\sigma'-\sigma)=G^2\left(\frac{z'}{h}-\frac{z}{h}\right)=\frac{G^2}{h}(z'-z)=\frac{G^2}{h}(x-z),$$

and substituting for (v) the observed temperature (θ') at the corresponding depth, the equations for (θ), $(\mathbf{W})$, and (ρ) become, respectively

$$\theta=\frac{u\displaystyle\int_z^{z_1}\left[1-\frac{K}{\frac{G^2}{h}(x-z)(\theta_o-\theta')}\right]\frac{F_1(x)dx}{h}+\frac{K}{h}\int_z^{z_1}\frac{\theta' F(x)dx}{\frac{G^2}{h}(x-z)(\theta_o-\theta')}+\frac{u}{h}\int_{z_0}^{z}F_1(x)dx}{\displaystyle\frac{1}{h}\int_{z_0}^{z_1}F_1(x)dx} \tag{16}$$

$$\mathbf{W}=\frac{\dfrac{K}{h}\displaystyle\int_{z}^{z_1}\frac{F(x)}{(\theta_o-\theta')}dx}{\dfrac{1}{h}\displaystyle\int_{z}^{z_1}\left[1-\frac{K}{\frac{G^2}{h}(x-z)(\theta_o-\theta')}\right]F_1(x)dx+\frac{1}{h}\int_{z_o}^{z}F_1(x)dx}\tag{17}$$

$$\rho=\frac{\dfrac{1}{h}\displaystyle\int_{z}^{z_1}\left[1-\frac{K}{\frac{G^2}{h}(x-z)\ (\theta_o-\theta')}\right]F_1(x)dx+\frac{1}{h}\int_{z_o}^{z}F_1(x)dx}{\dfrac{1}{h}\displaystyle\int_{z_o}^{z_1}F_1(x)dx}\tag{18}$$

Since

$$(\theta_o-\theta')=B(\sigma-\sigma_o)=\frac{B}{h}(x-z_o)=\frac{B}{h}x\tag{19}$$

$$\theta=\frac{u\displaystyle\int_{z_o}^{z_1}F_1(x)dx+\frac{Kh^2}{G^2}\int^{z_1}\frac{(\theta'-u)F_1(x)dx}{(x-z)Bx}}{\displaystyle\int_{z_o}^{z_1}F_1(x)dx}\tag{20}$$

$$\mathbf{W}=\frac{hK\displaystyle\int_{z}^{z_1}\frac{F_1(x)}{Bx}dx}{\displaystyle\int_{z_o}^{z_1}F(x)dx-\frac{Kh^2}{G^2}\int_{z}^{z_1}\frac{F_1(x)dx}{(x-z)Bx}}\tag{21}$$

$$\rho=\frac{\displaystyle\int_{z_o}^{z_1}F_1(x)dx-\frac{Kh^2}{G^2}\int_{z}^{z_1}\frac{F_1(x)dx}{(x-z)Bx}}{\displaystyle\int_{z_o}^{z_1}F_1(x)dx}\tag{22}$$

and

$$\theta=u-\frac{Kh^2}{G^2}\,\frac{\displaystyle\int_{z}^{z_1}\frac{(u-\theta')F_1(x)dx}{(x-z)Bx}}{\displaystyle\int_{z_o}^{z_1}F_1(x)dx}\tag{23}$$

From equation (1) page 205

$$\frac{\partial u}{\partial t}=\left\{\frac{hK\int_{z}^{z_1}\frac{F_1(x)}{Bx}dx}{\int_{z_o}^{z_1}F_1(x)dx-\frac{Kh^2}{G^2}\int_{z}^{z_1}\frac{F_1(x)dx}{(x-z)Bx}}\right\}\frac{\partial u}{\partial y}+\frac{F(R,T)}{\rho} \tag{24}$$

$$\frac{1}{\rho}=\frac{1}{1-\frac{\frac{Kh^2}{G^2}\int_{z}^{z_1}\frac{F_1(x)dx}{(x-z)Bx}}{\int_{z_o}^{z_1}F_1(x)dx}} \tag{25}$$

Since

$$\theta'=\theta_o-\frac{Bx}{h} \tag{26}$$

$$\frac{u-\theta'}{Bx}=\frac{u-\theta_o+\frac{Bx}{h}}{Bx}=\frac{u-\theta_o}{Bx}+\frac{1}{h} \tag{27}$$

Therefore

$$\theta=u-\frac{Kh^2}{G^2}\left[\frac{1}{\int_{z_o}^{z_1}F_1(x)dx}\right]\int_{z}^{z_1}\frac{F_1(x)}{(x-z)}\left[\frac{u-\theta_o}{Bx}+\frac{1}{h}\right]dx \tag{28}$$

and

$$\theta=u-\frac{Kh^2}{G^2}\left\{\frac{(u-\theta_o)\int_{z}^{z_1}\frac{F_1(x)dx}{(x-z)Bx}}{\int_{z_o}^{z_1}F_1(x)dx}+\frac{1}{h}\,\frac{\int_{z}^{z_1}\frac{F_1(x)}{x-z}dx}{\int_{z_o}^{z_1}F_1(x)dx}\right\} \tag{29}$$

INTERPRETATION OF CERTAIN INTEGRALS AND MODIFIED NOTATION

Assume that the least value of (z) capable of causing a downward displacement of a small water mass from the surface has the value (H). Then the expression for sums should begin with this value H instead of $z_o=0$. Accordingly, the lower limits of the integrals should be increased by this amount, and denoting any of the integrals by $\int_{z}^{z_1} f(x)dx$, the following modified values should be used

$$f(x-H)H+\int_{(z+H)}^{z}f(x)dx \tag{30}$$

or the closer approximation

$$f(x+H)H+f(x+2H)H+\int_{(z+2H)}^{z} f(x)dx \tag{31}$$

For brevity the following notation is introduced:

$$\int_{z}^{z_1}\frac{F_1(x)}{Bx}dx=\left\{H\frac{F_1(z+H)}{B(z+H)}+\int_{(z+H)}^{z_1}\frac{F_1(x)}{Bx}dx\right\}=P_1(z) \tag{32}$$

$$\int_{z_o}^{z_1}F_1(x)dx=HF_1(H)+\int_{H}^{z_1}F_1(x)dx=A_1 \tag{33}$$

$$\int_{z}^{z_1}\frac{F_1(x)dx}{Bx(x-z)}=\frac{F_1(z+H)}{B(z+H)}+\int_{(z+H)}^{z_1}\frac{F_1(x)dx}{Bx(x-z)}=P_2(z) \tag{34}$$

$$\int_{z}^{z_1}\frac{F_1(x)}{(x-z)}dx=F_1(z+H)+\int_{(z+H)}^{z_1}\frac{F_1(x)}{(x-z)}dx=P_3(z) \tag{35}$$

$$\int_{z}^{z_1}\frac{F_1(x)}{x}dx=H\frac{F_1(z+H)}{z+H}+\int_{(z+H)}^{z_1}\frac{F_1(x)}{x}dx=P_4(z) \tag{36}$$

$$\left\{\begin{array}{ll} A_2=hK, & \dfrac{A_2}{A_1}=A_4, \\ A_3=\dfrac{h^2K,}{G^2} & \dfrac{A_3}{A_1}=A_5 \end{array}\right\} \tag{37}$$

Expression of Basic Equations in the New Notation

Making use of the new notation the temperature equations take the following form, more convenient in numerical applications.

$$\theta=u-A_3\left\{(u-\theta_o)\frac{P_2(z)}{A_1}+\frac{P_3(z)}{hA_1}\right\} \tag{38}$$

$$\theta=u-A_5\left\{(u-\theta_o)P_2(z)+\frac{P_3(z)}{h}\right\} \tag{39}$$

$$\theta=u[1-A_5P_2(z)]-\frac{A_5}{h}P_3(z)+A_5\theta_oP_2(z) \tag{40}$$

$$u=\frac{\theta-A_5\theta_oP_2(z)+\dfrac{A_5}{h}P_3(z)}{1-A_5P_2(z)} \tag{41}$$

$$\frac{\partial u}{\partial t}=A_4\frac{P_1(z)}{1-A_5P_2(z)}\,\frac{\partial u}{\partial y}+\frac{F(R,T)}{\rho} \tag{42}$$

$$\frac{1}{\rho}=\frac{1}{1-A_5P_2(z)},\ A_5P_2(z)=(1-\rho) \tag{43}$$

Another form for the relation of (u) to (θ) will now be derived. Denote the ratio

$$\frac{(\theta-\theta')}{(x-z)} \text{ by } \frac{B'}{h}$$

Then

$$\frac{u-\theta'}{x-z}=\frac{B_1}{h}+\frac{u-\theta}{x-z}$$

and equation (23) for θ becomes

$$\theta=u-A_5\left\{\frac{1}{h}\int_z^{z_1}\frac{B_1F_1(x)}{Bx}dx+(u-\theta)\int_z^{z_1}\frac{F_1(x)}{(x-z)Bx}dx\right\} \tag{44}$$

$$\theta=u-\frac{A_5}{h}\int_z^{z_1}B'\frac{F_1(x)}{x}dx-A_5(u-\theta)\int_z^{z_1}\frac{F_1(x)}{(x-z)Bx}dx \tag{45}$$

where $B'=\frac{B_1}{B}$

Substituting a mean value $(\mathbf{B}')$ for the approximately constant quantity (B'), and using the new notation for the integrals, equation (44) reduces to the approximate forms

$$\theta=u-\frac{A_5\mathbf{B}'}{h}P_4(z)-A_5(u-\theta)P_2(z) \tag{46}$$

and

$$(u-\theta)=\frac{\frac{A_5\mathbf{B}'}{h}P_4(z)}{1-A_5P(z)} \tag{47}$$

APPLICATION OF THE THEORY OF TEMPERATURE DISTRIBUTION TO THE DISTRIBUTION OF OTHER PROPERTIES OF THE WATER

General Equations Derived for the Distribution of Any Property

Consider the distribution of any property, of which the observed value is (S), and the value corresponding to the ascending elements is $(\mathbf{S})$. The quantity S' is regarded as a variable corresponding to (x), the variable value of (z) at the depth in question. The number of elements descending from the surface and departing from the surface value by the amount $(S'-S_o)$ is assumed to be proportional to $F_1(x)$. This implies that $(S'-S_o)$ is the same for all elements for which $(\sigma'-\sigma_o)$ or x is the same. Thus in equation (23), replacing (θ) by (S), and (u) by $(\mathbf{S})$ results in the equation

$$S=\mathbf{S}-\frac{Kh^2}{G^2}\,\frac{\int_z^{z_1}\frac{(\mathbf{S}-S')F_1(x)}{(x-z)Bx}dx}{\int_{z_0}^{z_1}F_1(x)dx} \tag{48}$$

Similarly equation (24) becomes

$$\frac{\partial \mathbf{S}}{\partial t}=\left\{\frac{hK\int_{z}^{z_1}\frac{F_1(x)}{Bx}dx}{\int_{z_o}^{z_1}F_1(x)dx-\frac{Kh^2}{G^2}\int_{z}^{z_1}\frac{F_1(x)dx}{(x-z)Bx}}\right\}\frac{\partial \mathbf{S}}{\partial y}+\frac{F(T)}{\rho} \tag{49}$$

where $F(T)$ is the turbulence function of the same form as that used for temperatures.

Equations (48) and (49) hold for any property (S), if there is no agency directly modifying it, for example, as the absorption of radiation changes the temperature, and if the frequency or number of descending elements departing by any given amount from the mean or observed value at the surface is the same as the number having a corresponding variation in specific gravity. This is true of salinity, for example.

Special Equations Derived for the Distribution of Salinity

A special equation, more fundamental than the general equation (48), can be derived for the distribution of salinity from the effect of evaporation on the concentration of salts in the sea. For this purpose the following notation is used:

S_o = the initial surface salinity,

$(S_o+\Delta S)$ = the surface salinity at a time Δt later,

E = the rate of evaporation in the same units as were used in the temperature equations,

$S_o\Delta y$ = amount of salt in a volume element of unit area and thickness Δy.

Then

$$\frac{S_o\Delta y}{\Delta y-E\Delta t}=S_o+\Delta S_o, \tag{50}$$

$$\frac{S_o\Delta y-S_o(\Delta y-E\Delta t)}{(\Delta y-E\Delta t)\Delta t}=\frac{\Delta S_o}{\Delta t}=\frac{E}{\Delta y-E\Delta t}S_o$$

$$\frac{\Delta S_o}{\Delta t}=\left(\frac{S_o\Delta y}{\Delta y-E\Delta t}-S_o\right)\frac{1}{\Delta t} \tag{51}$$

$$E=\frac{K'}{L}=\frac{\lambda\phi\Delta y}{L\Delta t},\ \frac{\Delta t}{\Delta y}=\frac{\lambda\phi}{LE} \tag{52}$$

where (L) equals the latent heat of vaporization, and (K') equals (λK), the rate at which heat is removed from the surface by evaporation. Also from equation (50) we may write

$$S_o+\Delta S_o=\frac{S_o}{1-E\frac{\Delta t}{\Delta y}}=\frac{S_o}{1-\frac{\lambda\phi}{L}}=\frac{S_o}{1-\frac{\lambda(\theta_o-v)}{L}} \tag{53}$$

Since (L) is a large number, about 600, a close approximation is

$$S_o+\Delta S_o=\left[1+\frac{\lambda(\theta_o-v)}{L}\right]S_o=\left(1+\frac{\lambda Bx}{hL}\right)S_o \tag{54}$$

therefore

$$\frac{\Delta S_o}{\Delta t}=\left(\frac{\lambda Bx}{hL\Delta t}\right)S_o \tag{55}$$

But the $(S_o+\Delta S_o)$ is the value of (S') in equation (48) which therefore reduces to

$$S=\mathbf{S}-\frac{Kh^2}{G^2}\,\frac{\displaystyle\int_z^{z_1}\frac{\left[\mathbf{S}-S_o\left(1+\frac{\lambda Bx}{Lh}\right)\right]F_1(x)dx}{(x-z)Bx}}{\displaystyle\int_{z_o}^{z_1}F_1(x)dx}. \tag{56}$$

EXPRESSION IN NEW NOTATION OF EQUATIONS DERIVED FOR PROPERTIES OTHER THAN TEMPERATURE

For convenience these equations are expressed in the modified notation explained on pages 214 and 216. Equations (48), (49), and (56) are respectively (57), (58), and (59) in the new notation.

$$S=\mathbf{S}-A_5\mathbf{S}P_2(z)+A_5\int_z^{\infty}S'\frac{F_1(x)}{Bx(x-z)}dx \tag{57}$$

$$\frac{\partial\mathbf{S}}{\partial t}=A_4\left[\frac{P_1(z)}{\rho}\right]\frac{\partial\mathbf{S}}{\partial y}+\frac{F(T)}{\rho} \tag{58}$$

$$S=\mathbf{S}-A_5\left\{(\mathbf{S}-S_o)P_2(z)-\frac{\lambda S_o}{Lh}P_3(z)\right\} \tag{59}$$

Because (S') of the general equation (57) varies with (x), the integral expression was kept in the last term, but numerical applications would be facilitated by using a series of values of (z) with corresponding values of (S') and the function $P_2(z)$. The special equation (59) derived for salinity is free from this complication.

USEFUL TRANSFORMATIONS AND COMBINATIONS OF THE FOUR BASIC EQUATIONS; COMPUTATION OF RATES OF UPWELLING AND RATES OF EVAPORATION FROM THE SEA

Derivation of a Single Temperature Equation from the Pair of Equations (41) and (42) by Eliminating (u)

The expression

$$u=\frac{\theta}{\rho}-\left(\frac{1-\rho}{\rho}\right)\theta_o+\frac{A_5}{h}\,\frac{P_3(z)}{\rho}\tag{60}$$

is easily derived from equations (41) and (43), and for convenience will be used here instead of its equivalent (41). To facilitate the derivation of the fundamental equations the vertical current or "upwelling" was neglected. Now for the sake of generality this effect is included, accordingly with the aid of equation (43), equation (42) becomes

$$\frac{\partial u}{\partial t}=A_4\frac{P_1(z)}{\rho}\,\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial y}+\frac{F(R,T)}{\rho}\tag{61}$$

where (W) equals the upwelling velocity. At the surface, equation (60) reduces to

$$(u_o-\theta_o)=\frac{A_5}{h}\,\frac{P_3(z_o)}{\rho_o}\tag{62}$$

The partial differentiation of equation (60) with respect to (y) and (t) gives

$$\frac{\partial u}{\partial t}=\frac{1}{\rho}\left\{\frac{\partial}{\partial t}(\theta-\theta_o)-\left[\frac{\theta-\theta_o}{\rho}+\frac{A_5P_3(z)}{h\rho}\right]\frac{\partial\rho}{\partial t}+\frac{\rho\partial\theta_o}{\partial t}+\frac{A_5}{h}\,\frac{\partial P_3(z)}{\partial t}\right\}\tag{63}$$

and

$$\frac{\partial u}{\partial y}=\frac{1}{\rho}\left\{\frac{\partial\theta}{\partial y}-\left[\frac{\theta-\theta_o}{\rho}+\frac{A_5P_3(z)}{h\rho}\right]\frac{\partial\rho}{\partial y}+\frac{A_5\partial P_3(z)}{h\partial y}\right\}\tag{64}$$

Substituting the above values of $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial y}$ in equation (61) gives

$$\begin{aligned}&\frac{\partial\theta}{\partial t}-\left\{A_4P_1(z)\frac{\partial\theta}{\partial y}+W\frac{\partial\theta}{\partial y}+F(R,T)\right\}\\&=A_5\left\{P_2(z)\left[\frac{\partial\theta}{\partial t}-W\frac{\partial\theta}{\partial y}-F(R,T)+\frac{\partial\theta_o}{\partial t}\right]+(\theta_o-\theta)\left[\frac{\partial P_2(z)}{\partial t}\right.\right.\\&\left.-\frac{1}{h}\,\frac{\partial P_3(z)}{\partial t}\right]-\left[\frac{A_4P_1(z)}{1-A_5P_2(z)}+W\right]\frac{\partial P_2(z)}{\partial y}+\left.\frac{1}{h}\,\frac{\partial P_3(z)}{\partial y}\right\}+A_5^2\left\{-P_2(z)^2\frac{\partial\theta_o}{\partial t}\right.\\&-\frac{P_3(z)}{h}\,\frac{\partial P_2(z)}{\partial t}+\frac{P_2(z)}{h}\,\frac{\partial P_3(z)}{\partial t}+\left[\frac{A_4P_3(z)}{h[1-A_5P_2(z)]}\,\frac{\partial P_2(z)}{\partial y}\right]\\&\left.\times\left[P_1(z)+\frac{W}{A_4}[1-A_5P_2(z)]\right]\right\}.\end{aligned}\tag{65}$$

In dealing with mean annual values or other cases in which the derivative with respect to time can be neglected, equation (65) reduces to

$$-\left\{A_4P_1(z)\frac{\partial\theta}{\partial y}+W\frac{\partial\theta}{\partial y}+F(R,T)\right\}=A_5\left\{P_2(z)\left[-W\frac{\partial\theta}{\partial y}-F(R,T)\right]\right.$$
$$-\left[\frac{A_4P_1(z)}{1-A_5P_2(z)}+W\right]\frac{\partial P_2(z)}{\partial y}+\frac{1}{h}\frac{\partial P_3(z)}{\partial y}\Bigg\} \tag{66}$$
$$+A_5^2\left\{\left[\frac{A_4P_3(z)}{h[1-A_5P_2(z)]}\frac{\partial P_2(z)}{\partial y}\right]\left[P_1(z)+\frac{W}{A_4}[1-A_5P_2(z)]\right]\right\}$$

DERIVATION OF A SINGLE SALINITY EQUATION FROM THE PAIR OF EQUATIONS (58) AND (59) BY ELIMINATING ($\mathbf{S}$)

Solving equation (59) for ($\mathbf{S}$) and using equation (43) we obtain

$$\mathbf{S}=\frac{S}{\rho}+\left(1-\frac{1}{\rho}\right)S_o-\left(\frac{A_5}{h}\right)\frac{\lambda S_o}{L}\frac{P_3(z)}{\rho} \tag{67}$$

which becomes

$$(S_o-\mathbf{S}_o)=\frac{A_5}{h}\frac{\lambda S_o}{L}\frac{P_3(z_o)}{\rho_o} \tag{68}$$

for the surface.

Differentiating equation (67) partially with respect to (t) and (y) we get

$$\frac{\partial\mathbf{S}}{\partial t}=\frac{1}{\rho}\left\{\frac{\partial}{\partial t}(S-S_o)-\left[\frac{S-S_o}{\rho}-\frac{A_5}{h}\frac{\lambda S_o}{L}\frac{P_3(z)}{\rho}\right]\frac{\partial\rho}{\partial t}+\frac{\rho\partial S_o}{\partial t}-\frac{A_5}{h}\frac{\lambda S_o}{L}\frac{\partial P_3(z)}{\partial t}\right\} \tag{69}$$

and

$$\frac{\partial\mathbf{S}}{\partial y}=\frac{1}{\rho}\left\{\frac{\partial S}{\partial y}-\left[\frac{S-S_o}{\rho}-\frac{A_5}{h}\frac{\lambda S_o}{L}\frac{P_3(z)}{\rho}\right]\frac{\partial\rho}{\partial y}-\frac{A_5}{h}\frac{\lambda S_o}{L}\frac{\partial P_3(z)}{\partial y}\right\} \tag{70}$$

Introducing the upwelling velocity in equation (58) gives

$$\frac{\partial\mathbf{S}}{\partial t}=A_4\frac{P_1(z)}{\rho}\frac{\partial\mathbf{S}}{\partial y}+W\frac{\partial\mathbf{S}}{\partial y}+\frac{F(T)}{\rho} \tag{71}$$

Substituting the values of $\frac{\partial\mathbf{S}}{\partial t}$ and $\frac{\partial\mathbf{S}}{\partial y}$ in equation (71) we obtain

$$\frac{\partial S}{\partial t}-\left\{A_4P_1(z)\frac{\partial S}{\partial y}+W\frac{\partial S}{\partial y}+F(T)\right\} \tag{72}$$
$$=A_5\left\{P_2(z)\left[\frac{\partial S}{\partial t}-W\frac{W\partial S}{\partial y}-F(T)+\frac{\partial S_o}{\partial t}\right]\right.$$
$$+(S_o-S)\left[\frac{\partial P_2(z)}{\partial t}+\frac{\lambda S_o}{Lh}\frac{\partial P_3(z)}{\partial t}\right]$$
$$-\left[\frac{A_4P_1(z)}{1-A_5P_2(z)}+W\right]\frac{\partial P_2(z)}{\partial y}-\frac{\lambda S_o}{Lh}\frac{\partial P_3(z)}{\partial y}\Bigg\}$$
$$+A_5^2\left\{-P_2(z)\frac{^2\partial S_o}{\partial t}+\frac{\lambda S_o}{Lh}P_3(z)\frac{\partial P_2(z)}{\partial t}-\frac{\lambda S_o}{Lh}P_2(z)\frac{\partial P_3(z)}{\partial t}\right.$$
$$-\left[P_1(z)+\frac{W}{A_4}\left(1-A_5P_2(z)\right)\right]$$

$$\times\left(\frac{A_4P_3(z)\lambda S_o}{hL[1-A_5P_2(z)]}\right)\frac{\partial P_2(z)}{\partial y}\Bigg\} \tag{72}$$

In dealing with mean annual values or other cases in which the derivative with respect to time can be neglected, equation (72) reduces to

$$\begin{aligned}
&-\left\{A_4P_1(z)\frac{\partial S}{\partial y}+W\frac{\partial S}{\partial y}+F(T)\right\}\\
&=A_5\left\{P_2(z)\left[-W\frac{\partial S}{\partial y}-F(T)\right]-\frac{\lambda S_o}{Lh}\frac{\partial P_3(z)}{\partial y}\right\}\\
&+A_5^2\left\{-\left[\frac{A_4P_3(z)\lambda S_o}{hL[1-A_5P_2(z)]}\frac{\partial P_2(z)}{\partial y}\right]\left[P_1(z)+\frac{W}{A_4}\left(1-A_5P_2(z)\right)\right]\right\}
\end{aligned} \tag{73}$$

Methods of Computing the Rate of Evaporation from the Sea

Experience has indicated (page 224) that, except for a small portion of the upper layer within which the downward diffusion of the heavy surface elements is significant, the second member of equation (65) involving the factor A_5 can be neglected. Accordingly having computed (K) by fitting this simplified equation to the data (pages 224-229) an estimate of the coefficient (λ) will serve to determine the rate of evaporation (E) from the equation

$$E=\frac{K'}{L}=\frac{\lambda K}{L} \tag{52}$$

derived on page 217.

One method of estimating (λ) would be to apply the exact equation (65) to determine the different physical constants and functions. Then after substituting in the salinity equation (72) solve for (λ). In spite of the work required for such a lengthy procedure the computations can be made and the possibility of estimating (λ) and consequently ocean evaporation solely from serial observations of sea temperature and salinity is of theoretical interest.

However, the value of (λ) can be approximately estimated from observations on an evaporating pan containing sea water. Let $(\Delta\Theta_o)$ be the average temperature reduction for ocean surface elements in a layer of small thickness (Δy) in the time (Δt). Let $(\Delta\mathbf{S}_o)$ be the average accompanying increase in salinity of the same elements. The observed surface temperature would then be given by

$$\theta_o=\rho_o u_o+(1-\rho_o)\ (\theta_o-\Delta\Theta_o) \tag{74}$$

and the observed surface salinity would be given by

$$S_o=\rho_o\mathbf{S}_o+(1-\rho_o)\ (S_o+\Delta\mathbf{S}_o) \tag{75}$$

Solving equations (74) and (75) for the increments and dividing the salinity increment by the temperature increment gives

$$\frac{\Delta\mathbf{S}_o}{\Delta\Theta_o}=\frac{S_o-\mathbf{S}_o}{u_o-\theta_o}=-\frac{S_o-\mathbf{S}_o}{\theta_o-u_o} \tag{76}$$

Also dividing equation (68) by equation (62) gives

$$\frac{S_o - \mathbf{S}_o}{u_o - \theta_o} = \frac{\lambda S_o}{L} \tag{77}$$

Therefore

$$\lambda = \frac{L}{S_o}\left(\frac{\Delta \mathbf{S}_o}{\Delta \Theta_o}\right) \tag{78}$$

But $\Delta \mathbf{S}_o$ and $\Delta \Theta_o$ are simply the surface changes of sea water exposed to the meteorological conditions prevailing over the sea surface and having the temperature of the sea surface. Accordingly the ratio of these increments obtained from pan observations should approximate to the values corresponding to the sea surface if the pan temperature equals the sea surface temperature and the change of pan temperature due to the heating effect of solar radiation is subtracted from the observed change in pan temperature to obtain $(\Delta \Theta_o)$. The relatively large diurnal variation of pan temperatures could be eliminated by using a multiple of twenty-four hours for the time interval. Neglecting the effect of differences in back radiation from the two surfaces and convection through the air, differences in rate of evaporation will not effect the ratio $\left(\frac{\Delta \mathbf{S}_o}{\Delta \Theta_o}\right)$ since the numerator and denominator will be changed in the same proportion. In general (Richardson and Montgomery, 1929) evaporation is the main cooling factor. Accordingly, if the change in pan temperature due to the heating effect of solar radiation is eliminated, and the resultant temperature change is substituted for $\Delta \Theta_o$, and the salinity change is substituted for $\Delta \mathbf{S}_o$, the value of (λ) can be estimated approximately by substitution in equation (78). The rate of available solar radiation given by the simplified equation (84) affords a means of making this correction. Also, the correction would not be required for night observations.

If observations of wet- and dry-bulb temperatures in the air are available, the value of (λ) can be found without pan observations as follows. Denote by (R') the ratio of heat loss by conduction through the air to that lost by evaporation. Denoting by (B) the back radiation from the sea surface, we have

$$E = \frac{(K-B) - R'\lambda K}{L} = \frac{K(1-\lambda R') - B}{L} \tag{79}$$

and

$$E = \frac{K-B}{L(1+R')} \tag{80}$$

since (K) is the total rate of heat loss from the surface due to evaporation, back radiation, and convection (see page 206). Equating these two values of (E) and solving for λ results in the following equation

$$\lambda = \frac{K-B}{K(1+R')} = \frac{1-\frac{B}{K}}{1+R'} \tag{81}$$

But R' is, by definition, the "Bowen ratio" which can readily be computed from the surface water temperature, the wet-bulb air temperature, and the dry-bulb air temperature by Bowen's theoretical formula based on thermodynamics and the kinetic theory of gases (Bowen, 1926).

His formula is—

$$R' = .46\left(\frac{\theta_w - \theta_a}{P_w - P_a}\right)\left(\frac{P}{760}\right) \tag{82}$$

where

θ_w = surface water temperature
θ_a = air temperature
P_w = partial pressure of water vapor at the temperature θ_w
P_a = partial pressure of water vapor at the temperature θ_a

The back radiation can be estimated from Stefan's formula, (Cummings and Richardson, 1927) assuming the water to radiate as a black body. According to Richardson and Montgomery (1929) this result should be multiplied by a factor approximating .90 which gives

$$B = .90 \times 49.5 \times 10^{-10} \theta_o^4 \tag{83}$$

gram cal. per sq. cm. per hour where (θ_o) is the absolute surface temperature. The value of (K) can be estimated by means of the simplified approximate formula as explained on pages 224 and 229. Moreover, since the only observations that this method requires are serial temperatures and salinities, and wet- and dry-bulb temperatures of the air, only the usual oceanographic apparatus is necessary. Actual field tests should be made to determine what accuracy can be obtained by this method.

GENERAL EXPLANATION OF METHODS OF COMPUTING THE PHYSICAL CONSTANTS OF THE EQUATIONS FROM NUMERICAL DATA

Method of Applying the Approximate Simplified Equation

Besides deriving basic equations and reducing them to forms more convenient for use, various auxiliary mathematical problems must be solved in order to carry on further theoretical studies and make numerical applications. The simplified approximate form of the temperature equation remaining, after neglecting the second member of (65), has been exclusively used in preliminary investigations, involves the most essential factors in the theory, and apparently must be employed as the first step in the application of the exact equations of temperature and other properties of the water. The second member of equation (65) is multiplied by the factor (A_5) and may be regarded as a correction whose magnitude varies with the difference between the temperature (u) of the rising elements and the observed temperature (θ), or mean of the temperature, of the rising and sinking elements. This correction is evidently a maximum at the surface and decreases to zero at the depth where the downward diffusion of cold surface elements becomes zero. Since (A_5) varies inversely as the velocity of descent for a given difference between the specific gravity of the descending elements at any level and the average specific gravity at that level, the higher this velocity, the smaller will be (A_5) and, accordingly, the smaller will be the correction. Apparently in practical applications we are justified in neglecting this correction entirely except for depths small relative to the depth within which this particular phenomenon is significant. Accordingly consider first the special approximate equation

$$\frac{\partial \theta}{\partial t}=\frac{A_2}{A_1}P_1(z)\frac{\partial \theta}{\partial y}+W_1 f(y)\frac{\partial \theta}{\partial y}+F(R, T) \tag{84}$$

where the upwelling velocity (W) equals the constant (W_1) multiplied by the depth function $f(y)$.

Numerical applications require a suitable specific form of the solar radiation-turbulence function $F(R, T)$. This expression may be regarded as the sum of two others, the first is the rate at which a thin layer absorbs heat directly from the penetrating solar radiation; the second is the rate at which heat flows into and out of this layer because of turbulence. In depths exceeding a few meters, the direct solar radiation effect can be neglected and in general for an appropriate value of the constant (C) the difference ($\theta - C$) has been found to approximate closely to a simple exponential function of the depth (y) for depths exceeding five meters. Therefore, the usual expression $\left(\mu^2\frac{\partial^2 \theta}{\partial y^2}\right)$ for the time rate of temperature change due to heat flow becomes $\mu^2 C_1 a^2 e^{-ay}$ where

$$(\theta - C) = C_1 e^{-ay}, \tag{85}$$

(C_1) and (a) are positive constants determined empirically, and (μ^2) is the coefficient of turbulence, which corresponds to the coefficient of heat conduction in solids or undisturbed fluids.

The usual exponential law of absorption of radiation having a definite wave length is assumed to hold approximately for the sun's radiant energy. Accordingly the rate of temperature change at depth (y) due to direct absorption of solar radiation may be expressed by the equation

$$R = R_o a\frac{e^{-ay}}{1-e^{-ay_1}} \tag{86}$$

Where (R_o) is the rate at which solar radiation penetrates the surface, less the rate at which it is absorbed by the bottom, (a) is the absorption coefficient of total solar radiation, and (y_1) is the depth of the bottom. This form for (R) is convenient since the total rate of absorption of solar radiation within the layer from surface to bottom is given by the integral,

$$\int_0^{y_1} R\,dy = R_o \int_0^{y_1} \frac{a e^{-ay} dy}{1-e^{-ay_1}} = R_o \tag{87}$$

The rate at which solar radiation penetrates the surface is evidently

$$R_o + \frac{R_o e^{-ay_1}}{1-e^{-ay_1}} = \frac{R_o}{1-e^{-ay_1}}$$

Except for shallow bodies of water where y is less than about five meters the denominator is practically equal to unity and R_o is the rate at which solar radiation penetrates the surface. Also, the intensity of the radiation at any depth is $R_o e^{-ay}$. Thus in general $F(R, T)$ is the sum of two exponential functions of the depth y. Consider now the case in which only one of these—the expression for turbulence—is significant, then

$$\frac{\partial\theta}{\partial t} = \frac{A_2}{A_1} P_1(z) \frac{\partial\theta}{\partial y} + W_1 f(y) \frac{\partial\theta}{\partial y} + (\mu^2 C_1 a^2) e^{-ay} \tag{88}$$

where, according to Ekman's theory of upwelling in the sea (McEwen, 1918, page 402) we may use

$$f(y) = 1 - e^{-\alpha y} \cos \alpha y \tag{89}$$

as a reasonable approximation. The constant (α) depends upon the velocity of the wind producing the current. Although it appears to be impracticable to obtain a general solution of the differential equation (88), the derivatives can be evaluated from suitable observations by well-known graphical or numerical processes, and in the following explanation it is assumed that such computations have been made.

The procedure for numerical application to a body of water of moderate depth and having no upwelling will now be explained. Suppose for example, that the depth is 23 meters, that of Lake Mendota. First determine the constants in equation (85) as follows:

$$\begin{aligned}
\theta_{20} - \theta_{19} &= C_1[e^{-20a} - e^{-19a}] = \Delta\theta_{19} = C_1\, e^{-19a}(e^{-a} - 1) \\
\theta_{19} - \theta_{18} &= C_1[e^{-19a} - e^{-18a}] = \Delta\theta_{18} = C_1\, e^{-18a}(e^{-a} - 1) \\
\theta_{18} - \theta_{17} &= C_1[e^{-18a} - e^{-17a}] = \Delta\theta_{17} = C_1\, e^{-17a}(e^{-a} - 1) \\
\theta_{17} - \theta_{16} &= C_1[e^{-17a} - e^{-16a}] = \Delta\theta_{16} = C_1\, e^{-16a}(e^{-a} - 1)
\end{aligned}$$

$$\begin{aligned}
C + C_1\, e^{-19a} &= \theta_{19} \\
C + C_1\, e^{-18a} &= \theta_{18} \\
C + C_1\, e^{-17a} &= \theta_{17} \\
C + C_1\, e^{-16a} &= \theta_{16}
\end{aligned}$$

Therefore

$$\theta_i = C + \frac{\Delta\theta_i}{e^{-a}-1}$$

and values of θ_i plotted as ordinates on squared paper against values of $\Delta\theta_i$ as abscissae should fall on a straight line whose intercept on the vertical axis is the value of (C). From the slope relation we get

$$e^{-a} = 1 + \frac{1}{\text{slope}},$$

but a more accurate method is to plot $(\theta - C)$ as ordinates against (y) on semilogarithmic paper, and determine (a) from the slope of this line. Also C_1 is the intercept of this line on the vertical axis. In the interval from the bottom or near the bottom, to say 12 meters within which the term $\frac{A_2}{A_1}P_1(z)\frac{\partial\theta}{\partial y}$ is small and may be neglected in a first approximation, we may write

$$\sum_{12}^{21}\frac{\partial\theta}{\partial t} = \mu^2 C_1 a^2 \sum_{12}^{21} e^{-ay} \tag{90}$$

thus determining $(\mu^2 C_1 a^2)$ and μ^2. Through the point $y=0$ and the value of $(\mu^2 C_1 a^2)$ draw a straight line on the same logarithmic paper parallel to the graph of $(\theta - C)$. The agreement of this line with the points found by plotting $\frac{\partial\theta}{\partial t}$ against (y) for $12 < y < 21$ indicates the accuracy with which equation (88) fits the data in that interval. In case the graph of $(\theta - C)$ changes slope at a value of (y) between o and (12), another straight line should be fitted to this upper portion using, if necessary, a different value of (C). This simply means that the turbulence coefficient varies with the depth, and the part of the second graph drawn through the point $[o, \mu^2 C_1 a^2]$ should be replaced by another having this new slope and cutting the original line at the point corresponding to the value of (y) at which a marked change of slope occurs. Thus we can estimate approximately the value of $F(R, T)$ at all depths except at and near the surface where a more complicated function might be required owing to the combined action of radiation, turbulence, and surface disturbances. Next compute $\left[\frac{\partial\theta}{\partial t} - F(R, T)\right]$ which is the value of $\frac{A_2}{A_1}P_1(z)\frac{\partial\theta}{\partial y} = hKP_1(z)\frac{\partial\theta}{\partial y}\left(\frac{1}{A_1}\right)$ due to the downward diffusion of surface heat losses. Experience has shown that the frequency function $F_1(x)$ may be assumed to equal e^{-x^2}, which is that of the normal law of error (see page 265). Therefore dividing the above expression by $\frac{\partial\theta}{\partial y}$, results in the equation

$$hK\frac{\displaystyle\int_{z+H}^{\infty}\frac{e^{-x^2}}{Bx}dx+\frac{He^{-(z+H)^2}}{B(z+H)}}{\displaystyle\int_{H}^{\infty}e^{-x^2}dx+He^{-H^2}}=\frac{\dfrac{\partial\theta}{\partial t}-F(R,T)}{\dfrac{\partial\theta}{\partial y}} \tag{91}$$

For values of (z) corresponding to depths exceeding 1 or 2 meters the small quantity (H) may be neglected and the first member becomes

$$K\frac{h\displaystyle\int_{z}^{\infty}\frac{e^{-x^2}}{Bx}dx}{.885}$$

Since (B) is approximately constant and the value of this expression can be computed from the second member of equation (91), find by trial with the aid of table 2, a value of (h) that results in a series of values approximately proportional to the computed values, then (K) can be found by division of separate values or of sums. The rate of incoming radiation less the rate of absorption of radiation by the bottom is evidently

$$R_o=K+\int_0^{y_1}\frac{\partial\theta}{\partial t}dy \tag{92}$$

Another expression is

$$R_o=\int_0^{y_1}\left\{\frac{\partial\theta}{\partial t}-\frac{A_2}{A_1}P_1(z)\frac{\partial\theta}{\partial y}\right\}dy \tag{93}$$

$$=\int_0^{y_1}(\mu^2C_1a^2)e^{-ay}dy=\mu^2C_1a(1-e^{-ay_1})$$

where (y_1) is the depth of the water,

$$\int_0^{y_1}hP_1(z)\frac{\partial\theta}{\partial y}dy=-.885 \tag{94}$$

since

$$1.13\int_0^{y_1}hP_1(z)\frac{\delta\theta}{\delta y}dy \text{ must equal } -1, \tag{94}$$

therefore

$$\frac{A_2}{A_1}=1.13hK. \tag{95}$$

Having obtained the physical constants compute $\frac{\partial\theta}{\partial t}$ and compare with the observed values in order to test the validity of the mathematical formulation. Obtain a closer approximation by using $\left[\frac{\partial\theta}{\partial t}-\frac{A_2}{A_1}P_1(z)\frac{\partial\theta}{\partial y}\right]$ instead of $\left(\frac{\partial\theta}{\partial t}\right)$ in equation (90) and summing with the interval 1 or 2 to 21. Then recompute the quantities (h), (K), and (R_o).

Otherwise fit a straight line to the points found by plotting $\left(\frac{\partial\theta}{\partial t}\right)$ on semilogarithmic paper against (y) for the depths (12) to (20), and determine the intercept $(\mu^2C_1a^2)$ for $y=o$. Then find $(\mu^2C_1a^2)$ where (a) has the value obtained by plotting $(\theta-C)$ and draw a line through this point for $y=o$, having the slope corresponding to the new value of (a).

Experience has indicated that the downward diffusion of surface cooling in the ocean is practically limited to a depth of 100 meters. A depth unit of 10 meters has been found convenient for such ocean investigations. Denote the upwelling velocity (W) by

$$W=W_1f(y) \tag{96}$$

Compute the constants (C_1), (C) and (a) by the method already described. For values of (y) between (5) and (10), the term $\frac{A_2}{A_1}P_1(z)\frac{\partial\theta}{\partial y}$ may be neglected in a first approximation, therefore

$$\sum_{y=5}^{y=10}\frac{\partial\theta}{\partial t}-W_1\sum_{y=5}^{y=10}f(y)\frac{\partial\theta}{\partial y}=T\sum_{y=5}^{y=10}\frac{ae^{-ay}}{1-e^{-10a}} \tag{97}$$

where

$$\frac{Ta}{1-e^{-10a}}=\mu^2C_1a^2 \tag{98}$$

Assume a value of (W_1) and compute the corresponding value of (T). The expression

$$T\frac{ae^{-ay}}{1-e^{-10a}}=\mu^2\frac{\partial^2\theta}{\partial y^2} \tag{99}$$

is the rate of temperature change at the level (y) due to turbulence. Assume the same law to hold for $(y)=4$, 3, 2, and 1. Draw a straight line on semilogarithmic paper (where the abscissa is (y)) through the point $y=0$ and $T\frac{a}{1-e^{-10a}}$ parallel to the graph of $(\theta-C)$ on the same paper. Then plot the values of $\left[\frac{\partial\theta}{\partial t}-W_1f(y)\frac{\partial\theta}{\partial y}\right]$ as ordinates against (y) as abscissae for $y=5$, 6, 7, 8, 9, and 10. These points should fall on the same line and their departure from this line will indicate an appropriate revision of the assumed value of (W_1). Having thus determined the best estimate of (W_1), read off the values of $Ta\dfrac{e^{-ay}}{1-e^{-10a}}$ from the graph and subtract them from $\left[\dfrac{\partial\theta}{\partial t}-W_1 f(y)\dfrac{\partial\theta}{\partial y}\right]$, for $y=0$, 1, 2, etc. These differences are due to the downward diffusion of surface cooling and should equal $\dfrac{hK}{A_1}P_1(z)\dfrac{\partial\theta}{\partial y}$.

Plot these differences as ordinates on semilogarithmic paper against depths as abscissae. According to experience the points will fall approximately on a straight line. Draw this line, then find by trial a value of h such that the values of the expression $h\left[\displaystyle\int_z^\infty \frac{e^{-x^2}}{x}\,dx\,\frac{\partial\theta}{\partial y}\right]$, plotted as ordinates on the same paper against (y) as abscissae, determine a line parallel to the first. Having thus found the value of (h), closer approximations to the other constants may be found. Also the available solar radiation penetrating the surface may be computed from the equation

$$R_o=\int_0^{10}\frac{\partial\theta}{\partial t}dy+W_1\int_0^{10}f(y)\frac{\partial\theta}{\partial y}dy+K \tag{100}$$

The value of $F(R, T)$ can now be expressed by the equation

$$F(R, T)=\left[R_o\frac{a_o e^{-a_o y}}{1-e^{-10a_o}}\right]+\left[\mu^2\frac{\partial^2\theta}{\partial y^2}\right] \tag{101}$$

which provides an estimate of the absorption coefficient a_o.

Also in the trial computations, as well as in computations of corrections to preliminary values, equation (88) may be divided by e^{-ay} after computing (a). Then from assumed or trial values of (W_1) and (h) we can compute $(\mu^2 C_1 a^2)$ and (K) by plotting on squared paper the equation

$$\left[\frac{\dfrac{\partial\theta}{\partial t}-W_1 f(y)\dfrac{\partial\theta}{\partial y}}{e^{-ay}}\right]=K\left[\frac{hP_1(z)}{e^{-ay}}\frac{\partial\theta}{\partial y}+\mu^2 C_1 a^2\right] \tag{102}$$

where the first member is an ordinate, and the coefficient of (K) is an abscissa.

METHODS OF APPLYING THE EXACT EQUATION

After estimating the constants of equation (88) compute (H) from the equation

$$\int_H^z \left\{ \int_z^\infty \frac{e^{-x^2}}{x} dx \right\} dz + H \int_H^z \frac{e^{-x^2}}{x} dx = -\frac{1}{2} \left\{ hP_1(z) \frac{\partial \theta}{\partial y} \Big]_0 + hP_1(z) \frac{\partial \theta}{\partial y} \Big]_1 \right\} \qquad (103)$$

where (z) corresponds to the depth 1. If (z) corresponds to the depth (2), the second member is

$$-\tfrac{1}{2}\left(hP_1(z)\frac{\partial \theta}{\partial y}\Big]_0 + 2hP_1(z)\frac{\partial \theta}{\partial y}\Big]_1 + hP_1(z)\frac{\partial \theta}{\partial y}\Big]_2 \right)$$

or

$$-\tfrac{1}{3}\left(hP_1(z)\frac{\partial \theta}{\partial y}\Big]_0 + 4\left[hP_1(z)\frac{\partial \theta}{\partial y}\Big]_1 + hP_1(z)\frac{\partial \theta}{\partial y}\Big]_2 \right) \right.$$

If $(H) > 0.001$ the first integral above,

$$\int_H^z \left\{ \int_z^\infty \frac{e^{-x^2}}{x} dx \right\} dz$$

is equivalent to

$$\int_{.001}^z \left[\int_z^\infty \frac{e^{-x^2}}{x} dx \right] dz - \int_{.001}^H \left[\int_H^\infty \frac{e^{-x^2}}{x} dx \right] dH$$

therefore the first member of equation (103) can be evaluated by means of tables 1 and 3.

From equation (60) we get the approximate equations

$$u = \theta + A_5 r [1 + A_5 r] \left[\frac{P_3(z)}{hr} - (\theta_o - \theta) \right] \qquad (104)$$

$$u = \theta + A_5 P_3(z) [1 + A_5 r] \left[\frac{1}{h} - (\theta_o - \theta) \frac{r}{P_3(z)} \right] \qquad (105)$$

where $(r) = P_2(z)$, and third and higher powers of $(A_5 r)$ are neglected. The first is more convenient for smaller values of (y) and the second for larger values. Neglecting $(A_5 r)$ in the squared brackets and differentiating equation (104) with respect to (t), then with respect to (y), gives

$$\frac{\partial u}{\partial t} = \left\{ \frac{\partial \theta}{\partial t} + A_5 \frac{\partial}{\partial t} \left[r \left(\frac{P_3(z)}{hr} - (\theta_o - \theta) \right) \right] \right\} \qquad (106)$$

and

$$\frac{\partial u}{\partial y}=\left\{\frac{\partial \theta}{\partial y}+A_5\frac{\partial}{\partial y}\left[r\left(\frac{P_3(z)}{hr}-(\theta_o-\theta)\right)\right]\right\} \tag{107}$$

Substitute these derivatives in equation (61) thus obtaining

$$\frac{\partial \theta}{\partial t}=-[A_4P_1(z)+W]\frac{\partial \theta}{\partial y}-A'e^{-ay} \tag{108}$$

$$=-A_5\left\{\frac{\partial}{\partial t}\left[r\left(\frac{P_3(z)}{hr}-(\theta_o-\theta)\right)\right]-(A_4P_1(z)+W)\frac{\partial}{\partial y}\left[r\left(\frac{P_3(z)}{hr}-(\theta_o-\theta)\right)\right]\right.$$
$$\left.-rA'e^{-ay}-rA_4P_1(z)\frac{\partial \theta}{\partial y}\right\}$$

which is appropriate for small values of (y). Neglecting (A_5r) in the squared brackets, the derivatives of equation (105) are

$$\frac{\partial u}{\partial t}=\left\{\frac{\partial \theta}{\partial t}+A_5\frac{\partial}{\partial t}\left[P_3(z)\left(\frac{1}{h}-(\theta_o-\theta)\frac{r}{P_3(z)}\right)\right]\right\} \tag{109}$$

and

$$\frac{\partial u}{\partial y}=\left\{\frac{\partial \theta}{\partial y}+A_5\frac{\partial}{\partial y}\left[P_3(z)\left(\frac{1}{h}-(\theta_o-\theta)\frac{r}{P_3(z)}\right)\right]\right\} \tag{110}$$

Substituting these values in equation (61) gives

$$\frac{\partial \theta}{\partial t}-\left(A_4P_1(z)+W\frac{\partial \theta}{\partial y}-A'e^{-ay}\right)=-A_5\left\{\frac{\partial}{\partial t}\left[P_3(z)\left(\frac{1}{h}-(\theta_o-\theta)\frac{r}{P_3(z)}\right)\right]\right.$$
$$-(A_4P_1(z)+W)\frac{\partial}{\partial y}\left[P_3(z)\left(\frac{1}{h}-(\theta_o-\theta)\frac{r}{P_3(z)}\right)\right]-rA'e^{-ay} \tag{111}$$
$$\left.-rA_4P_1(z)\frac{\partial \theta}{\partial y}\right\}$$

which is adapted to large values of (y). Compute the coefficient of (A_5) in equations (108) and (111) using tabulations of the functions on pages 263 to 291, then solve for A. At the surface, equation (60) becomes

$$u_o=\theta_o+\frac{A_5P_3(z_o)}{h\rho_o}=\theta_o+A_5\phi_o \text{ where } \phi_o \text{ is known approximately.} \tag{112}$$

Compute $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \phi}{\partial y}$

Then:

$$\frac{\partial \theta}{\partial t}-\left[A_4\frac{P_1(z)}{\rho}+W\right]\frac{\partial \theta}{\partial y}-\frac{A'}{\rho}e^{-ay}=-A_5\left\{\frac{\partial \phi}{\partial t}-\left[A_4\frac{P_1(z)}{\rho}+W\right]\frac{\partial \phi}{\partial y}\right\} \tag{113}$$

from which a closer approximation to the constants can be found. Thus all values are found that are needed to substitute in equation (72) in order to compute (λ).

An alternative procedure can be based upon

$$u=\theta+\frac{A_5}{h}P_5(z)[1+A_5r] \tag{114}$$

derived from equation (45) where

$$P_5(z)=\int_z^{z_1} B'\frac{e^{-x^2}}{x}dx \tag{115}$$

and $B'=\frac{B_1}{B}$ (see page 216). The result is

$$\left\{\frac{\partial\theta}{\partial t}-[A_4P_1(z)+W]\frac{\partial\theta}{\partial y}-A'e^{-ay}\right\}+\frac{W}{h}\frac{\partial P_5(z)}{\partial y}+rA'e^{-ay} \tag{116}$$

$$=-A_5\left\{\left[\frac{\partial P_5(z)}{h\partial t}-\frac{A_4P_1(z)+W}{h}\frac{\partial P_5(z)}{\partial y}-rA'e^{-ay}\right]-A_4P_1(z)r\frac{\partial\theta}{\partial y}\right\}$$

$$-A_5^2\left\{\frac{\partial[rP_5(z)]}{h\partial t}-\frac{A_4P_1(z)+W}{h}\frac{\partial[rP_5(z)]}{\partial y}\right\}$$

In the first approximation to the value of (A_5) the (A_5^2) term can be neglected. Values of (u) corresponding to (θ) can now be computed by means of equation (60), and the derivatives of these values of (u) can be computed and substituted in equation (61). Thus closer approximations to the quantities first found can be computed.

Methods of Correcting for Effect of Radiation Absorbed by the Bottom in Shallow Water

If the water is less than four or five meters deep, an appreciable amount of solar heat may be directly absorbed by the bottom. Accordingly, the water will be heated directly by solar radiation which it absorbs and will be heated indirectly because of contact with the heated bottom. Thus a thin layer of the water in contact with the bottom becomes heated and these lighter elements tend to rise. Assume that different elements become heated by different amounts before rising and proceed as was done with the settling of the cold heavy surface elements.

The following additional notation will be used in deriving the correction formula.

$\int_0^{y_b} Rdy=M_1=$ rate at which heat is absorbed by the water of depth y_b.

$\int_{y_b}^{\infty} Rdy=M_2=$ rate at which the heat is absorbed by the bottom.

$R_o=M_1+M_2=$ rate at which solar energy penetrates the surface.

$$z=h(\sigma-\sigma_o),\ z'=h'(\sigma_b-\sigma),\ z'=h'(\sigma_b-\sigma_o)$$

$$B=\frac{\theta_o-\theta}{\sigma-\sigma_o},\ B'=\frac{\theta-\theta_b}{\sigma_b-\sigma}$$

Neglect as before the difference between the observed temperature and that of the elements that are *rising*, (conditions are reversed here).

Then the approximate simplified form of the correction to the rate of temperature change becomes

$$-h'K'\left[\frac{\displaystyle\int_{z'}^{\infty}\frac{F_1(z')}{B'z'}dz'}{\displaystyle\int_{z_b'}^{\infty}F_1(z')dz'}\right]\frac{\partial\theta}{\partial y}=-h'K'\frac{P_1(z')}{A_1}\frac{\partial\theta}{\partial y} \qquad (117)$$

where $P_1(z')$ is found by substituting (z') and (B') for (x) and (B) in equation (32) and the limits of the integral are $(z'+H)$ and ∞. The limits for the integral in (A_1) are (H) and ∞ (see equation 33). The temperature equation corresponding to no upwelling, but including the effect of bottom heating, is therefore

$$\frac{\partial\theta}{\partial t}=\left[hK\frac{P_1(z)}{A_1}-h'K'\frac{P_1(z')}{A_1}\right]\frac{\partial\theta}{\partial y}+A'e^{-ay} \qquad (118)$$

Also

$$R_o=K+\int_o^{y_b}\frac{\partial\theta}{\partial t}dy+M_2-K' \qquad (119)$$

therefore

$$M_1=K+\int_o^{y_b}\frac{\partial\theta}{\partial t}dy-K' \qquad (120)$$

In applying equation (118) to shallow water consider first the upper half in which $P_1(z')$ can be neglected. Compute (a) by plotting $(\theta-C)$ against the depth (y) on semilogarithmic paper. Denote A' by $M_1\frac{a}{1-e^{-ay_b}}$ and divide the equation by $\frac{ae^{-ay}}{1-e^{-ay_b}}$ thus obtaining

$$\left[\frac{\dfrac{\partial\theta}{\partial t}}{\dfrac{ae^{-ay}}{1-e^{-ay_b}}}\right]=\left[\frac{h\dfrac{P_1(z)}{A_1}\dfrac{\partial\theta}{\partial y}}{\dfrac{ae^{-ay}}{1-e^{-ay_b}}}\right]K+M_1 \qquad (121)$$

Plot on squared paper, values of the first member as ordinates against values of the coefficient of (K) as abscissae, thus obtaining (K) and M_1. Using the value of (K) thus found, write

$$\left[\frac{\dfrac{\partial\theta}{\partial t}}{\dfrac{ae^{-ay}}{1-e^{-ay_b}}}-K\left(\frac{\dfrac{hP_1(z)}{A_1}\dfrac{\partial\theta}{\partial y}}{\dfrac{ae^{-ay}}{1-e^{-ay_b}}}\right)\right]=-K'\left[\frac{\dfrac{h'P_1(z')}{A_1}\dfrac{\partial\theta}{\partial y}}{\dfrac{ae^{-ay}}{1-e^{-ay_b}}}\right]+M_1 \quad (122)$$

and plot on squared paper to obtain (K) and another estimate of M_1.

Methods of Solving the Equations in Order to Predict the Temperature and Salinity from Given Initial Conditions

In addition to methods of computing the physical constants in the equations of temperature, salinity, and other properties of the water, there is the problem of predicting the values of these quantities from given initial conditions. A general solution of these partial differential equations would provide the answer to this problem, but such a solution has not been obtained. There may indeed be no general solution of equations of this type, but various numerical and graphical methods are available for approximating as closely as desired to the particular solution for any given initial conditions. The following brief statement of numerical methods of solving the equations of temperature distribution in a fresh-water lake is presented to illustrate how the problem of prediction may be attacked. The same methods can be extended to the more complicated problem involving other properties of the water.

Consider the pair of equations

$$\theta=u-\left(\frac{A_3}{A_1}\right)\int_z^\infty\frac{(u-\theta')e^{-x^2}}{(x-z)Bx}dx \quad (123)$$

$$\frac{\partial u}{\partial t}=K\left(\frac{h}{A_1}\right)\frac{\displaystyle\int_z^\infty\frac{e^{-x^2}}{Bx}dx}{\rho}\frac{\partial u}{\partial y}+\frac{F(R,T)}{\rho} \quad (124)$$

where

$$\rho=1-\frac{A_3}{A_1}\int_z^\infty\frac{e^{-x^2}}{(x-z)Bx}dx \quad (125)$$

In terms of the notation defined on page 215 these reduce to

$$\theta = \rho u + \left[\theta_o P_2(z) - \frac{P_3(z)}{h}\right] A_5 \tag{126}$$

$$u = \theta + \left[(\theta - \theta_o) P_2(z) + \frac{P_3(z)}{h}\right] \frac{A_5}{\rho} \text{ and} \tag{127}$$

$$\frac{\partial u}{\partial t} = K\left(\frac{h}{A_1}\right) \frac{P_1(z)}{\rho} \frac{\partial u}{\partial y} + \frac{F(R, T)}{\rho} \tag{128}$$

where

$$\rho = 1 - A_5 P_2(z) \tag{129}$$

Assuming $A_5 P_2(z)$ to be small (see p. 223) equation (127) can be expanded into the rapidly converging series,

$$u = \theta + \left[(\theta - \theta_o) P_2(z) + \frac{P_3(z)}{h}\right] A_5 + \left[(\theta - \theta_2) P_2(z) + \frac{P_3(z)}{h}\right] [A_5 P_2(z)] A_5 + \ldots \ldots \tag{130}$$

At the surface equations (126), (127), and (130) are respectively

$$\theta_o = u_o - \frac{P_3(z_o)}{h \rho_o} A_5 \tag{131}$$

$$u_o = \theta_o + \frac{P_3(z_o)}{h \rho_o} A_5 \tag{132}$$

and

$$u_o = \theta_o + \frac{P_3(z_o)}{h} A_5 + \frac{P_3(z_o)}{h} \left(A_5 P_2(z_o)\right) A_5 + \ldots\ldots\ldots \tag{133}$$

To solve the pair of simultaneous equations (127) and (128) substitute the initial values of (θ) in (127) for each depth. Compute $\frac{\partial u}{\partial y}$ by numerical or graphical differentiation and substitute in equation (128). Then $\frac{\partial u}{\partial t} \Delta t = \Delta u$, the approximate increment of (u) at each depth corresponding to the time increment (Δt). Substitute $(u + \Delta u)$ in equation (126) to determine the value of $(\theta + \Delta \theta)$. Substitute $(\theta + \Delta \theta)$ in equation (127), compute the derivative $\frac{\partial (u + \Delta u)}{\partial y}$, substitute in (128), and compute the next increment $\Delta u = \frac{\partial (u + \Delta u)}{\partial t} \Delta t$ etc. Continuing this step-by-step method results in the approximate temperature at each depth and at the end of each of a succession of time intervals. The same schedule of computations is described symbolically as follows, where the subscript to the left of a letter is numerically equal to the the number of time intervals, and the superscript denotes the order of the approximation.

$${}_{o}u = \varphi_1({}_{o}\theta), \text{ compute } \frac{\partial_o u}{\partial y} \text{ by numerical differentiation,}$$

$${}_{1}u' - {}_{o}u' = \frac{\partial_o u}{\partial t}\Delta t = \varphi_2\left({}_{o}\theta, \frac{\partial_o u}{\partial y}\right)\Delta t, \text{ compute } {}_{1}u' \text{ and } \frac{\partial_1 u'}{\partial y},$$

$${}_{1}\theta' = \varphi_3({}_{1}u', {}_{o}\theta),\ {}_{2}u' - {}_{1}u' = \frac{\partial_1 u'}{\partial t}\Delta t = \varphi_2\left({}_{1}\theta', \frac{\partial_1 u'}{\partial y}\right)\Delta t,$$

$${}_{2}\theta' = \varphi_3({}_{2}u', {}_{1}\theta'), \text{ compute } \frac{\partial_2 u'}{\partial y} \text{ by numerical differentiation.}$$

$${}_{3}u' - {}_{2}u' = \frac{\partial_2 u'}{\partial t}\Delta t = \varphi_3\left({}_{2}\theta', \frac{\partial_2 u'}{\partial y}\right)\Delta t, \text{ etc.}$$

The following modification of the above method expressed in the same notation will result in a closer approximation. First find the value of (u) and its derivatives at a time $\left(\frac{1}{2}\Delta t\right)$ later than the initial time. Then compute the temperatures at the time (Δt) later by means of the time derivative at a time $\left(\frac{1}{2}\Delta t\right)$ later, since this derivative is a closer approximation to the slope of the chord from zero time to the time (Δt) later. This procedure is expressed symbolically as follows:

$${}_{o}u = \varphi_1({}_{o}\theta), \text{ compute } \frac{\partial_o u}{\partial y} \text{ by numerical differentiation,}$$

$${}_{\frac{1}{2}}u' - {}_{o}u = \frac{1}{2}\frac{\partial_o u}{\partial t}\Delta t = \varphi_2\left({}_{o}\theta, \frac{\partial_o u}{\partial y}\right)\frac{\Delta t}{2}, \text{ compute } {}_{\frac{1}{2}}u' \text{ and } \frac{\partial_{\frac{1}{2}} u'}{\partial y}.$$

$${}_{\frac{1}{2}}\theta' = \varphi_3({}_{\frac{1}{2}}u', {}_{o}\theta),\ {}_{1}u' - {}_{o}u \frac{\partial_{\frac{1}{2}} u'}{\partial t}\Delta t = \varphi_2\left({}_{\frac{1}{2}}\theta', \frac{\partial_{\frac{1}{2}} u'}{\partial y}\right)\Delta t$$

$${}_{1}\theta' = \varphi_3({}_{1}u', {}_{\frac{1}{2}}\theta')$$

$${}_{1\frac{1}{2}}u' - {}_{\frac{1}{2}}u' = \frac{\partial u'}{\partial t}\Delta t = \varphi_2\left({}_{1}\theta', \frac{\partial_1 u'}{\partial y}\right)\Delta t$$

$${}_{1\frac{1}{2}}\theta' = \varphi_3({}_{1\frac{1}{2}}u', {}_{1}\theta'),\ {}_{2}u' - {}_{1}u' = \frac{\partial_{1\frac{1}{2}} u'}{\partial t}\Delta t = \varphi_2\left({}_{1\frac{1}{2}}\theta', \frac{\partial u}{\partial y}\right)\Delta t,$$

$${}_{2}\theta' = \varphi_3({}_{2}u', {}_{1\frac{1}{2}}\theta') \text{ etc.}$$

Thus after estimating the derivatives corresponding to the time $\frac{\Delta t}{2}$ and thereby computing (u), its derivatives, and (θ) for the time (Δt), find the derivatives corresponding to the time (Δt). This time derivative is then used to determine (u), its derivatives, and (θ) for the time $\left(\frac{3}{2}\Delta t\right)$. Then advance from (Δt) to $(2\Delta t)$. Next advance from $\left(\frac{3}{2}\Delta t\right)$

to $\left(\frac{5}{2}\Delta t\right)$, and so on. Except for computations made for the first interval, $\left(\frac{\Delta t}{2}\right)$ the interval (Δt) is used throughout.

Again using the first two terms of the expansion in a Taylor's series, and the average value of $F(R, T) = Q$ for the time interval (Δt), the increment equation expressed in the same notation is

$$ {}_{t+\Delta t}u = {}_t u + \left(\frac{\partial_t u}{\partial t}\right)\Delta t + \frac{1}{2}\left(\frac{\partial^2{}_t u}{\partial t^2}\right)(\Delta t)^2. $$

Differentiating equation (128) with respect to (t) and with respect to (y) we get

$$ \frac{\partial^2 u}{\partial t^2} = \left(K\frac{h}{A_1}\right)^2\left(\frac{P_1(z)}{\rho}\right)\left[\frac{P_1(z)}{\rho}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{P_1(z)}{\rho}\right)\right] $$
$$ + \left(K\frac{h}{A_1}\right)\frac{P_1(z)}{\rho}\frac{\partial}{\partial y}\left(\frac{Q}{\rho}\right) + \left(K\frac{h}{A_1}\right)\frac{\partial u}{\partial y}\frac{\partial}{\partial t}\left(\frac{P_1(z)}{\rho}\right) + \frac{\partial}{\partial t}\left(\frac{Q}{\rho}\right) = $$
$$ \varphi_4\left[\theta, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial t}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}\right] $$

Thus the increments of (u) corresponding to (Δt) may be computed to a closer approximation as follows:

$$ {}_1u' - {}_0u' = \varphi_2\left({}_0\theta, \frac{\partial_0 u}{\partial y}\Delta t\right) + \frac{1}{2}\varphi_4\left({}_0\theta, \frac{\partial_0 \theta}{\partial y}, \frac{\partial_0 \theta}{\partial t}, \frac{\partial_0 u}{\partial y}, \frac{\partial^2{}_0 u}{\partial y^2}\right)(\Delta t)^2 $$

Probably the previous method is most practicable, even though in some cases it may be necessary to compute $\frac{\partial_{\frac{1}{2}} u}{\partial y}$ from values of ${}_{\frac{1}{2}}u$ given by the more accurate equation,

$$ {}_{\frac{1}{2}}u' - {}_0u = \varphi_2\left({}_0\theta, \frac{\partial_0 u}{\partial y}\right)\frac{\Delta t}{2} + \frac{1}{2}\varphi_4\left({}_0\theta, \frac{\partial_0 \theta}{\partial y}, \frac{\partial_0 \theta}{\partial t}, \frac{\partial_0 u}{\partial y}, \frac{\partial^2{}_0 u}{\partial y^2}\right)\left(\frac{\Delta t}{2}\right)^2 $$

in order to obtain the requisite accuracy for the starting point corresponding to $\left(\frac{\Delta t}{2}\right)$.

ILLUSTRATIVE COMPUTATION OF THE PHYSICAL CONSTANTS OF THE APPROXIMATE SIMPLIFIED EQUATION

Numerical Applications to Serial Temperatures of Lake Mendota, in Madison, Wisconsin

The primary purpose of this paper is to present the development of the theory here set forth, methods of applying it, and tables to facilitate the computation. In order to illustrate some of these methods and to show how the theory works out in practice, details are presented of the computations for three very different cases selected from the large number that were computed along with the development of the theory.

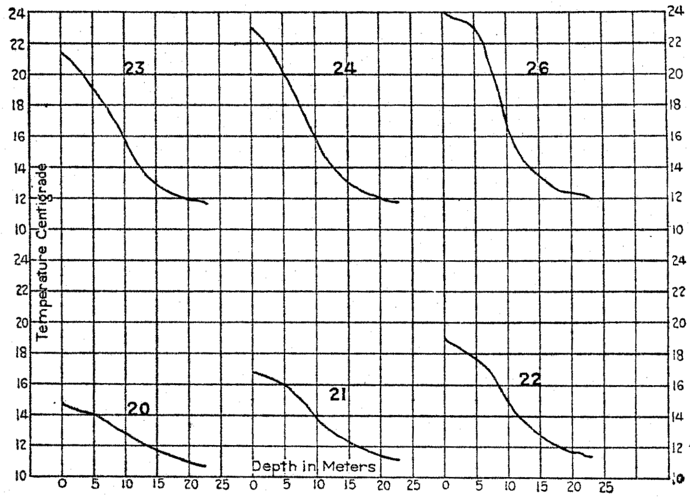

Fig. 4.—Relation to depth of weekly averages of serial temperatures of Lake Mendota, in Madison, Wisconsin.

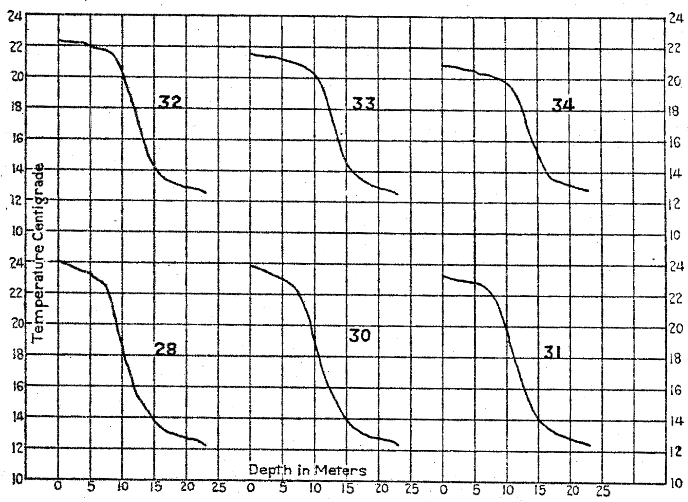

Fig. 5.—Relation to depth of weekly averages of serial temperatures of Lake Mendota, in Madison, Wisconsin.

Under the direction of Dr. E. A. Birge, thorough and comprehensive investigations of Lake Mendota, in Madison, Wisconsin, have been carried out since 1895. These investigations included temperature observations at each meter from the surface to twenty-three meters at the bottom. I am indebted to Doctor Birge for the use of a table of weekly means of these temperatures for the interval 1895 to 1915. On the average, each temperature is the mean of about eighty observations. The variation of these means with respect to depth and time is remarkably regular, and the data are especially adapted to theoretical studies. The relation of temperature to depth is shown graphically by figures 4 and 5 for the interval May 24 to September 15. The numbers 20, 21, etc., designate weeks May 24–31, June 1–9, etc.

The computations carried out as explained on pages 223 to 229 are presented in table 2 and figure 6. The derivatives, columns 5 and 6, have the values that would result from differentiating a second degree parabola fitted to five consecutive values of the temperature. They were computed by the convenient method of moments explained by Von Sanden (1923, pp. 114–120) which reduces in this case to the process illustrated by the following examples. Find the time derivative $\left(\frac{\partial \theta}{\partial t}\right)$ at the surface for week 20, using one month as the unit of time. First arrange the five temperatures in their time order with the temperature for week (20) at the middle thus—12°, 13°3, 14°8, 16°8, 19°0. Then multiply the first temperature by (-1), the second by (-2), the fourth by $(+2)$, and the last by $(+1)$. The sum (14) divided by (8) equals the derivative corresponding to a time unit of one week or the interval from each temperature to the next. The derivative corresponding to one month as a unit equals $14 \div 2 = 7.00$. Similarly the derivative $\frac{\partial \theta}{\partial y}$ for the same week at the depth (2) meters is found by arranging the temperatures in the order of depth beginning with the surface thus—14°8, 14°5, 14°4, 14°2, 14°1. Use the same multiplier as before and obtain the products $-14°8 - 29°0 - 0° + 28°4 + 14°1 = -1°3$. The sum $(-1°3)$ divided by (8) equals the derivative $(-.162)$ corresponding to a depth unit of one meter. The value of (C) was found to be 9°8 by the method explained on page 225. Plotting the differences $(\theta - 9°8)$ as ordinates on semilogarithmic paper against depths (y) as abscissae determined the points according to which the straight line V, figure 6 was drawn. From the slope of this line the value of (a) was found to be (.098), according to page 226 and pages 292 to 300. The value of $(\mu^2 C_1 a^2)$ was found to be $34.70 \div 2.064 = 16.8$ from equation (90), page (226). Corresponding to the change in slope of line V at the abscissa $(y=10)$ the part of line VI between $(y=0)$ and $(y=10)$ has been redrawn as explained on page (226).

TABLE 2

Computations of the constants in the approximate simplified equation from Lake Mendota serial temperatures for the week May 24 to 31, using one month as the time unit and one meter as the depth unit

	(1)	(2)	(3)	(4)	(5)	(6)	(7)[1]	(8)[2]	(9)
y	θ	$(\theta_o-\theta)$	σ	$(\sigma-\sigma_o)$	$\frac{\partial\theta}{\partial t}$	$\frac{\partial\theta}{\partial y}$	e^{-ay}	$\mu^2C_1a^2e^{-ay}$ from the graph	(5)−(8)
0	14.8	0	.99916	0	7.0	(−.072)	1.0000	12.0	−5.00
1	14.5	.3	.99920	.00004	7.2	(−.209)	.9499	11.3	−4.10
2	14.4	.4	.99922	.00006	7.35	−.162	.8834	10.7	−3.35
3	14.2	.6	.99924	.00008	7.4	−.138	.8303	10.0	−2.6
4	14.1	.7	.99926	.00010	7.15	−.125	.7803	9.5	−2.35
5	14.0	.8	.99927	.00011	7.2	−.163	.7334	9.0	−1.80
6	13.8	1.0	.99930	.00014	6.85	−.225	.6893	8.3	−1.45
7	13.5	1.3	.99934	.00018	6.65	−.250	.6479	7.8	−1.15
8	13.3	1.5	.99937	.00021	6.2	−.250	.6096	7.4	−1.20
9	13.0	1.8	.99940	.00024	5.6	−.275	.5723	7.0	−1.40
10	12.8	2.0	.99943	.00027	4.85	−.250	.5379	6.5	−1.65
11	12.5	2.3	.99947	.00031	4.40	−.238	.3403	5.9	−1.50
12	12.3	2.5	.99949	.00033	4.15	−.212	.3085	5.4	−1.25
13	12.1	2.7	.99951	.00035	3.90	−.200	.2797	4.9	−1.00
14	11.9	2.9	.99954	.00038	3.80	−.200	.2536	4.4	−0.60
15	11.7	3.1	.99956	.00040	3.65	−.200	.2299	4.1	−0.45
16	11.5	3.3	.99958	.00042	3.50	−.187	.2085	3.65	−0.15
17	11.3	3.5	.99960	.00044	3.40	−.150	.1890	3.3	+0.10
18	11.2	3.6	.99961	.00045	3.20	−.125	.1713	3.0	....
19	11.1	3.7	.99962	.00046	3.05	−.138	.1554	2.7	....
20	10.9	3.9	.99964	.00048	3.00	−.137	.1408	2.5	....
21	10.8	4.0	.99965	.00049	3.05	−.112	.1277	2.2	....
22	10.7	4.1	.99966	.00050	3.00	−.112	.1158	2.0	....
23	10.6	4.2	.99967	.00051	2.95	−.100	.1050	1.8	....

	(10)	(11)	(12)	(13)	(14)
y	(9)÷(6) which equals $1.13Kh\int_z^\infty \frac{e^{-z^2}}{Bx}dx$	$hP_1(z)\mathbf{B}$ from table 1, assuming $h=2000$	(10)÷(11) equals $\frac{1.13K}{\mathbf{B}}$	$(11)\frac{\partial\theta}{\partial y}$	$\frac{1.13K}{\mathbf{B}}(13)$
0	....	....	....	....	....
1	....	....	....	....	....
2	....	....	....	....	....
3	....	....	....	....	....
4	18.8	2681	.00701	−335.0	−1.75
5	11.0	2512	.00396	−409.5	−2.13
6	6.45	2054	.00315	−462.0	−2.41
7	4.60	1595	.00288	−398.7	−2.08
8	4.80	1334	.00360	−333.8	−1.74
9	5.09	1115	.00457	−306.5	−1.60
10	6.60	928	.00712	−232.0	−1.21
11	6.30	731	.00860	−174.0	−0.91
12	5.90	647	.00912	−193.3	−1.01
13	5.00	572	.00875	−175.0	−0.91
14	3.00	475	.00632	−126.3	−0.66
15	2.25	419	.00538	−107.7	−0.56
16	.80	370	.00216	− 69.2	−0.36
17	....	326	.00000	− 48.9	−0.25
18	....	305	....	− 38.1	−0.20
19	....	286	....	− 39.5	−0.21
20	....	268	....	− 36.7	−0.19
21	....	234	....	− 26.2	−0.14
22	....	219	....	− 24.5	−0.13
23	....	205	....	− 20.5	−0.11

[1] (a) has the value .062 for $y\overline{<}10$, and the value .098 for $y>10$.
[2] ($\mu^2C_1a^2$) has the value 12 for $y\overline{<}10$, and the value 16.8 for $y>10$.

TABLE 2 (*Continued*)

y	(15) (14)+(8) equals computed value of (5)	(16) (4)÷(2) equals $\frac{1}{B}$	(17) $10^6\times$[means of (16)]	(18) 2000 $\times$(4)$=z$	(19) $\int_z^\infty \frac{e^{-x^2}}{x}\,dx$	(20) Δ(19)
0	....	(.000145)	139.0	.000	....	....
1	....	.000133	141.5	.080	2.2402	.4013
2	....	.000150	141.5	.120	1.8389	.2821
3	....	.000133	137.5	.160	1.5568	.2162
4	7.75	.000142	139.5	.200	1.3406	.0848
5	6.87	.000137	138.5	.220	1.2558	.2286
6	5.89	.000140	139.0	.280	1.0272	.2298
7	5.72	.000138	139.0	.360	0.7974	.1305
8	5.66	.000140	136.5	.420	.6669	.1096
9	5.40	.000133	134.0	.480	.5573	.0934
10	5.29	.000135	134.5	.540	.4639	.0984
11	4.99	.000134	133.0	.620	.3655	.0421
12	4.39	.000132	130.5	.660	.3234	.0376
13	3.99	.000129	130.0	.700	.2858	.0479
14	3.74	.000131	130.0	.760	.2379	.0283
15	3.54	.000129	128.0	.800	.2096	.0245
16	3.29	.000127	126.0	.840	.1851	.0221
17	3.05	.000125	125.0	.880	.1630	.0107
18	2.80	.000125	124.5	.900	.1523	.0092
19	2.49	.000124	123.5	.920	.1431	.0177
20	2.31	.000123	122.5	.960	.1254	.0079
21	2.06	.000122	121.5	.980	.1175	.0080
22	1.87	.000121	121.0	1.000	.1095	.0068
23	1.69	.000121	121.0	1.020	.1027	.1027

y	(21) (20)$\frac{h}{10^6}$(17)	(22) Σ(21) from bottom $=hP_1(z)$	(23) (22)$\frac{\partial\theta}{\partial y}$	(24) (9)÷(23) $=1.13K$	(25) (24)+(8) equals computed value of $\frac{\partial\theta}{\partial t}$	(26) 1.13K(23) = more accurate value of (14)
0	....	....	(−.3574)	(14.0)	−1.66	−13.66
1	.1136	.6146	−.1283	31.9	−6.4	−4.91
2	.0800	.5010	−.0811	41.3	7.6	−3.10
3	.0595	.4210	−.0581	44.7	7.8	−2.22
4	.0236	.3615	−.0451	52.1	7.8	−1.73
5	.0639	.3379	−.0550	32.8	6.9	−2.10
6	.0641	.2740	−.0616	23.6	5.9	−2.36
7	.0363	.2099	−.0525	21.9	5.8	−2.01
8	.0299	.1736	−.0444	27.0	5.7	−1.70
9	.0250	.1437	−.0395	35.4	5.5	−1.51
10	.0264	.1187	−.0296	55.8	5.4	−1.13
11	.01120	.0923	−.0220	68.2	5.1	−0.84
12	.00982	.0811	−.0172	72.7	4.7	−0.66
13	.01243	.0713	−.0143	69.9	4.4	−0.55
14	.00735	.0589	−.0118	50.9	4.0	−0.45
15	.00627	.0515	−.0103	43.7	3.7	−0.39
16	.00556	.0452	−.0085	17.7	3.3	−0.33
17	.00268	.0397	−.0060	....	3.1	−0.23
18	.00230	.0370	−.0046	....	2.8	−0.18
19	.00438	.0347	−.0048	....	2.5	−0.18
20	.00193	.0303	−.0042	....	2.3	−0.16
21	.00194	.0284	−.0032	....	2.1	−0.12
22	.00165	.0265	−.0030	....	1.9	−0.11
23	.02482	.0248	−.0025	....	1.7	−0.10

[1] (a) has the value .062 for $y\overline{\overline{<}}10$, and the value .098 for $y>10$.

[2] ($\mu^2C_1a^2$) has the value 12 for $y\overline{\overline{<}}10$, and the value 16.8 for $y>10$.

The entries in column 11 can be read directly from table 2 for any trial value of (h). Approximate values of $(1.13K)$ can be found by multiplying the entries of column 12 by the average value of (B) which in this case is 7700. A comparison of the computed values of $\left(\frac{\partial \theta}{\partial t}\right)$ entered in column 15 with the observed ones tests the validity of the equation.

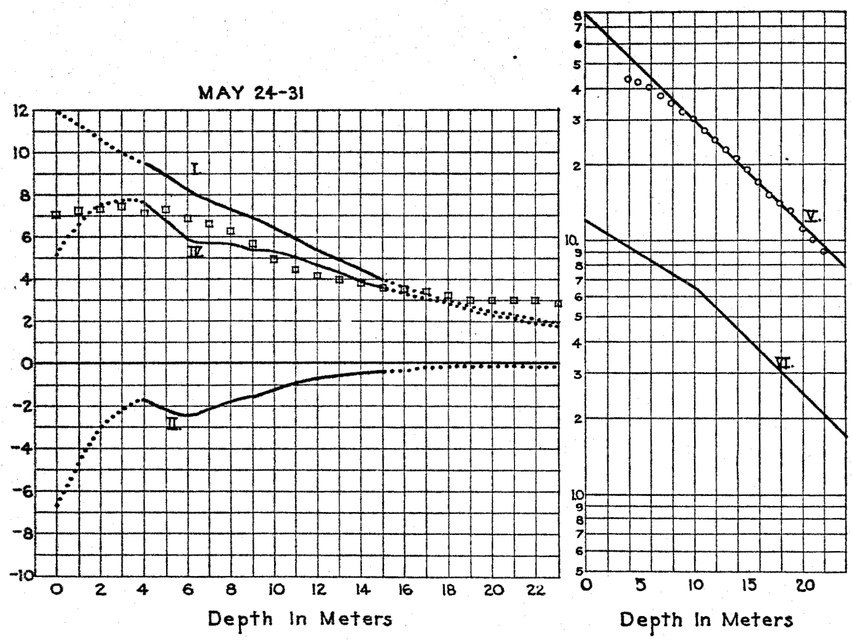

Fig. 6.—Analysis of time rates of change of temperature in Lake Mendota, at different depths, during week number 20, May 24–31.

I. Effect of solar radiation and turbulence.
II. Effect of downward diffusion of surface cooling.
IV. Sum of curves I and II equals the theoretical rate of temperature change.
[·] Observed rate of temperature change.
V. Graph of difference between the observed temperature and the constant 9°8.
VI. Graph parallel to V drawn to compute the effect of solar radiation and turbulence, represented by I.

After making the preliminary computations indicated by the first fifteen columns according to pages 223 to 228 a closer approximation may be found by taking into account the variation of (B) with the depth and using table 1. According to equation (94), page 227, one-half the first and last terms plus the remaining terms of column 23 have the value $-.885$. The value of the first term was thus found to be $(-.3574)$. The ratios entered in column 24 are independent and more accurate estimates of $(1.13K)$. The mean of these ratios equals (43.1) and the ratio of the mean of column 9 to the mean of column 23 equals (38.3) which is a better estimate. Accordingly, the value of K equals $\frac{38.3}{1.13}=33.9$. Curve IV of figure 6 was plotted from the computed values of $\frac{\partial\theta}{\partial t}$ entered in column 25 and agrees well with the observed values $\boxed{\cdot}$ except at the surface where the actual conditions depart most from the conditions justifying the use of the approximate equation (84).

The radiation penetrating the surface is

$$R_o=\int_0^{23}\frac{\partial\theta}{\partial t}dy+K=113.1+33.9=147.0$$

Substituting in equation (85) the values of (C) and (a) already found gave (8.0) as the value of (C_1). Also this value agrees with the ordinate of line V for $y=0$. Therefore, (μ^2), the coefficient of turbulence below the depth (10) equals

$$\mu^2=\frac{\mu^2C_1a^2}{C_1a^2}=\frac{16.81}{8.0(.098)^2}=219.$$

The rate of loss of heat and rate at which solar radiation penetrates the surface expressed in depth of evaporation are respectively

$$\frac{K}{6}=\frac{33.9}{6}=5.65\frac{\text{cm.}}{\text{month}}$$

and

$$\frac{R_o}{6}=\frac{147.0}{6}=24.5\frac{\text{cm.}}{\text{month}}$$

where the latent heat of vaporization of water is assumed to be 600.

The results of applying the same method of computation to the series of weeks May 24–31 to September 9–14, or weeks 20 to 34, are presented in table 3. From May when stratification is least to September when stratification is especially developed, the turbulence μ^2 decreases, as would be expected. The estimated evaporation increases during the same interval according to results usually found by direct pan observations. A rough check on the theoretical evaporation rates is afforded by table 3*A*, which presents evaporation rates estimated for Lakes Michigan and Huron by Freeman, (1926, p. 145) and Horton and Grunsky, (1927, p. 342) using various methods involving pan observations, meteorological observations, and difference between inflow and outflow. According to Horton, (1927, p. 185) the mean monthly evaporation from May to October estimated from the difference between outflow and inflow of Lakes Michigan and Huron equals $\frac{14.96}{6}\times2.54$

TABLE 3

Summary of results of computations applied to Lake Mendota serial temperatures observed between May and September

Time	Week Number	a	h	K	$\int_0^{23} \frac{\partial\theta}{\partial t} dy = N_1$	$K+N_1=R_1$	$1.13K$	μ^2	Theoretical evaporation in cm. per month uncorrected for convection and back radiation $\frac{K}{6}$	$\frac{R_1}{6}$ Evaporating power in cm. per month of theoretical penetrating radiation
May 24–31	20	.098	2000	34.73	113.11	147.0	38.3	219	5.65	24.5
June 1–8	21	.148	1250	27.3	101.24	128.5	30.9	84	4.55	21.4
June 9–15	22	.185	900	25.5	83.42	108.9	28.8	27.6	4.25	18.1
June 16–23	23	.221	800	48.5	64.96	113.5	54.9	15.2	8.10	18.9
June 24–30	24	.221	200	17.4	53.70	71.1	19.6	11.1	2.9	11.9
July 1–8	25	.244	200	20.5	46.43	66.9	23.1	9.2	3.4	11.1
July 9–15	26	.250	300	32.0	39.2	71.2	36.1	11.4	5.3	11.9
July 16–23	27	.255	400	34.2	25.81	60.0	38.6	9.5	5.7	10.0
July 24–31	28	.257	700	41.5	13.60	55.1	46.95	7.9	6.9	9.2
Aug. 1–8	29	.279	700	53.8	6.89	60.7	60.8	4.5	9.0	10.1
Aug. 9–15	30	.287	700	82.5	5.24	87.7	93.1	5.6	13.8	14.6
Aug. 16–23	31	.272	700	52.0	−1.63	50.4	58.8	6.4	8.7	8.4
Aug. 24–31	32	.279	2000	95.0	9.89	85.1	107.5	6.2	15.8	14.2
Sept. 1–8	33	.296	3000	92.2	−15.04	77.2	104.1	5.5	15.4	12.8
Sept. 9–15	34	.250	3100	73.2	−24.51	48.7	82.8	4.1	12.2	8.1

TABLE 3-A

Summary of estimates of rate of evaporation in cm. per month from Lakes Michigan and Huron for comparison with theoretical values for Lake Mendota taken from table 3

Month	Michigan-Huron		Lake Mendota				
	Freeman	Grunsky	McEwen Theoretical values for				
	cm./mo.	cm./mo.	First week cm./mo.	Second week cm./mo.	Third week cm./mo.	Fourth week cm./mo.	Mean cm./mo.
Jan.	7.62	.40					
Feb.	6.35	.36					
March	4.57	.63					
April	2.54	1.78					
May	.76	4.90	5.65				5.65
June	1.02	9.50	4.5	4.2	8.1	2.9	4.90
July	4.06	12.50	3.4	5.3	5.7	6.9	5.30
August	7.63	11.80	9.0	13.8	8.7	15.8	11.80
Sept.	9.40	6.80	15.4	12.2			13.80
Oct.	9.15	3.38					
Nov.	8.64	1.02					
Dec.	8.14	.53					
Year	69.8	53.5					
May to Sept.	22.9	45.5					41.45

$=6.34\frac{\text{cm.}}{\text{m}}$. From relations of evaporation rates to air temperature, humidity, wind, and surface-water temperature he obtained the value $9.65\ \frac{\text{cm.}}{\text{mo.}}$ which he regards as a better estimate, while according to Freeman, (average of values May to October, table 3*A*) the rate is $5.34\ \frac{\text{cm.}}{\text{mo.}}$. The average of the theoretical values for Lake Mendota for the same period is $8.3\ \frac{\text{cm.}}{\text{mo.}}$ which is reduced to $7.0\ \frac{\text{cm.}}{\text{mo.}}$ by assuming an average value of 20 per cent for the Bowen ratio (see pages 220-223) and agrees well with the values found by other methods for Lakes Michigan and Huron.

After reading a manuscript by Richardson (1929) while this paper was in press I requested him to estimate the monthly rate of evaporation from Lake Mendota, using the method explained in his manuscript. According to this method, he uses the solar radiation incident upon the exterior of the earth's atmosphere per month reduced to its value at the earth's surface, back radiation from the water surface, sensible heat, an average value of .22 for R^1, the Bowen Ratio (see pages 220 to 223), and 584 for (L) the latent heat of evaporation. His results are tabulated in the first line of the following table and mine are entered in the second line after correcting the last column of table 3A by using the same values of (R) and (L) in order to make the results comparable. The correction factor is .842.

THEORETICAL EVAPORATION CM PER MONTH FROM LAKE MENDOTA

May	June	July	August	September	Mean
—1.98	7.63	9.87	10.40	10.60	7.30 (Richardson)
4.76	4.13	4.46	9.93	11.60	6.98 (McEwen)

NUMERICAL APPLICATIONS TO SERIAL TEMPERATURES OF WATER IN A TANK ABOUT SIX FEET SQUARE AND FIVE FEET DEEP

During the summer of 1922, a rectangular concrete tank at the Scripps Institution, having for one side a part of the sea wall, was made water tight by coating the inside with a special kind of black paint. The tank is about six feet square and five feet deep. Its top is at the general level of the ground near the sea wall. Equipment was provided for accurately measuring any small change of water level and for measuring the volume of water supplied; thus two independent methods were available for measuring the evaporation loss. By means of a hand pump and a small suction pipe which could be held at any desired level, water was pumped from that level through a small reservoir in which the bulb of a thermometer was inserted. Water entered the suction pipe between two horizontal disks about one-third of a centimeter apart, thus insuring a flow precisely from the desired level. An intensive series of observations of water temperatures at different levels in the tank was made during the week August 7 to 15, 1922, at hourly intervals during the day. These were accompanied by measurements of the rate of evaporation, and wet- and dry-bulb air temperatures. These data are well suited to testing the theory presented in this paper by a relatively small and shallow body of water in which an appreciable amount of heat is absorbed by the bottom of the container. The relation of temperature to depth below the surface of the water in the tank is shown for certain times of the day by figure 7, based upon the intensive observations made from August 7 to 15.

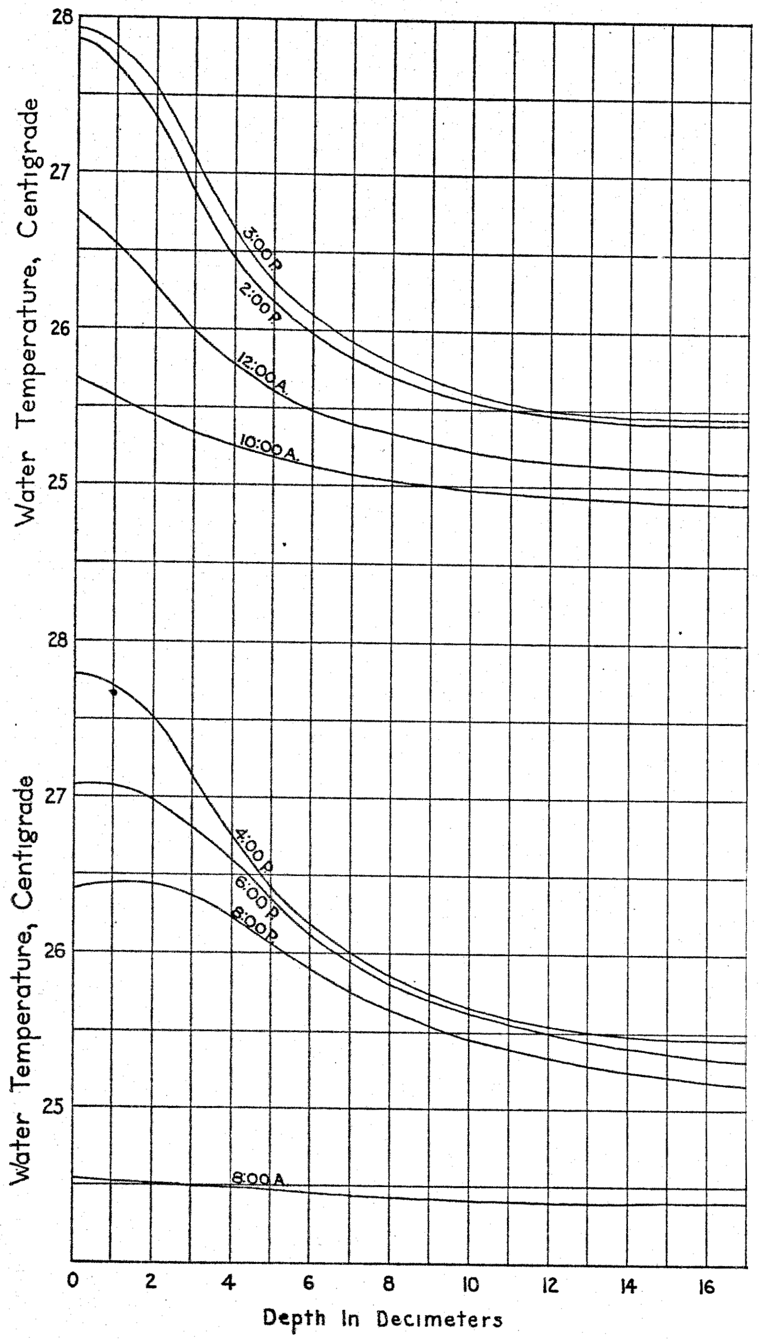

Fig. 7.—Relation to depth of hourly averages of serial temperatures in an open tank of fresh water at the Scripps Institution during August, 1922.

TABLE 4

Computation of the constants in the approximate simplified equation from tank temperatures for the noon hour during August, 1922, using one hour as the time unit and ten cm. as the depth unit

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
y	θ	$(\theta_o-\theta_y)$	σ	$(\sigma-\sigma_o)$	$\frac{\partial\theta}{\partial t}$	$\frac{\partial\theta}{\partial y}$	$\frac{ae^{-ay}}{1-e^{-17a}}$	$(5)\div(7)$ $=Y$	$(6)\div(7)$	$hP_1(z)B$	$\frac{(10)}{A_1\mathbf{B}}$
0	26.74	0	.99661	0	.54	−.214	.2347	2.302	− .912	....	....
1	26.53	.21	.99666	.00005	.56	−.244	.1865	3.002	−1.308	3457.9	1.016
2	26.27	.47	.99674	.00013	.51	−.275	.1482	3.44	−1.856	2048.2	.602
3	25.99	.75	.99681	.00020	.39	−.225	.1177	3.315	−1.913	1438.6	.423
4	25.77	.97	.99687	.00026	.29	−.194	.0935	3.102	−2.075	1091.6	.321
5	25.60	1.14	.99692	.00031	.23	−.152	.0743	3.097	−2.046	875.9	.257
6	25.48	1.26	.99695	.00034	.18	−.106	.0590	3.051	−1.797	761.2	.224
7	25.40	1.34	.99697	.00036	.19	−.075	.0469	4.052	−1.600	699	.205
8	25.33	1.41	.99698	.00037	.18	−.068	.0373	4.83	−1.823	611	.197
9	25.27	1.47	.99700	.00039	.15	−.056	.0296	5.07	−1.892	581	.180
10	25.23	1.51	.997016	.00040	.13	−.044	.0235	5.53	−1.873	558	.171
11	25.18	1.56	.997029	.00041	.13	−.033	.0187	6.95	−1.765	535	.164
12	25.16	1.58	.997034	.00042	.13	−.026	.0149	8.72	−1.745	511	.157
13	25.14	1.60	.99704	.00043	.12	−.017	.0118	10.17	−1.441	487	.150
14	25.12	1.62	.997045	.00044	.13	−.012	.0094	13.83	−1.277	487	.143
15	25.11	1.63	.997047	.00044	.12	−.012	.0074	16.22	−1.622	487	.143
16	25.10	1.64	.99705	.00044	.13	−.012	.0059	22.03	−2.033	487	.143
17	25.10	1.64	.99705	.00044	.13	−.012	.0047	27.68	−2.552	487	.143

TABLE 4 (*Continued*)

	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)
y	$(9)\times(11)=X$	KX	$Y-KX=Y'$	$(\sigma_{17}-\sigma)$	$h^1P_1(z^1)\mathbf{B}$	$\frac{(16)}{A_1\mathbf{B}}$	$-(9)\ (17)$ $=X^1$	$K^1(17)\frac{\partial\theta}{\partial y}$	$K(11)\frac{\partial\theta}{\partial y}$	$Y-K^1X^1$ $=Y''$
0	....	....	....	.00041	....	....	....	....	(.635)	....
1	−1.328	−2.79	5.792	.00039	....	....	....	....	.5205	....
2	−1.117	−2.347	5.787	.00031	....	....	....	....	.3478	....
3	− .809	−1.699	5.014	.00024	....	....	....	....	.2000	....
4	− .666	−1.399	4.501	.00018	....	....	....	....	.1309	....
5	− .526	−1.105	4.202	.00013	....	....	....	....	.0821	....
6	− .402	− .846	3.897	.00010	....	....	....	....	.0498	....
7	− .328	− .689	4.741	.00008	....	....	....	....	.0323	....
8	− .359	− .754	5.584	.00007	150	.044	.0803	−.00138	.0281	4.793
9	− .340	− .714	5.784	.00005	1,140	.335	.643	−.0086	.0212	5.041
10	− .320	− .672	6.202	.000034	4,476	1.313	2.46	−.0266	.0158	4.398
11	− .289	− .607	7.557	.000021	12,143	3.560	6.28	−.0540	.0113	4.06
12	− .274	− .575	9.295	.000016	17,613	5.180	9.04	−.0620	.0086	4.56
13	− .216	− .454	10.624	.00001	28,212	8.28	11.93	−.0648	.0053	4.68
14	− .182	− .382	14.212	.000005	44,977	13.20	16.84	−.0729	.0036	6.09
15	− .232	− .487	16.707	.000003	57,632	16.90	27.40	−.0934	.0036	3.60
16	− .291	− .611	22.641	0	....	....	....	....	.0036	....
17	− .365	− .766	28.446	0	....	....	....	....	.0036	....

TABLE 5

Summary of results of computations applied to tank temperature observations at the Scripps Institution during August, 1922, for the period from 10 A. M. to 6 P. M.

1	2	3	4	5		6	7	8	9
Time	h	Estimated by fitting straight line K	h^1	$\int_o^{17} \frac{\partial\theta}{\partial t} dy$ N_1	K^1	a	$\frac{ae^{-17a}}{1-e^{-17a}}$	$M_2 = M_1(7)$	R_o Penetrating radiation $N_1+K+M_2-K^1$
10 A. M.	1500	2.4	25,000	3.22	.31	.19	.0078	.0351	5.34
Noon	1500	2.1	25,000	3.91	.46	.23	.0047	.0240	5.67
2 P. M.	1500	1.8	25,000	3.53	.24	.29	.0022	.0106	5.10
4 P. M.	1500	2.67	25,000	0	.026	.30	.0018	.0056	2.65
6 P. M.	500	1.5	25,000	−1.84	0	.23	.0047	0	−.34

1	10	11	12	13	14	15	16	17
						Evaporation in $\frac{\text{mm.}}{\text{hour}}$		
Time	Estimated by fitting straight line M_1	R_o Penetrating radiation M_1+M_2	$\frac{K}{6}$	Bowen ratio $=R^1$	$\frac{K}{6(1+R^1)}$	Theoretical $0.94 \times \frac{K}{6(1+R^1)}$	Observed	Evaporating power of penetrating radiation in $\frac{\text{mm.}}{\text{hour}}$
10 A. M.	4.5	4.81	.40	.21	.33	.31	.18	.91
Noon	5.1	5.56	.35	.18	.30	.28	.20	.95
2 P. M.	4.82	5.06	.30	.17	.26	.24	.26	.85
4 P. M.	3.10	3.13	.44	.21	.36	.34	.35	.44
6 P. M.	0	0	.25	.25	.20	.19	.28	−.06

Computations of the constants in the approximate simplified equation (118) were made for several periods of the day. In order to illustrate the method, the computations for the noon hour are summarized in table 4 and figure 8. The general method explained on pages 223 to 234 was followed in making these computations. Line V in figure 8 was

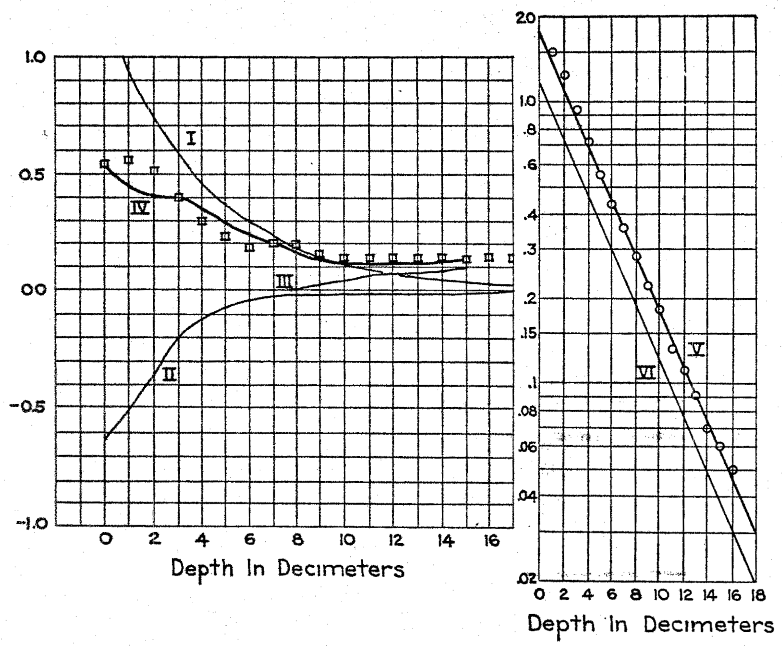

Fig. 8.—Analysis of time rates of change of temperatures in the tank of water at the Scripps Institution during the noon hour.

I. Effect of solar radiation and turbulence.
II. Effect of downward diffusion of surface cooling.
III. Effect of upward diffusion of heat absorbed at the bottom.
IV. Sum of curves I, II, and III equals the theoretical rate of temperature change.
$\boxed{\cdot}$ Observed rate of temperature change.
V. Graph of difference between the observed temperature and the constant 25°05.
VI. Graph parallel to V drawn to compute the effect of solar radiation and turbulence represented by I.

determined by plotting the values of $(\theta^\circ - 25^\circ 05)$ on semilogarithmic paper against depths as abscissae. The corresponding value of (a) equals (0.23) from which column 7 of table 4 was computed. The value of (h) was found by trial to be 1500 and $\dfrac{1}{A_1\mathbf{B}} = 1.13 \times .00026 = .000294$. Neglecting the bottom effect on the upper layers, the variable parts, column 8 and column 12 of equation (121) were plotted on cross-section paper.

The values $K=2.1$ and $M_1=5.1$ were estimated from this graph. Then column 16 was computed from column 15 and table 2, using the value $h^1=25000$ found by trial. The variable parts, columns 14 and 18 of equation (122), were then plotted on cross-section paper and the values $K^1=0.46$ and $M_1=5.1$ computed. The corresponding value of (M_2) was found to be (.024) from the expression $\frac{M_1 a e^{-ay_b}}{1-e^{-ay_b}}$. Line VI of figure 8 was drawn parallel to line V through the ordinate $\frac{M_1 a}{1-e^{-ay_b}}$ $=5.1\times.2347=1.19$ corresponding to zero depth. The ordinates of line VI are equal to the product of 5.1 by the entries in column 7 of table 4.

In figure 8 the effect of solar radiation and turbulence is represented by line I, plotted from the product of 5.1 by the entries in column 7. The cooling effect of surface evaporation, convection through the air, and back radiation, is represented by line II, plotted from column 20 and the heating effect of radiation absorbed by the bottom is represented by line III, plotted from column 19. The sum of these three effects is represented by line IV which is therefore a graph of the theoretical values of $\frac{\partial\theta}{\partial t}$. The points $\boxed{\cdot}$ represent the observed values and agree well with the theoretical curve.

The results of similar computations for several hours of the day, as well as theoretical and observed evaporation rates, are summarized in table 5. The values of $\frac{K}{6}$ entered in column 12 would be the theoretical evaporation rates if the whole surface cooling effect were due to evaporation. They were first corrected for convection through the air by means of values of the Bowen ratio R^1, (Bowen, 1926) and entered in column 14. The back radiation estimated from the surface-water temperature was found according to equation (83) to be about 6 per cent, accordingly, column 14 multiplied by 94 per cent gives the theoretical evaporation rates entered in column 15 which are seen to agree well with the directly observed values in column 16. The average of the five theoretical values is .272 which is only 6 per cent greater than .254, the average of the observed values.

Numerical Applications to Serial Temperatures and Salinities of the Pacific Ocean in the Coronado Island Region, near San Diego, California. Computation of Upwelling Velocity

Observations taken at irregular intervals in the deep-water region a few miles west of the Coronado Islands (Michael and McEwen, 1915, 1916) have been used as a basis for obtaining monthly averages of temperature and salinity. Owing to the variation in conditions from one year to another and the unequal distribution of the observations within a year, these averages based upon an eight-year period can hardly be expected to give more than a rough idea of the typical seasonal cycle. However, since these observations have been used before (McEwen, 1918) as a basis for theoretical estimates of turbulence and rates of upwelling, they were selected to illustrate the new theory

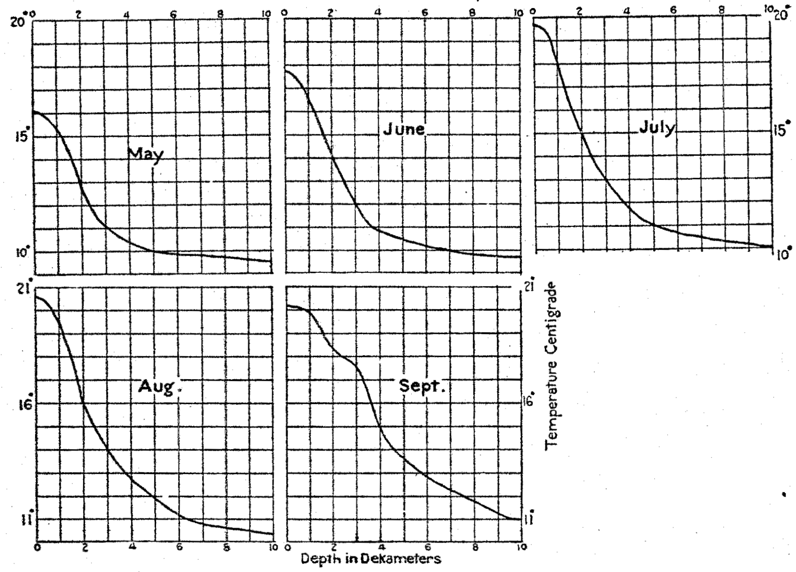

Fig. 9.—Relation to depth of monthly averages of serial ocean temperatures near San Diego.

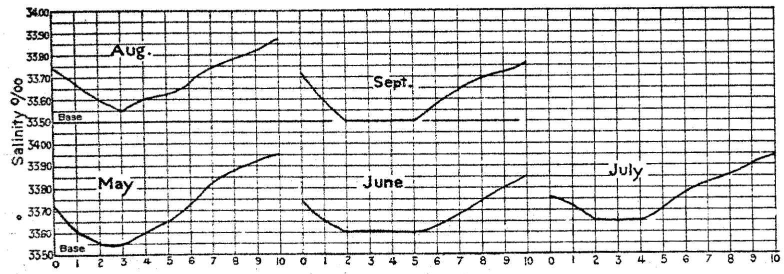

Fig. 10.—Relation to depth of monthly averages of serial ocean salinities near San Diego.

in order that results might be compared with the older, less fundamental treatment. During the past ten years many observations have been made in a region about thirty miles north of the Coronado Island region. These observations were quite regularly distributed throughout the period from spring to autumn and were made frequently enough to indicate the monthly change during each year. The interpretation of these observations and an extensive accumulation of suitable lake and reservoir observations will be undertaken later with the aid of available dynamical methods.

The monthly averages of temperature and salinity in the Coronado Island region used as a basis for the following computations are shown in figures 9 and 10. The computations for the month of July, made in accordance with the explanation on pages 223 to 229, are presented in table 6 and figure 11. The entries in columns 3 and 4 are proportional to densities and differences in density and were computed from the temperature and salinity according to Knudsen, (1901).

The values of $\frac{10^5}{B}$ entered in column 5 were computed with the aid of table 10, as explained on pages 301 to 304. The following computation for the depth (5) illustrates the process. In table 10, part 1, under $\theta_o = 19°$ are entered the values of $10^5(\sigma - \sigma_o)$ corresponding to temperature reductions $(\theta_o - v) = 0$, 1, 2, etc. entered at the left and right. Correct these values of $10^5(\sigma - \sigma_o)$ to correspond to $\theta_o = 19.°7$ of table 6 by adding the increments entered in table 10, part 3 under $\Delta\theta_o = 0.7$. Assume $D = 4$, which corresponds to $\lambda = .95$, and find the corresponding corrections to $10^5(\sigma - \sigma_o)$ in table 10, part 3, under the value (4). Thus the corrected values of $10^5(\sigma - \sigma_o)$ are 0, $25.1 + .7 + 4.0 = 29.8$, $48.7 + 1.4 - 8.0 = 58.1$, etc. Also in table 10, part 1, under $\theta_o = 19°$ are entered the values of $\left(\frac{10^5}{B}\right)$ corresponding to temperature reductions $(\theta_o - v) = 0$, 1, 2, etc., entered at the left and right. These values of $\left(\frac{10^5}{B}\right)$ are corrected to the temperature $\theta_o = 19.7$ and $D = 4$ by means of part 2, of table 10. The corrected values are $25.5 + 0.7 + 4.0 = 30.2$, $25.1 + 0.7 + 4.0 = 29.8$, $24.3 + 0.7 + 4.0 = 29.0$, etc. This correction is nearly a constant, (4.7) for all values of $(\theta_o - v)$. After plotting these corrected values of $\left(\frac{10^5}{B}\right)$ against the corrected values of $10^5(\sigma - \sigma_o)$, read off from the graph the values of $\left(\frac{10^5}{B}\right)$ corresponding to the values of $10^5(\sigma - \sigma_o)$ entered in column 4 of table 6, thus obtaining column 5. Corresponding values of $(\sigma_o - v)$ can be obtained in the same way and are 0, 1.25, 3.7, 5.3, etc. As a check, any entry in column 4 divided by the corresponding value of $(\sigma_o - v)$ should equal $\left(\frac{10^5}{B}\right)$. The results may also be checked with the aid of the formula of page 301 which in this case becomes

$$10^5(\sigma - \sigma_o) = f(\theta_o - v) + 4(\theta_o - v).$$

Line V of figure 11 was plotted as explained on pages 224 and 226. The values found for the constants in equation (85) are $C = 9.5$, $C_1 = 4.2$, $a = .202$ (for $y > 6$), and $a = .359$ (for $y < 6$).

TABLE 6

Computation of the constants in the simplified approximate equation including the upwelling velocity, from July data on temperature and salinity in the Coronado Island region

y	(1) θ	(2) S	(3) $(\sigma-1)10^5$	(4) $10^5(\sigma-\sigma_o)$	(5) $\frac{10^5}{B}$	(6) $(\theta-C)$	(7) $\frac{\partial\theta}{\partial t}$	(8) $\frac{\partial\theta}{\partial y}$
0	19.7	33.75	2390.7	0	30.2	10.2	1.61	− .306
1	18.2	.72	2426.7	36.0	29.6	8.7	2.31	−3.49
2	15.0	.65	2495.0	104.3	28.2	5.5	.974	−2.41
3	13.0	.65	2536.4	145.7	27.7	3.5	.804	−1.45
4	11.8	.65	2559.4	168.7	27.2	2.3	1.10	− .98
5	11.0	.70	2577.8	187.1	26.8	1.5	.804	− .51
6	10.7	.77	2588.6	197.9	26.6	1.2	.526	− .36
7	10.5	.82	2596.0	205.3	26.4	1.0	.414	− .27
8	10.3	.85	2601.7	211.0	26.3	0.8	.392	− .16
9	10.2	.90	2607.3	216.6	26.2	0.7	.326	− .13
10	10.0	.94	2614.2	223.5	26.1	0.5	.260	− .11

y	(9) $\frac{ae^{-ay}}{1-e^{-10a}}$ $a=.359$	$a=.206$	(10) $f(y)\frac{\partial\theta}{\partial y}$ for $D=7.5$	(11) $\frac{Tae^{-ay}}{1-e^{-10a}}$ from graph	(11′) Should = (11) $T(9)=15.46\times(9)$ for $a=.206$	$T(9)=24.63(9)$ for $a=.36$	(12) 1.5 (10)
0	.370	....	0	9.1	....	9.12	0
1	.258	....	−1.39	6.3	....	6.35	−2.082
2	.180	....	−1.71	4.4	....	4.44	−2.563
3	.126	....	−1.32	3.1	....	3.10	−1.980
4	.088	....	−0.98	2.15	....	2.17	−1.470
5	.061	....	−0.51	1.50	....	1.50	− .765
6	.043	.069	−0.36	1.05	1.061	1.06	− .540
7	....	.056	−0.27	.85	.863	....	− .405
8	....	.045	−0.16	.70	.701	....	− .240
9	....	.037	−0.13	.57	.572	....	− .195
10	....	.030	−0.11	.46	.465	....	− .165

y	(13) $\frac{\partial\theta}{\partial t}-(12)$	(14) (13) −(11′)	(15) $h\int_z^\infty \frac{e^{-x^2}}{x}$	(16) $\frac{\partial\theta}{\partial y}(15)$	(17) $.00029\frac{\partial\theta}{\partial y}(15)$	(18) 3.8(17)	(19) (11′) +(18) +(12)
0	1.61	−7.51	....	....	....	....	....
1	4.39	−1.96	469.2	−1633.0	−.482	−1.83	2.44
2	3.54	− .90	261.3	−630.0	−.140	−.54	1.34
3	2.78	− .32	197	−286.0	−.083	−.32	.80
4	2.57	.40	171	−168.0	−.049	−.19	.51
5	1.57	.07	153	− 78.0	−.023	−.09	.65
6	1.07	.01	143	− 51.5	−.015	−.06	.56
7	.82	− .043	137	− 37.0	−.011	−.04	.42
8	.63	− .071	133	− 21.3	−.006	−.02	.44
9	.52	− .052	128	− 16.6	−.005	−.02	.35
10	.42	− .045	124	− 13.6	−.004	−.02	.28

The coefficients in equation (47) are

$$\sum_{y=5}^{y=10}\frac{\partial\theta}{\partial t}=2.722,\ \sum_{y=5}^{y=10}f(y)\frac{\partial\theta}{\partial y}=-1.54,$$

$$\sum_{y=5}^{y=10}\frac{ae^{-ay}}{1-e^{-10a}}=.3258,\text{ and }\frac{a}{1-e^{-10a}}=.2329$$

for $a=.202$. Equation (47) therefore becomes $T=8.36+4.73W_1$.

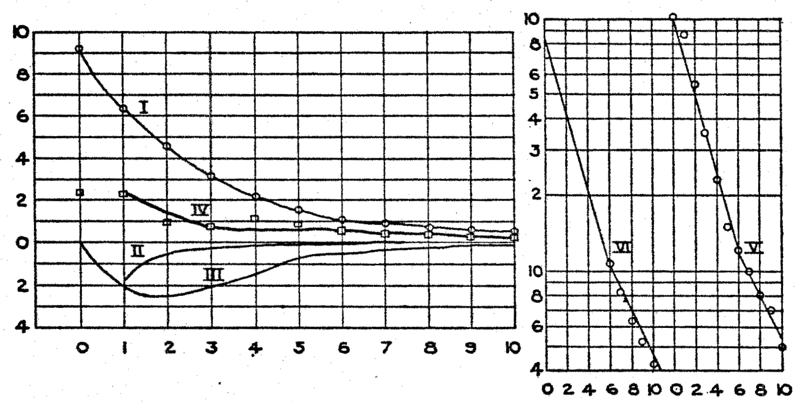

Fig. 11.—Analysis of time rate of change of temperatures in the Pacific Ocean near San Diego during July.

I. Effect of solar radiation and turbulence.
II. Effect of downward diffusion of surface cooling.
III. Effect of upwelling.
IV. Sum of curves I, II, and III equals the theoretical rate of temperature change.
$\boxed{\cdot}$ Observed rate of temperature change.
V. Graph of difference between observed temperature and the constant 9°.5.
VI. Graph parallel to V drawn to compute the effect of solar radiation and turbulence represented by I.

Substituting in equation (88) the value of $\mu^2C_1a^2$ from equation (98) results in

$$\frac{\partial\theta}{\partial t}=\frac{A_2}{A_1}P_1(z)\frac{\partial\theta}{\partial y}+W_1f(y)\frac{\partial\theta}{\partial y}+\frac{Ta}{1-e^{-ay_1}}e^{-ay} \qquad (134)$$

where, in this case $y_1=10$.

Substituting the expression $(a_1+a_2W_1)$ for (T) where the coefficients a_1 and a_2 are known from equation (97) we have

$$\frac{\partial\theta}{\partial t}=\frac{hK}{A_1}P_1(z)\frac{\partial\theta}{\partial y}+W_1f(y)\frac{\partial\theta}{\partial y}+\frac{(a_1+a_2W_1)}{1-e^{-ay_1}}aTe^{-ay} \qquad (135)$$

which takes the form

$$\left\{\ \right\}_1 = \left\{\ \right\}_2 W_1 + \frac{K}{A_1}\left[\frac{1}{\mathbf{B}} h \int_z^{\infty} \frac{e^{-x^2}}{x} dx\right]\frac{\partial \theta}{\partial y} \tag{136}$$

where the last term can be neglected for values of (y) greater than about (5), the coefficients $\left\{\ \right\}_1$ and $\left\{\ \right\}_2$ are known, and $\left(\frac{1}{\mathbf{B}}\right)$ is a mean value for $\left(\frac{1}{B}\right)$. Thus a trial value (1.5) was found for (W_1) using .00029 for $\left(\frac{1}{\mathbf{B}}\right)$.

The corresponding value of (T) is 15.46, and line VI of figure 11 is accordingly drawn through the point $y=0$, $15.46 \times .2329 = 3.60$ parallel to the lower part of line V. Since the slope of line V changes at the depth (6) the slope of line VI changes there also. The entries in column 11 of table 6 were read off from this graph. The entries of column 11′ were computed for $y \geqq 6$ as indicated in the table using the value 15.46 for (T). The value 24.63 of (T) for $y=6$ was found from the fact that at the depth (6), $15.46 \times .0686 = 1.06 = 24.63 \times .043$, referring to column 9. The computed entries in column 11′ should equal those in column 11 read off from the graph. Using the value $h=200$ found by trial, the entries in column 15 were computed from column 4 with the aid of table 2. The corresponding value of $\frac{K}{A_1}$ in equation (153) was found to be (3.8) from which the entries in column 18 were computed.

In figure 11 curve I representing the effect of conduction was plotted from column 11′, curve II representing the surface loss of heat was plotted from column 18, curve III representing the effect of upwelling was plotted from column 12, and curve IV the theoretical value of $\frac{\partial \theta}{\partial t}$ represents the sum of the curves I, II, and III; the numerical values are entered in column 19. The points representing the observed values of $\frac{\partial \theta}{\partial t}$ agree well with the theoretical curve.

The result of similar computations for each of the five months from May to September are summarized in table 7. The coefficient of turbulence (μ^2) is less between the surface and depth (6) dekameters than between 6 and 10 dekameters. This accords with what would be expected from the relatively greater stratification in the upper levels. The rates of solar radiation at the surface measured by a pyranometer in the same general region have been reduced to evaporating power in cm. per month and entered for comparison. They are seen to be of the same order as the theoretical ones estimated from ocean temperatures.

The mean annual rate of evaporation from the sea at the latitude of San Diego has been estimated to be 7.5 cm. per month (Schmidt, 1915, p. 121) and 9.2 cm. per month (Wüst, 1920, p. 83). These values are consistent with the theoretical rates for summer months, entered in the last column of table 7. Their average (13.2) should exceed the mean annual value.

The average upwelling velocity (W_1) was found to be 1.5 dekameters per month. Earlier estimates (McEwen, 1919) from the same data,

TABLE 7

Summary of the results of applying the simplified approximate equation including the upwelling velocity to data on temperature and salinity in the Coronado Island region

Month	C	h	$\frac{K'}{A'}$	K	$\int_0^{10} \frac{\partial\theta}{\partial t} dy = N_1$	W_1	$-\int_0^{10} W_1 f(y) \frac{\partial\theta}{\partial y} dy = N_2$	Radiation $= (N_1+N_2+K) = R_o$
May	9.4	400	12.2	10.8	3.63	2	10.08	24.5
June	9.6	500	15.5	13.8	6.66	1	5.94	26.4
July	9.5	200	3.8	3.5	8.56	1.5	10.32	22.3
Aug.	10.15	250	14.7	13.0	10.61	2	13.56	37.2
Sept.	10.3	500	8.8	7.8	13.22	1	7.48	28.5

Month	$\frac{R_1 a}{1-e^{-ay_1}}$		a		C_1		a^2C_1		μ^2		Evaporating power of total loss of heat from the surface $= \frac{10K}{6}$
	$y>6$	$y<6$	$y>6$	$y<6$	$y>6$	$y<6$	$y>6$	$y<6$	$y>6$	$y<6$	
May	3.6	12.0	.347	.575	3.7	9.0	.445	2.98	8.1	4.0	18.0
June	5.2	20.0	.401	.684	6.0	16.0	.965	7.5	5.4	2.7	23.0
July	3.6	9.1	.202	.359	4.1	10.0	.168	1.29	21.4	7.0	5.8
Aug.	22		.434		14.0		2.64		8.3		21.6
Sept.	12.5		.296		14.0		2.64		10.2		13.0

	Evaporating power of theoretical penetrating radiation $= \frac{10R}{6}$	Evaporating power of observed incident radiation at—			Theoretical evaporation assuming the Bowen ratio to be .22
		Pasadena	Murray Lake in San Diego	Scripps Institution, La Jolla	
May	40.9				14.8
June	44.0	31.4			18.9
July	37.2		33.1	28.9	4.8
Aug.	62.0				17.7
Sept.	47.5				10.7

but obtained by a less fundamental treatment that involved wind velocities, were about twice as large. As stated before, these observations are not so well suited to the application of such methods as later, more continuous and regularly distributed ones, and the results are probably only rough approximations.

SUPPLEMENTARY EXAMINATION OF RESULTS AND SUGGESTIONS RELATIVE TO FURTHER APPLICATIONS OF THE THEORY

A theory of vertical gradients of temperature and concentration of dissolved substances has been formulated, using as a basis the sinking of small relatively heavy surface masses of water and a compensating ascent of lighter masses. The increase of density at the surface caused by evaporation, conduction, and convection through the air, and back radiation, suggested the idea of deducing the behavior of a mechanism involving a vertical interchange of water masses having different properties. The mathematical theory of such a mechanism of the downward diffusion of surface cooling led to a pair of simultaneous differential equations expressing causal relations between turbulence, rate of surface cooling, rate at which solar radiation penetrates the surface, vertical distribution of temperature and salinity, and the rate of vertical flow of the water.

These equations express the result of a statistical study of a multitude of small (not molecular) units or individual masses and apply to the *gross* or resultant behavior of the system in which the behavior of a single element is unknown. This basic idea is familiar to all students of statistical mechanics which is proving so essential in modern theoretical physics and chemistry.

Methods have been worked out for applying these equations to numerical data, but their general solution has not been attempted. Four integrals appearing in these equations and depending upon the vertical variation of specific gravity have been tabulated to facilitate the application of these equations.

Numerical results have been obtained by applying the theory to serial temperatures of Lake Mendota, Wisconsin, to serial temperatures in a tank of water, and to serial temperatures and salinities in the Pacific Ocean near San Diego, California. From such observations the theory provided a means of estimating the rate of penetration of solar radiation through the water surface, the rate of surface cooling, and, therefore, an estimate of evaporation, and the rate of vertical flow of the water. The upwelling velocity during summer in the Pacific near San Diego was thus estimated to be about fifteen meters per month, and the rate of penetration of solar radiation thus found was in good agreement with results obtained by independent methods.

By applying the same mechanism of downward diffusion to the distribution of salinity, and combining the resulting equations with those pertaining to temperature, a method has been worked out for estimating separately the surface cooling due to evaporation and other causes. This method of determining the rate of evaporation from the ocean solely by means of serial observations of temperature and salinity down to the hundred-meter level, while theoretically possible, has not yet been applied. A more practicable method is presented which involves much less computation but requires wet- and dry-bulb temperatures of the air as well as serial observations of the water in order to estimate the "Bowen ratio," or ratio of loss of heat by conduction and convection through the air to that lost by evaporation. This simpler method should yield reasonably accurate results. A simpler approximate method is also presented of deducing ocean evaporation from the rate of surface cooling, computed from serial observations, and supplementary observations on an evaporating pan containing sea water.

Numerical or graphical integration of the equations after the various physical magnitudes have been found as indicated above should reproduce the subsequent changes in temperatures and salinities from their initial values. Details of how such computations may be made are presented.

Applying the theory to numerous observations representative of widely different conditions has produced results consistent with other investigations. In addition to yielding numerical results, as accurate as are usually found in "field" problems where the complexity of the phenomena precludes great precision, an accuracy approaching that of the physical laboratory resulted from the small-scale tank experiment.

The amount of labor involved in making the computations may at first appear unreasonably great, but the phenomena are complex. The formulation of the theory is worth while if, as already indicated, it agrees reasonably well with facts. And the work of facilitating its application can doubtless be carried much farther than has been done. Moreover, as information accumulates about the values of the different physical constants, the trial and error process of computing will be shortened.

In addition to applying the theory to available data, as was done in the illustrative examples, and thus determining the various physical constants under widely different conditions, other problems whose solution requires a treatment of the general type presented in this paper are outlined below.

1. Determination of "normal" temperatures and salinities, that is, values corresponding to the latitude and time of year in the absence of a general flow of the water. After computing the constants of the temperature equation (equation 84) proceed to predict the temperatures and salinities as explained on pages 234 to 237, assuming the velocity W_1 to be zero, and thus estimate normal values.

2. Determination of the concentration at different depths and times of various chemicals from given external conditions and the physical laws of diffusion independent of organic influences. Comparison of the actual distribution which is determined not only by inorganic conditions and laws, but also by living organisms that may either consume or produce the particular substance in question, should result in information regarding the interrelation between these organisms and their environment.

3. A great deal of attention has been given, especially within the past twenty years, to problems of turbulence in the atmosphere. Great progress has resulted in our knowledge of frictional resistance, diffusion, conduction, and the laws of the variation of wind velocity with height. It may be that the theory of this paper will find application to the lower part of the atmosphere overlying relatively warmer sea or land surfaces, and supplement general theories already developed. Thus it may contribute to our knowledge of this zone of contact between the atmosphere and the surface of the earth.

The development of this theory of vertical velocity and the distribution of temperature and salinity in the ocean was suggested during the preparation of an earlier, more empirical paper (McEwen, 1918). Also in developing this more fundamental theory, accompanying studies of the less complicated phenomena in fresh-water lakes and reservoirs proved to be an invaluable aid.

Methods of determining oceanic circulation from forces producing it use as a basis a space distribution of simultaneous observations. For example, in applying the Bjerknes dynamical method, a set of serial observations is required corresponding as nearly as possible to the same time. Similarly the circulation computed from wind velocity according to Ekman's theory occurs simultaneously with the wind which is assumed to continue until a steady condition is reached.

On the other hand, the method derived in this paper for computing circulation from its effect upon the vertical distribution of temperature and other properties of the water requires the repetition at successive time intervals of a vertical series of observations. By means of the Bjerknes dynamical method, circulation is inferred from the space distribution of temperature and salinity. The method of this paper infers changes in the distribution of temperature and salinity due to circulation. Consequently there is a possibility of combining the two methods and thus predicting the successive changes that follow given initial conditions.

According to plans already formulated, oceanographical and meteorological observations in the less disturbed region of the North Pacific high-pressure area will be available for applications of this theory. Such studies should prove helpful in the search for indices of use in investigations of long-range weather forecasting.

I am greatly indebted to my colleague, Dr. N. W. Cummings of the San Bernardino Union Junior College for helpful suggestions in preparing the final draft of the manuscript. It is a pleasure to acknowledge my obligation to Dean B. M. Woods of the University of California for his critical reading of the completed manuscript and for his editorial assistance. I also wish to express my appreciation to Dr. A. F. Gorton, Associate in Meteorology at the Scripps Institution of Oceanography, for assistance in reading the proofs and for final suggestions.

APPENDIX

TABULATIONS OF FUNCTIONS NEEDED IN NUMERICAL APPLICATIONS OF THE EQUATIONS. (THE FREQUENCY FUNCTION $F_1(x)$ IS ASSUMED TO BE e^{-x^2}.)

General Statement and Table 1 of the Integral, $\int_z^\infty \frac{e^{-x^2}}{x}dx$

Tabulated values of certain functions involved in the various equations of temperature and other properties of the water must be available in order to make numerical applications. These are tabulated here in a form and to a degree of accuracy suitable for such applications. A brief statement of the methods used and the precision reached in the computations is given for each table.

With the aid of the equation

$$\int_z^\infty \frac{e^{-x^2}}{x}dx = \frac{1}{2}\int_{z^2}^\infty \frac{e^{-x}}{x}dx \tag{1}$$

formulae and tables of the integral in the second member can be used for computing table 1 of the integral, $\int_z^\infty \frac{e^{-x^2}}{x}dx$

According to Glaisher (1870, pp. 367-369) the function

$$Ei(x) = \int_\infty^{-x} \frac{e^{-u}}{u}du = .57721 + \frac{1}{4}\log_e x^4 + x + \frac{1}{2}\frac{x^2}{1\cdot 2} + \frac{1}{3}\frac{x^3}{\underline{|3}} + \frac{1}{4}\frac{x^4}{\underline{|4}} + \dots \tag{2}$$

$$Ei(x) = e^x\left\{\frac{1}{x} + \frac{1}{x^2} + \frac{1\cdot 2}{x^3} + \frac{\underline{|3}}{x^4} + \frac{\underline{|4}}{x^5} + \dots\right\} \tag{3}$$

Therefore interchanging the limits and substituting $(-z)$ for (x) gives the exponential integral

$$-Ei(-z) = \int_z^\infty \frac{e^{-x}}{x}dx = -\log_e z + \Big[-.5772 + Z - \frac{Z^2}{2\underline{|2}} + \frac{Z^3}{3\underline{|3}} - \frac{Z^4}{4\underline{|4}} + \frac{Z^5}{5\underline{|5}} - \dots\Big] \tag{4}$$

and

$$-Ei(-z) = \left[\frac{1}{Z} - \frac{1}{Z^2} + \frac{\underline{|2}}{Z^3} - \frac{\underline{|3}}{Z^4} + \frac{\underline{|4}}{Z^5} - \frac{\underline{|5}}{Z^6} + \dots\right]e^{-z} \tag{5}$$

Equation (3) is adapted to small values of (z). Equation (5) is adapted to large values of (z). This function is tabulated (Glaisher, 1870, p. 385) for values of (z) increasing by intervals of 0.1 from 1.0 to 5.0, and intervals of 1 from 5.0 to 15.0. On page 380 the function is tabulated for values of (Z) increasing by intervals of .01 from .01 to 1.0. In a later paper (Miller and Rosebrugh, 1903, pp. 80–81) the function

$$B=\int_z^\infty \frac{e^{-x}}{x}dx+\log_e Z \tag{6}$$

is tabulated for values of (z) increasing by intervals of .001 from .001 to 0.100 and the integral $\int_z^\infty \frac{e^{-x^2}}{x}dx$ is tabulated for values of (z) increasing by intervals of .001 from 0.100 to 1.000. For values of (z) between 1.00 and 2.00 the interval is .01. According to equation (3), (B) can be computed from the series

$$B=\left[-.5772+Z-\frac{z^2}{2\,\underline{|2}}+\frac{z^3}{3\,\underline{|3}}-\frac{z^4}{4\underline{|4}}+\frac{z^5}{5\underline{|5}}-\ \ldots\right] \tag{7}$$

From equations (1) and (6) we get

$$\int_z^\infty \frac{e^{-x^2}}{x}dx=-\log_e z+\frac{B}{z} \tag{8}$$

where (B) corresponds to (z^2)

and

$$\int_z^\infty \frac{e^{-x^2}}{x}dx=-\log_e z+\left[-.28861+\frac{z^2}{2}-\frac{z^4}{8}+\frac{z^6}{36}-\frac{z^8}{192}+\ \ldots\right] \tag{9}$$

which are suitable for values of (z) not too large. From equations (1) and (5) we get

$$\int_z^\infty \frac{e^{-x^2}}{x}dx=\frac{1}{2}\left[+\frac{1}{z^2}-\frac{1}{z^4}+\frac{\underline{|2}}{z^6}-\frac{\underline{|3}}{z^8}+\frac{\underline{|4}}{z^{10}}-\frac{\underline{|5}}{z^{12}}\right]e^{-z^2} \tag{10}$$

which is adapted to large values of (z).

Table 1 was first computed by using well-known methods of numerical integration and interpolation in connection with tables of the exponential function (Fowle, 1914, pp. 48–53) and the normal probability integral (Pearson, 1914, pp. 1–11). Later, after having access to tables of the exponential integral, $-E(-z)$, independent computations of table 1 were made which revealed a maximum error of .001

in the previous results. Tables of the exponential function and natural logarithms (Hayashi, 1926) were very useful for making these computations which were carried to four decimal places for this paper.

Pearson's tables of the incomplete gamma function $I(u, \rho)$ and certain of its expansions in series were especially useful for making computations involving the more general frequency function $\frac{e^{-x^2}}{x^a}$, since it can easily be shown that

$$\int_z^\infty \frac{e^{-x^2}}{x^{1+a}}dx=\frac{1}{a}\left\{\frac{e^{-z^2}}{z^a}-\Gamma(\rho+1)[1-I(u,\rho)]\right\} \tag{11}$$

where $\rho=-\frac{a}{z}$ and $u=\frac{z^2}{\sqrt{1+\rho}}$. However, experience in numerous applications has shown that the normal frequency function e^{-x^2} is a suitable value of $F_1(x)$. Accordingly this value was assumed in computing these tables.

TABLE 1

z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$
.001	6.6191	.025	3.4005	.049	2.7297	.073	2.3320
.002	5.9255	.026	3.3640	.050	2.7087	.074	2.3183
.003	5.5204	.027	3.3277	.051	2.6896	.075	2.3046
.004	5.2330	.028	3.2913	.052	2.6706	.076	2.2917
.005	5.0097	.029	3.2549	.053	2.6615	.077	2.2788
.006	4.8276	.030	3.2185	.054	2.6325	.078	2.2659
.007	4.6732	.031	3.1877	.055	2.6134	.079	2.2530
.008	4.5399	.032	3.1569	.056	2.5961	.080	2.2402
.009	4.4220	.033	3.1261	.057	2.5787	081	2.2281
.010	4.3166	.034	3.0953	.058	2.5613	.082	2.2160
.011	4.2229	.035	3.0646	.059	2.5440	.083	2.2040
.012	4.1334	.036	3.0379	.060	2.5266	.084	2.1919
.013	4.0541	.037	3.0112	.061	2.5107	.085	2.1899
.014	3.9800	.038	2.9845	.062	2.4947	.086	2.1686
.015	3.9110	.039	2.9578	.063	2.4788	.087	2.1574
.016	3.8456	.040	2.9311	.064	2.4628	.088	2.1462
.017	3.7860	.041	2.9076	.065	2.4468	.089	2.1350
.018	3.7286	.042	2.8842	.066	2.4321	.090	2.1237
.019	3.6749	.043	2.8607	.067	2.4173	.091	2.1130
.020	3.6236	.044	2.8373	.068	2.4026	.092	2.1022
.021	3.5790	.045	2.8138	.069	2.3878	.093	2.0915
.022	3.5343	.046	2.7928	.070	2.3737	.094	2.0807
.023	3.4898	.047	2.7717	.071	2.3594	.095	2.0699
.024	3.4451	.048	2.7507	.072	2.7453	.096	2.1598

TABLE 1 (*Continued*)

z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x}dx$
.097	2.0496	.145	1.6533	.193	1.3751	.61	.3764
.098	2.0394	.146	1.6466	.194	1.3701	.62	.3655
.099	2.0292	.147	1.6390	.195	1.3651	.63	.3546
.100	2.0190	.148	1.6331	.196	1.3602	.64	.3437
.101	2.0094	.149	1.6264	.197	1.3553	.65	.3327
.102	1.9999	.150	1.6197	.198	1.3504	.66	.3234
.103	1.9903	.151	1.6134	.199	1.3455	.67	.3140
.104	1.9807	.152	1.6071	.200	1.3406	.68	.3046
.105	1.9711	.153	1.6008	.21	1.2983	.69	.2952
.106	1.9618	.154	1.5945	.22	1.2558	.70	.2858
.107	1.9527	.155	1.5882	.23	1.2135	.71	.2776
.108	1.9434	.156	1.5819	.24	1.1710	.72	.2695
.109	1.9342	.157	1.5756	.25	1.1287	.73	.2613
.110	1.9249	.158	1.5694	.26	1.0948	.74	.2531
.111	1.9161	.159	1.5631	.27	1.0607	.75	.2450
.112	1.9074	.160	1.5568	.28	1.0272	.76	.2379
.113	1.8987	.161	1.5507	.29	.9933	.77	.2308
.114	1.8898	.162	1.5447	.30	.9594	.78	.2238
.115	1.8811	.163	1.5385	.31	.9314	.79	.2167
.116	1.8727	.164	1.5325	.32	.9037	.80	.2096
.117	1.8642	.165	1.5265	.33	.8760	.81	.2035
.118	1.8558	.166	1.5208	.34	.8483	.82	.1974
.119	1.8473	.167	1.5150	.35	.8207	.83	.1913
.120	1.8389	.168	1.5092	.36	.7974	.84	.1851
.121	1.8309	.169	1.5035	.37	.7742	.85	.1790
.122	1.8227	.170	1.4978	.38	.7510	.86	.1737
.123	1.8146	.171	1.4922	.39	.7277	.87	.1683
.124	1.8065	.172	1.4866	.40	.7045	.88	.1630
.125	1.7983	.173	1.4810	.41	.6857	.89	.1576
.126	1.7907	.174	1.4754	.42	.6669	.90	.1523
.127	1.7830	.175	1.4699	.43	.6481	.91	.1477
.128	1.7754	.176	1.4643	.44	.6293	.92	.1431
.129	1.7677	.177	1.4588	.45	.6105	.93	.1386
.130	1.7599	.178	1.4533	.46	.5928	.94	.1340
.131	1.7525	.179	1.4477	.47	.5751	.95	.1294
.132	1.7451	.180	1.4423	.48	.5573	.96	.1254
.133	1.7376	.181	1.4369	.49	.5396	.97	.1214
.134	1.7302	.182	1.4315	.50	.5219	.98	.1175
.135	1.7227	.183	1.4262	.51	.5074	.99	.1135
.136	1.7156	.184	1.4208	.52	.4929	1.00	.1095
.137	1.7085	.185	1.4155	.53	.4784	1.01	.1061
.138	1.7013	.186	1.4103	.54	.4639	1.02	.1027
.139	1.6943	.187	1.4052	.55	.4494	1.03	.0993
.140	1.6872	.188	1.4001	.56	.4370	1.04	.0960
.141	1.6804	.189	1.3950	.57	.4246	1.05	.0926
.142	1.6736	.190	1.3900	.58	.4122	1.06	.0896
.143	1.6668	.191	1.3850	.59	.3998	1.07	.0866
.144	1.6600	.192	1.3800	.60	.3873	1.08	.0837

TABLE 1 (*Continued*

z	$\int_z^\infty \frac{e^{-x^2}}{x} dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x} dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x} dx$	z	$\int_z^\infty \frac{e^{-x^2}}{x} dx$
1.09	.0807	1.42	.0239	1.75	.0059	2.08	.0012
1.10	.0777	1.43	.0230	1.76	.0057	2.09	.0011
1.11	.0752	1.44	.0221	1.77	.0055	2.10	.0010
1.12	.0728	1.45	.0211	1.78	.0052	2.11	.0010
1.13	.0703	1.46	.0204	1.79	.0050	2.12	.0009
1.14	.0678	1.47	.0196	1.80	.0048	2.13	.0009
1.15	.0654	1.48	.0188	1.81	.0046	2.14	.0008
1.16	.0632	1.49	.0181	1.82	.0043	2.15	.0007
1.17	.0611	1.50	.0173	1.83	.0041	2.16	.0007
1.18	.0589	1.51	.0166	1.84	.0039	2.17	.0007
1.19	.0567	1.52	.0160	1.85	.0037	2.18	.0007
1.20	.0546	1.53	.0153	1.86	.0035	2.19	.0006
1.21	.0527	1.54	.0147	1.87	.0034	2.20	.0006
1.22	.0509	1.55	.0140	1.88	.0032	2.21	.0006
1.23	.0491	1.56	.0135	1.89	.0031	2.22	.0006
1.24	.0473	1.57	.0130	1.90	.0029	2.23	.0005
1.25	.0454	1.58	.0125	1.91	.0028	2.24	.0005
1.26	.0439	1.59	.0120	1.92	.0027	2.25	.0005
1.27	.0423	1.60	.0115	1.93	.0026	2.26	.0005
1.28	.0408	1.61	.0110	1.94	.0024	2.27	.0005
1.29	.0393	1.62	.0106	1.95	.0023	2.28	.0004
1.30	.0377	1.63	.0101	1.96	.0022	2.29	.0004
1.31	.0364	1.64	.0097	1.97	.0021	2.30	.0004
1.32	.0351	1.65	.0092	1.98	.0020	2.31	.0004
1.33	.0338	1.66	.0088	1.99	.0019	2.32	.0004
1.34	.0325	1.67	.0085	2.00	.0018	2.33	.0003
1.35	.0312	1.68	.0081	2.01	.0017	2.34	.0003
1.36	.0301	1.69	.0078	2.02	.0016	2.35	.0003
1.37	.0290	1.70	.0074	2.03	.0015	2.36	.0003
1.38	.0280	1.71	.0071	2.04	.0015	2.37	.0003
1.39	.0269	1.72	.0068	2.05	.0014	2.38	.0002
1.40	.0258	1.73	.0065	2.06	.0013	2.39	.0002
1.41	.0249	1.74	.0062	2.07	.0012	2.40	.00025

TABLE 2

Tabulation of $h\int_{z}^{\infty}\frac{e^{-x^2}}{x}dx$ with respect to $(\sigma_y-\sigma_o)$, where $z=h(\sigma_y-\sigma_o)$

For convenience in preliminary trial computations the function $h\int_{z}^{\infty}\frac{e^{-x^2}}{x}dx$ has been tabulated with respect to the difference $(\sigma_y-\sigma_o)$. The computations were based upon table 1 and the error is about one part in a thousand.

$(\sigma_y-\sigma_o)$ \ h	100	200	300	400	500	600	700	800
.00001	661.9	1185	1656	2093	2505	2896	3271	3632
.00002	592.5	1046	1448	1816	2158	2480	2786	3077
.00003	552.0	965	1326	1653	1956	2237	2506	2756
.00004	523.3	908	1240	1538	1812	2067	2304	2526
.00005	501.0	863	1173	1449	1700	1931	2145	2345
.00006	482.8	827	1118	1378	1609	1823	2019	2201
.00007	467.3	796	1074	1316	1532	1730	1911	2077
.00008	454.0	769	1033	1263	1465	1650	1817	1970
.00009	442.2	745	998	1215	1407	1579	1735	1877
.00010	431.7	724	965	1172	1354	1516	1661	1792
.00011	422.3	707	938	1135	1307	1459	1595	1717
.00012	413.3	689	911	1100	1263	1407	1534	1648
.00013	405.4	673	887	1068	1223	1360	1479	1585
.00014	398.0	658	865	1038	1186	1315	1428	1526
.00015	391.1	643	844	1010	1152	1274	1380	1471
.00016	384.5	631	825	985	1120	1236	1344	1420
.00017	378.6	619	807	961	1090	1200	1293	1373
.00018	372.9	607	790	938	1062	1166	1254	1328
.00019	367.5	597	773	917	1035	1134	1216	1285
.00020	362.4	586	758	896	1009	1103	1181	1246
.00021	357.9	577	743	877	985	1074	1148	1208
.00022	353.4	567	729	858	962	1047	1116	1172
.00023	349.0	558	716	841	940	1021	1086	1137
.00024	344.5	550	703	824	919	996	1057	1104
.00025	340.1	541.9	691.6	807.9	899.6	971.8	1029.6	1072.6
.00026	336.4	534.3	680.0	792.5	880.4	949.6	1002.5	1045.4
.00027	332.8	526.6	668.6	777.6	861.9	927.5	977.0	1018.3
.00028	329.1	519.4	657.8	763.2	844.2	906.4	952.3	991.0
.00029	325.5	512.4	647.4	749.4	826.9	885.9	929.6	963.9
.00030	321.9	505.5	637.3	735.9	809.9	865.8	908.8	936.7
.00031	318.8	499.1	627.7	722.9	794.4	846.7	887.9	909.6
.00032	315.7	492.7	618.1	710.4	779.0	828.4	867.2	886.6
.00033	312.6	486.6	609.0	698.4	763.9	810.4	846.4	864.8
.00034	309.5	480.7	600.1	686.6	749.6	796.4	825.6	843.1
.00035	306.5	474.8	592.0	675.0	735.0	778.0	804.8	821.5

TABLE 2 (*Continued*)

$(\sigma_y - \sigma_o)$ \ h	100	200	300	400	500	600	700	800
.00036	303.8	469.2	583.6	664.1	721.1	762.0	787.1	801.1
.00037	301.1	463.7	575.4	653.5	707.7	746.4	769.5	780.7
.00038	298.4	458.4	567.4	643.1	694.7	731.2	752.3	760.9
.00039	295.8	453.6	559.6	632.9	682.1	716.4	735.5	741.7
.00040	293.1	448.2	552.0	623.0	670.0	702.0	719.0	723.0
.00041	290.8	443.3	544.7	613.4	659.0	688.4	703.4	705.1
.00042	288.4	438.6	537.5	604.0	648.0	675.0	687.8	687.7
.00043	286.1	433.9	530.5	594.8	637.2	661.8	672.4	670.7
.00044	283.7	429.3	523.7	585.8	626.6	648.8	657.2	654.1
.00045	281.4	424.9	517.0	577.0	616.0	636.0	642.0	638.0
.00046	279.3	420.5	510.6	568.2	605.2	622.8	625.4	622.1
.00047	277.2	416.2	504.2	559.8	594.6	610.2	609.6	606.7
.00048	275.1	412.0	498.0	551.6	584.4	598.0	594.8	591.9
.00049	273.0	407.9	492.0	543.6	574.6	586.2	581.0	577.7
.00050	270.9	403.9	486.0	536.0	565.0	575.0	568.0	564.0
.00051	269.0	400.0	480.2	529.2	556.0	565.0	557.7	551.0
.00052	267.1	396.1	474.4	522.4	547.7	555.2	547.5	538.6
.00053	265.1	392.4	468.8	515.6	539.1	545.4	537.3	526.6
.00054	263.2	388.7	463.4	508.8	530.5	535.6	527.1	515.0
.00055	261.3	385.1	458.0	502.0	522.0	526.0	517.0	504.0
.00056	259.6	381.5	452.8	494.9	513.0	516.2	506.7	491.9
.00057	257.9	378.1	447.8	487.9	504.4	506.6	496.5	480.1
.00058	256.1	374.7	442.8	481.1	496.0	497.2	486.5	468.5
.00059	254.4	371.3	437.8	474.5	487.8	488.0	476.7	457.1
.00060	252.7	368.0	433.0	468.4	480.0	479.0	467.0	446.0
.00061	251.1	364.7	428.2	461.8	473.0	470.2	457.6	435.0
.00062	249.5	361.4	423.6	455.6	465.9	461.6	448.2	424.2
.00063	247.9	358.2	419.0	449.6	458.9	453.2	439.0	413.8
.00064	246.3	355.1	414.4	443.7	452.0	445.0	430.0	403.8
.00065	244.7	352.1	410.0	438.0	445.0	437.0	421.0	394.0
.00066	243.2	349.1	405.6	432.6	437.6	429.4	412.0	384.9
.00067	241.7	346.2	401.4	427.2	430.5	421.8	403.2	375.9
.00068	240.3	343.3	397.2	421.8	423.6	414.4	394.6	367.1
.00069	238.8	340.5	393.0	416.4	416.9	407.2	386.2	358.5
.00070	237.4	337.7	389.0	411.0	410.5	400.0	378.0	350.0
.00071	235.9	334.8	385.2	405.3	404.9	393.0	370.1	341.6
.00072	234.6	332.0	381.4	399.7	399.3	386.2	362.3	333.4
.00073	233.2	329.2	377.6	394.3	393.6	379.4	354.7	325.4
.00074	231.8	326.5	373.8	389.1	387.8	372.6	347.3	317.6
.00075	230.5	323.9	370.0	384.0	382.0	366.0	340.0	310.0
.00076	229.2	321.4	366.1	379.3	375.4	358.9	333.0	302.7
.00077	227.9	318.9	362.3	374.7	369.2	352.1	326.0	295.5
.00078	226.6	313.4	358.5	370.1	363.2	345.7	319.2	288.5
.00079	225.3	314.0	354.7	365.5	357.4	339.7	312.6	281.7
.00080	224.0	311.6	351.0	361.0	352.0	334.0	306.0	275.0
.00081	222.8	309.1	347.2	356.4	347.5	330.2	299.4	268.6
.00082	221.6	306.8	343.4	352.0	342.9	326.0	293.0	262.2
.00083	220.4	304.4	339.8	347.6	338.3	321.4	286.8	256.0

TABLE 2 (*Continued*)

$(\sigma_y - \sigma_o)$ \ h	100	200	300	400	500	600	700	800
.00084	219.2	302.1	336.4	343.2	333.7	316.4	280.8	250.0
.00085	219.0	299.8	333.0	339.0	329.0	311.0	275.0	244.0
.00086	216.9	297.5	330.0	334.9	324.0	303.6	269.8	238.2
.00087	215.7	295.2	327.0	330.9	319.2	296.6	264.6	232.6
00088	214.6	293.0	324.0	326.9	314.4	290.0	259.4	227.0
.00089	213.5	290.7	321.0	322.9	309.6	283.8	254.2	221.4
.00090	212.4	288.6	318.0	319.0	305.0	278.0	249.0	216.0
.00091	211.3	286.4	315.0	315.1	300.6	273.3	243.6	210.5
.00092	210.2	284.3	312.0	311.3	296.2	268.7	238.2	205.2
.00093	209.1	282.2	309.0	307.5	291.8	264.1	233.0	200.0
.00094	208.1	280.1	306.0	303.7	287.4	259.5	228.0	194.9
.00095	207.0	278.1	303.0	300.0	283.0	255.0	223.0	190.0
.00096	206.0	275.9	299.8	296.2	278.3	250.4	218.2	185.3
.00097	205.0	273.9	296.8	292.4	273.7	246.0	213.4	180.7
.00098	203.9	271.9	293.8	288.8	269.3	241.6	208.8	176.3
.00099	202.9	269.9	290.8	285.4	265.1	237.2	204.4	171.9
.00100	201.9	268.1	288.0	282.0	261.0	233.0	200.0	167.7
.00101	200.9	266.3	285.3	279.0	257.4	228.9	195.8	163.6
.00102	200.0	264.6	282.7	276.0	253.8	224.9	191.8	159.7
.00103	199.0	263.0	280.1	273.0	250.2	220.9	187.8	155.8
.00104	198.1	261.3	277.5	270.0	246.6	216.9	183.8	151.9
.00105	197.1	259.7	275.0	267.0	243.0	213.0	180.0	148.2
.00106	196.2	258.0	272.7	264.0	239.2	209.0	176.3	144.4
.00107	195.3	256.3	270.3	261.0	235.6	205.2	172.7	140.8
.00108	194.3	254.6	267.9	258.0	232.0	201.4	169.1	137.2
.00109	193.4	252.9	265.5	255.0	228.4	197.6	165.5	133.8
.00110	192.5	251.2	263.0	252.0	225.0	194.0	162.0	130.4
.00111	191.6	249.5	260.4	248.9	221.7	190.4	158.4	127.1
.00112	190.7	247.8	257.9	245.9	218.5	187.0	155.0	123.8
.00113	189.9	246.1	255.5	242.9	215.3	183.6	151.6	120.7
.00114	189.0	244.4	253.2	239.9	212.1	180.2	148.2	117.6
.00115	188.1	242.7	251.0	237.0	209.0	177.0	145.0	114.6
.00116	187.3	241.0	248.7	234.2	205.8	174.0	141.9	111.6
.00117	186.4	239.3	246.5	231.4	202.8	171.0	138.9	108.8
.00118	185.6	237.6	244.3	228.6	199.7	168.0	135.9	106.0
.00119	184.7	235.9	242.1	225.8	196.7	165.0	132.9	103.2
.00120	183.9	234.2	240.0	223.0	193.8	162.0	130.0	100.5
.00121	183.1	232.3	237.8	220.0	191.0	158.8	127.1	97.7
.00122	182.3	230.6	235.6	217.2	188.2	155.8	124.3	95.0
.00123	181.5	228.1	233.4	214.4	185.5	152.8	121.5	92.5
.00124	180.7	227.2	231.2	211.6	182.7	149.8	118.7	90.0
.00125	179.8	225.7	229.0	209.0	180.0	147.0	116.0	87.6
.00126	179.1	224.3	226.8	206.6	177.2	144.4	113.4	85.4
.00127	178.3	222.9	224.6	204.2	174.5	141.8	110.8	83.2
.00128	177.5	221.6	222.4	201.8	171.8	139.2	108.2	81.1
.00129	176.8	220.2	220.2	199.4	169.1	136.6	105.6	78.9
.00130	176.0	218.9	218.0	197.0	166.4	134.0	103.0	76.8
.00131	175.2	217.5	216.2	194.8	164.0	131.6	100.8	74.8

TABLE 2 (*Continued*)

$(\sigma_y - \sigma_o)$ \ h	100	200	300	400	500	600	700	800
.00132	174.5	216.1	214.4	192.6	161.7	129.2	98.6	72.8
.00133	173.8	214.8	212.6	190.4	159.3	126.8	96.5	70.9
.00134	173.0	213.4	210.8	188.2	157.0	124.4	94.3	68.9
.00135	172.3	212.1	209.0	186.0	154.7	122.0	92.2	67.0
.00136	171.6	210.7	207.2	183.8	152.3	119.8	90.2	65.2
.00137	170.9	209.4	205.4	181.6	149.9	117.6	88.2	63.4
.00138	170.1	208.0	203.6	179.4	147.6	115.4	86.2	61.7
.00139	169.4	206.7	201.8	177.2	145.2	113.2	84.2	59.9
.00140	168.7	205.4	200.0	175.0	142.9	111.0	82.2	58.2
.00141	168.0	204.0	198.4	173.0	140.8	109.0	80.3	56.6
.00142	167.4	202.6	196.8	171.0	138.8	107.0	78.5	55.1
.00143	166.7	201.3	195.2	169.0	136.7	105.0	76.7	53.6
.00144	166.0	199.9	193.6	167.0	134.7	103.0	74.9	52.1
.00145	165.3	198.6	192.0	165.0	132.7	101.0	73.1	50.6
.00146	164.7	197.2	190.2	163.0	130.6	99.0	71.4	49.2
.00147	163.9	195.8	188.4	161.0	128.6	97.1	69.7	47.8
.00148	163.3	194.5	186.6	159.0	126.5	95.2	68.1	46.4
.00149	162.6	193.1	184.8	157.0	124.5	93.3	66.4	45.0
.00150	162.0	191.8	183.0	155.0	122.5	91.4	64.8	43.7
.00151	161.3	190.7	181.4	153.2	120.7	89.7	63.3	42.5
.00152	160.7	189.6	179.8	151.4	119.0	88.1	61.9	41.3
.00153	160.1	188.5	178.2	149.6	117.2	86.4	60.4	40.1
.00154	159.5	187.4	176.6	147.8	115.5	84.8	59.0	38.9
.00155	158.8	186.3	175.0	146.0	113.8	83.2	57.6	37.8
.00156	158.2	185.1	173.4	144.2	112.0	81.6	56.2	36.7
.00157	157.6	184.0	171.8	142.4	110.2	80.0	54.9	35.7
.00158	156.9	182.9	170.2	140.6	108.4	78.4	53.6	34.6
.00159	156.3	181.8	168.6	138.8	106.6	76.8	52.3	33.6
.00160	155.7	180.7	167.0	137.0	104.8	75.3	51.0	32.6
.00161	155.1	179.6	165.4	135.4	103.2	73.8	49.8	31.7
.00162	154.5	178.5	163.8	133.8	101.7	72.4	48.6	30.8
.00163	153.8	177.4	162.2	132.2	100.2	70.9	47.4	29.9
.00164	153.2	176.3	160.6	130.6	98.7	69.5	46.2	29.0
.00165	152.6	175.2	159.0	129.0	97.2	68.1	45.0	28.2
.00166	152.1	174.1	157.6	127.6	95.6	66.8	43.9	27.4
.00167	151.5	173.0	156.2	126.2	94.1	65.5	42.9	26.6
.00168	150.9	171.9	154.8	124.8	92.5	64.2	41.8	25.8
.00169	150.3	170.8	153.4	123.4	91.0	62.9	40.8	25.0
.00170	149.8	169.7	152.0	122.0	89.5	61.7	39.8	24.2
.00171	149.2	168.5	150.8	120.4	88.1	60.4	38.8	23.4
.00172	148.7	167.4	149.6	118.8	86.8	59.2	37.8	22.7
.00173	148.1	166.3	148.4	117.2	85.4	58.0	36.9	22.0
.00174	147.5	165.2	147.2	115.6	84.1	56.8	35.9	21.3
.00175	147.0	164.1	146.0	114.0	82.8	55.6	35.0	20.6
.00176	146.4	163.1	144.6	112.8	81.4	54.5	34.1	20.0
.00177	145.9	162.2	143.2	111.6	80.1	53.4	33.2	19.4
.00178	145.3	161.3	141.8	110.4	78.8	52.3	32.4	18.8
.00179	144.8	160.4	140.4	109.2	77.5	51.2	31.5	18.2

TABLE 2 (*Continued*)

$(\sigma_y - \sigma_0)$ \ h	100	200	300	400	500	600	700	800
.00180	144.2	159.5	139.0	108.0	76.2	50.2	30.7	17.7
.00181	143.7	158.5	137.8	106.6	75.0	49.2	29.9	17.1
.00182	143.1	157.6	136.6	105.3	73.9	48.2	29.2	16.6
.00183	142.6	156.6	135.4	103.9	72.7	47.2	28.4	16.1
.00184	142.1	155.7	134.2	102.6	71.6	46.2	27.7	15.6
.00185	141.6	154.8	133.0	101.3	70.5	45.2	27.0	15.1
.00186	141.0	153.8	131.8	100.0	69.3	44.3	26.3	14.6
.00187	140.5	152.9	130.6	98.8	68.1	43.4	25.6	14.1
.00188	140.0	152.0	129.4	97.6	67.0	42.5	25.0	13.7
.00189	139.5	151.1	128.2	96.4	65.8	41.6	24.3	13.2
.00190	139.0	150.2	127.0	95.2	64.7	40.8	23.7	12.8
.00191	138.5	149.2	126.0	94.0	63.7	39.9	23.1	12.4
.00192	138.0	148.3	125.0	92.9	62.7	39.1	22.5	12.0
.00193	137.5	147.4	124.0	91.8	61.7	38.3	21.9	11.6
.00194	137.0	146.5	123.0	90.7	60.7	37.5	21.3	11.2
.00195	136.5	145.6	122.0	89.6	59.8	36.7	20.7	10.9
.00196	136.0	144.6	120.8	88.4	58.8	35.9	20.18	10.56
.00197	135.5	143.7	119.6	87.2	57.8	35.14	19.66	10.22
.00198	135.0	142.7	118.4	86.1	56.8	34.36	19.14	9.88
.00199	134.5	141.8	117.2	84.9	55.8	33.58	18.62	9.54
.00200	134.1	140.9	116.0	83.8	54.8	32.80	18.10	9.20
.00201	133.7	140.1	115.0	82.8	53.94	32.14	17.64	8.92
.00202	133.2	139.3	114.0	81.8	53.08	31.48	17.18	8.64
.00203	132.8	138.6	113.0	80.9	52.22	30.82	16.72	8.36
.00204	132.3	137.8	112.0	79.9	51.36	30.16	16.26	8.08
.00205	131.9	137.1	111.0	79.0	50.50	29.50	15.80	7.80
.00206	131.5	136.3	110.0	78.0	49.66	28.88	15.38	7.56
.00207	131.1	135.6	109.1	77.0	48.82	28.26	14.96	7.32
.00208	130.6	134.8	108.2	76.0	47.98	27.64	14.54	7.08
.00209	130.2	134.1	107.3	75.0	47.14	27.02	14.12	6.84
.00210	129.8	133.4	106.4	74.1	46.30	26.40	13.70	6.60
.00211	129.4	132.6	105.4	73.1	45.56	25.84	13.34	6.38
.00212	129.0	131.8	104.4	72.2	44.82	25.28	12.98	6.16
.00213	128.5	131.1	103.4	71.3	44.08	24.72	12.62	5.94
.00214	128.1	130.3	102.4	70.4	43.34	24.16	12.26	5.72
.00215	127.7	129.6	101.5	69.5	42.60	23.60	11.90	5.50
.00216	127.3	128.8	100.6	68.6	41.86	23.10	11.58	5.32
.00217	126.9	128.1	99.7	67.7	41.12	22.60	11.26	5.14
.00218	126.4	127.3	98.8	66.9	40.38	22.10	10.94	4.96
.00219	126.0	126.6	97.9	66.0	39.64	21.60	10 62	4.78
.00220	125.6	125.9	97.0	65.2	38.90	21.10	10.30	4.60
.00221	125.2	125.1	96.16	64.34	38.28	20.64	10.04	4.44
.00222	124.7	124.3	95.32	63.48	37.66	20.18	9.78	4.28
.00223	124.3	123.6	94.48	62.62	37.04	19.72	9.52	4.12
.00224	123.8	122.8	93.64	61.76	36.42	19.26	9.26	3.96
.00225	123.4	122.1	92.80	60.90	35.80	18.80	9.00	3.80
.00226	123.0	121.4	91.96	60.18	35.18	18.40	8.76	3.66
.00227	122.6	120.7	91.12	59.46	34.56	18.00	8.52	3.52

TABLE 2 (*Continued*)

$(\sigma_y - \sigma_o)$ \ h	100	200	300	400	500	600	700	800
.00228	122.1	120.0	90.28	58.74	33.94	17.60	8.28	3.38
.00229	121.7	119.3	89.44	58.02	33.32	17.20	8.04	3.24
.00230	121.3	118.6	88.60	57.30	32.70	16.80	7.80	3.10
.00231	120.9	117.8	87.78	56.56	32.16	16.42	7.58	3.00
.00232	120.5	117.16	86.96	55.82	31.62	16.04	7.36	2.90
.00233	120.0	116.44	86.14	55.08	31.08	15.66	7.14	2.80
.00234	119.6	115.7	85.32	54.34	30.54	15.28	6.92	2.70
.00235	119.2	115.00	84.50	53.60	30.00	14.90	6.70	2.60
.00236	118.7	114.3	83.76	52.92	29.46	14.58	6.50	2.52
.00237	118.3	113.6	83.02	52.24	28.92	14.26	6.30	2.44
.00238	117.8	112.9	82.28	51.56	28.38	13.94	6.10	2.36
.00239	117.4	112.20	81.54	50.88	27.84	13.62	5.90	2.28
.00240	117.0	111.5	80.80	50.20	27.30	13.30	5.70	2.20
.00241	116.6	110.7	80.08	49.56	26.84	13.00	5.54	2.12
.00242	116.2	110.0	79.36	48.92	26.38	12.70	5.38	2.04
.00243	115.8	109.3	78.64	48.28	25.92	12.40	5.22	1.96
.00244	115.3	108.6	77.92	47.64	25.46	12.10	5.06	1.88
.00245	115.0	107.9	77.20	47.00	25.00	11.80	4.90	1.80
.00246	114.6	107.2	76.46	46.36	24.56	11.52	4.76	1.72
.00247	114.2	106.5	75.72	45.72	24.12	11.24	4.62	1.64
.00248	113.8	105.8	74.98	45.08	23.68	10.96	4.48	1.56
.00249	113.3	105.1	74.24	44.44	23.24	10.68	4.34	1.48
.00250	112.9	104.4	73.50	43.80	22.80	10.40	4.20	1.40
.00251	112.6	103.8	72.8	43.3				
.00252	112.2	103.2	72.2	42.7				
.00253	111.9	102.6	71.5	42.2				
.00254	111.5	102.1	70.9	41.6				
.00255	111.2	101.5	70.2	41.1				
.00256	110.9	100.9	69.6	40.5				
.00257	110.5	100.3	69.0	40.0				
.00258	110.2	99.7	68.3	39.5				
.00259	109.8	99.1	67.9	39.0				
.00260	109.5	98.6	67.1	38.4				
.00261	109.2	98.0	66.5	37.9				
.00262	108.8	97.4	65.9	37.4				
.00263	108.5	96.8	65.2	36.9				
.00264	108.1	96.2	64.6	36.4				
.00265	107.8	95.7	64.0	35.9				
.00266	107.5	95.0	63.3	35.4				
.00267	107.1	94.4	62.7	34.9				
.00268	106.8	93.9	62.2	34.4				
.00269	106.4	93.3	61.6	34.0				
.00270	106.1	92.8	61.0	33.5				
.00271	105.8	92.2	60.4	33.1				
.00272	105.4	91.7	59.8	32.5				
.00273	105.1	91.1	59.3	32.1				
.00274	104.7	90.6	58.7	31.7				
.00275	104.4	90.1	58.1	31.2				

TABLE 2 (*Continued*)

$(\sigma_y-\sigma_o)$ \ h	100	200	300	400	500	600	700	800
.00276	104.1	89.6	57.7	30.8				
.00277	103.7	89.0	57.1	30.4				
.00278	103.4	88.5	56.6	30.0				
.00279	103.0	88.0	56.2	29.6				
.00280	102.7	87.4	55.5	29.2				
.00281	102.4	86.9	55.0	28.8				
.00282	102.0	86.4	54.4	28.4				
.00283	101.7	85.8	53.9	28.0				
.00284	101.3	85.4	53.4	27.5				
.00285	101.0	84.9	52.9	27.2				
.00286	100.7	84.4	52.4	26.8				
.00287	100.3	83.9	51.9	26.4				
.00288	100.0	83.4	51.5	26.0				
.00289	99.6	82.9	51.0	25.6				
.00290	99.3	82.4	50.5	25.3				
.00291	99.0	81.9	50.0	25.0				
.00292	98.6	81.4	49.5	24.6				
.00293	98.3	80.9	49.0	24.3				
.00294	97.9	80.4	48.5	24.0				
.00295	97.6	79.9	48.0	23.7				
.00296	97.3	79.5	47.6	23.4				
.00297	96.9	79.0	47.1	23.1				
.00298	96.6	78.5	46.6	22.8				
.00299	96.2	78.0	46.2	22.5				
.00300	95.9	77.5	45.7	22.2				

$(\sigma_y-\sigma_o)$ \ $h=$	1000	1500	2000	2500	3000
.00001	4318	5867	7249	8503	9657
.00002	3624	4828	5863	6773	7581
.00003	3219	4221	5054	5763	6373
.00004	2931	3790	4481	5049	5519
.00005	2709	3457	4039	4497	4859
.00006	2527	3186	3679	4049	4329
.00007	2374	2957	3376	3677	3894
.00008	2240	2759	3115	3351	3512
.00009	2124	2585	2886	3086	3182
.00010	2019	2429	2681	2821	2877
.00011	1925	2291	2511	2609	2628
.00012	1839	2164	2341	2397	2392
.00013	1760	2048	2189	2224	2183
.00014	1688	1947	2053	2051	2000
.00015	1619	1851	1918	1906	1831
.00016	1557	1756	1807	1761	1672
.00017	1499	1667	1696	1643	1522
.00018	1443	1591	1595	1526	1391

700	800

	3000
	9657
	7581
	6373
	5519
	4859
	4329
	3894
	3512
	3182
	2877
	2628
	2392
	2183
	2000
	1831
	1672
	1522
	1391

Table 2 (*Continued*)

$(\sigma_y - \sigma_o)$ \ $h=$	1000	1500	2000	2500	3000
.00019	1390	1514	1502	1415	1293
.00020	1340	1438	1409	1304	1162
.00021	1298	1376	1333	1214	1063
.00022	1255	1314	1258	1123	970
.00023	1213	1251	1185	1046	886
.00024	1171	1196	1114	968	808
.00025	1129	1143	1043	900	735
.00026	1095	1091	985	832	671
.00027	1061	1042	928	773	611
.00028	1027	1000	874	715	555
.00029	993	958	824	663	505
.00030	959	915	775	612	457
.00031	931	875	731	568	416
.00032	904	836	687	524	376
.00033	876	796	647	486	341
.00034	848	761	609	448	308
.00035	820.7	729	572	414	278
.00036	797.4	696.9	538.3	382.4	251.4
.00037	774.2	666.1	506.0	352.6	226.9
.00038	751.0	636.5	475.5	324.6	204.1
.00039	727.7	608.1	446.5	298.4	183.0
.00040	704.5	581.0	419.0	274.0	163.8
.00041	685.7	555.9	394.0	252.6	147.5
.00042	666.9	531.7	370.0	232.4	132.4
.00043	648.1	508.3	347.2	213.4	118.5
.00044	629.3	485.7	325.6	195.6	105.6
.00045	610.5	464.0	305.0	179.0	93.9
.00046	592.8	443.1	286.0	164.2	84.0
.00047	575.1	423.1	267.8	150.3	74.9
.00048	557.3	403.9	250.6	137.3	66.5
.00049	539.6	385.5	234.4	125.2	58.8
.00050	521.9	368.0	219.0	114.0	51.9
.00051	507.4	351.8	205.1	104.2	46.2
.00052	492.9	336.0	191.9	95.1	41.0
.00053	478.4	320.8	179.3	86.5	36.2
.00054	463.9	306.2	167.3	78.6	31.8
.00055	449.4	292.0	156.0	71.2	27.9
.00056	437.0	278.3	145.5	64.7	24.7
.00057	424.6	265.2	135.6	58.7	21.7
.00058	412.2	252.6	126.2	53.1	19.0
.00059	399.8	240.5	117.3	48.0	16.6
.00060	387.3	229.0	109.0	43.3	14.4
.00061	376.4	218.3	101.6	39.3	12.7
.00062	365.5	208.0	94.5	35.6	11.1
.00063	354.6	198.1	87.8	32.1	9.68
.00064	343.7	188.5	81.5	28.9	8.38
.00065	332.7	179.3	75.6	26.0	7.20
.00066	323.4	170.5	70.2	23.2	6.25
.00067	314.0	162.0	65.1	20.7	5.39

TABLE 2 (*Continued*)

h / $(\sigma_o - \sigma_y)$	1000	1500	2000	2500	3000
.00068	304.6	153.9	60.3	18.5	4.61
.00069	295.2	146.2	55.8	16.6	3.91
.00070	285.8	138.9	51.6	15.0	3.30
.00071	277.6	132.1	47.7	14.3	2.87
.00072	269.5	125.5	44.1	13.5	2.47
.00073	261.3	119.2	40.7	12.7	2.11
.00074	253.1	113.2	37.5	11.8	1.79
.00075	245.0	107.4	34.6	10.8	1.50
.00076	237.9	101.8	32.0	9.2	1.25
.00077	230.8	96.4	29.6	7.79	1.03
.00078	223.8	91.3	27.3	6.53	0.85
.00079	216.7	86.5	25.2	5.43	0.71
.00080	209.6	81.9	23.2	4.50	0.60
.00081	203.5	77.6	21.3	3.96	
.00082	197.4	73.5	19.5	3.48	
.00083	191.3	69.6	17.9	3.04	
.00084	185.1	65.9	16.4	2.64	
.00085	179.0	62.4	15.0	2.30	
.00086	173.7	59.0	13.8	2.06	
.00087	168.3	55.8	12.6	1.84	
.00088	163.0	52.7	11.5	1.64	
.00089	157.6	49.7	10.5	1.46	
.00090	152.3	47.0	9.6	1.30	
.00091	147.7	44.4	8.8	1.16	
.00092	143.1	41.9	8.0	1.04	
.00093	138.6	39.6	7.3	.94	
.00094	134.0	37.3	6.6	.86	
.00095	129.4	35.2	6.0	.80	
.00096	125.4	33.1	5.4		
.00097	121.4	31.2	4.9		
.00098	117.5	29.4	4.4		
.00099	113.5	27.6	4.0		
.00100	109.5	26.0	3.6		
.00101	106.1	24.5	3.3		
.00102	102.7	23.1	3.0		
.00103	99.3	21.7	2.7		
.00104	96.0	20.4	2.4		
.00105	92.6	19.2	2.2		
.00106	89.6	18.1	2.0		
.00107	86.6	17.0	1.9		
.00108	83.7	15.9	1.7		
.00109	80.7	14.9	1.5		
.00110	77.7	14.00	1.40		
.00111	75.2	13.10	1.26		
.00112	72.8	12.26	1.14		
.00113	70.3	11.46	1.02		
.00114	67.8	10.70	0.90		
.00115	65.4	10.00	0.80		
.00116	63.2	9.39	0.70		

[Vo

3000
4.61
3.91
3.30
2.87
2.47
2.11
1.79
1.50
1.25
1.03
0.85
0.71
0.60

TABLE 2 (*Continued*)

$h=$ / $(\sigma_y-\sigma_o)$	1000	1500	2000	2500	3000
.00117	61.1	8.81	0.62		
.00118	58.9	8.25	0.54		
.00119	56.7	7.71	0.46		
.00120	54.6	7.20	0.40		
.00121	52.7	6.70			
.00122	50.9	6.22			
.00123	49.1	5.78			
.00124	47.3	5.38			
.00125	45.4	5.00			
.00126	43.9	4.70			
.00127	42.3	4.42			
.00128	40.8	4.14			
.00129	39.3	3.86			
.00130	37.7	3.60			
.00131	36.4	3.32			
.00132	35.1	3.06			
.00133	33.8	2.82			
.00134	32.5	2.60			
.00135	31.2	2.40			
.00136	30.1	2.25			
.00137	29.0	2.11			
.00138	28.0	1.97			
.00139	26.9	1.83			
.00140	25.8	1.70			
.00141	24.9	1.56			
.00142	23.9	1.42			
.00143	23.0	1.30			
.00144	22.1	1.20			
.00145	21.1	1.10			
.00146	20.4	1.03			
.00147	19.6	0.97			
.00148	18.8	0.91			
.00149	18.1	0.85			
.00150	17.3	0.80			
.00151	17.6	0.77			
.00152	16.0	0.73			
.00153	15.3	0.69			
.00154	14.7	0.65			
.00155	14.0	0.60			
.00156	13.5	0.55			
.00157	13.0	0.49			
.00158	12.5	0.43			
.00159	12.0	0.37			
.00160	11.5	0.30			
.00161	11.0				
.00162	10.6				
.00163	10.1				
.00164	9.7				
.00165	9.2				

TABLE 2 (*Continued*)

$(\sigma_y-\sigma_o)$	$h=1000$	$(\sigma_y-\sigma_o)$	$h=1000$	$(\sigma_y-\sigma_o)$	$h=1000$
.00166	8.8	.00191	2.8	.00216	0.7
.00167	8.5	.00192	2.7	.00217	0.7
.00168	8.1	.00193	2.6	.00218	0.7
.00169	7.8	.00194	2.4	.00219	0.6
.00170	7.4	.00195	2.3	.00220	0.6
.00171	7.1	.00196	2.2	.00221	0.6
.00172	6.8	.00197	2.1	.00222	0.6
.00173	6.5	.00198	2.0	.00223	0.5
.00174	6.2	.00199	1.9	.00224	0.5
.00175	5.9	.00200	1.8	.00225	0.5
.00176	5.7	.00201	1.7	.00226	0.5
.00177	5.5	.00202	1.6	.00227	0.5
.00178	5.2	.00203	1.5	.00228	0.4
.00179	5.0	.00204	1.5	.00229	0.4
.00180	4.8	.00205	1.4	.00230	0.4
.00181	4.6	.00206	1.3	.00231	0.4
.00182	4.3	.00207	1.2	.00232	0.4
.00183	4.1	.00208	1.2	.00233	0.3
.00184	3.9	.00209	1.2	.00234	0.3
.00185	3.7	.00210	1.0	.00235	0.3
.00186	3.5	.00211	1.0	.00236	0.3
.00187	3.4	.00212	0.9	.00237	0.3
.00188	3.2	.00213	0.9	.00238	0.2
.00189	3.1	.00214	0.8	.00239	0.2
.00190	2.9	.00215	0.7	.00240	0.2

$$\text{TABLE 3.}\quad P_o(z) = \int_{.001}^{z} \left[\int_{z}^{\infty} \frac{e^{-x^2}}{x} dx \right] dz$$

The double integral $P_o(z)$ is needed for computing (H), (see page 230) and for applying the integral from

$$\int_{y'}^{y} \frac{\partial\theta}{\partial t} dy = 1.13 \left\{ \frac{B_1}{B_2} \int_{H}^{(z+H)} \left(\int_{(z+H)}^{\infty} \frac{e^{-x^2}}{x} dx \right) dz - \frac{B'_1}{B'_2} \int_{H}^{z'+H} \left(\int_{z'+H}^{\infty} \frac{e^{-x^2}}{x} dx \right) dz' \right. \tag{12}$$

$$\left. + H \int_{z'+H}^{z+H} \frac{e^{-(z+H)^2}}{(z+H)} d(z+H) \right\} K + W_1 \int_{y'}^{y} f(y) \frac{\partial\theta}{\partial y} dy + \mu^2 C_1 a (e^{-ay'} - e^{-ay})$$

of equation (88), where (B_1) is the mean value of (B) for $x < (z+H)$, and (B_2) is the mean for $x > (z+H)$.

For values of (z) between .001 and 0.200 the value of $P_o(z)$ was computed from the expression

$$P_o(z) = \int_{.001}^{z} \left[\int_{z}^{\infty} \frac{e^{-x^2}}{x} dx \right] dz = -z(\log_e z - 1) - .007908 + \int_{.001}^{z} \frac{B}{z} dz \tag{13}$$

found by integrating the equation (8) between the limits .001 and (z). For greater values of (z), numerical integration of the tabulated values of $\int_{z}^{\infty} e^{-x^2} dx$ provided successive sums to be added to $P(.2)$, thus completing the tabulation.

$$\text{Tabulation of } P_o(z) = \int_{.001}^{z} \left[\int_{z}^{\infty} \frac{e^{-x^2}}{x} dx \right] dz$$

z	$P_o(z)$	z	$P_o(z)$	z	$P_o(z)$	z	$P_o(z)$
.001	.00000	.018	.07750	.035	.1347	.052	.1833
.002	.00623	.019	.08120	.036	.1377	.053	.1860
.003	.01194	.020	.08485	.037	.1408	.054	.1886
.004	.01809	.021	.0884	.038	.1438	.055	.1912
.005	.02243	.022	.0920	.039	.1477	.056	.1939
.006	.02761	.023	.0955	.040	.1497	.057	.1965
.007	.03209	.024	.0990	.041	.1526	.058	.1991
.008	.03670	.025	.1024	.042	.1555	.059	.2017
.009	.04118	.026	.1058	.043	.1584	.060	.2042
.010	.04555	.027	.1091	.044	.1612	.061	.2067
.011	.04981	.028	.1125	.045	.1641	.062	.2092
.012	.05399	.029	.1157	.046	.1669	.063	.2117
.013	.05808	.030	.1189	.047	.1697	.064	.2142
.014	.06210	.031	.1222	.048	.1725	.065	.2166
.015	.06605	.032	.1253	.049	.1752	.066	.2191
.016	.06993	.033	.1285	.050	.1779	.067	.2215
.017	.07374	.034	.1316	.051	.1806	.068	.2239

TABLE 3 (*Continued*)

z	$P_o(z)$	z	$P_o(z)$	z	$P_o(z)$	z	$P_o(z)$
.069	.2263	.119	.3314	.169	.4150	.39	.6487
.070	.2287	.120	.3332	.170	.4165	.40	.6559
.071	.2311	.121	.3351	.171	.4181	.41	.6628
.072	.2334	.122	.3369	.172	.4196	.42	.6696
.073	.2359	.123	.3387	.173	.4211	.43	.6761
.074	.2382	.124	.3405	.174	.4225	.44	.6825
.075	.2405	.125	.3423	.175	.4240	.45	.6887
.076	.2428	.126	.3441	.176	.4255	.46	.6948
.077	.2446	.127	.3459	.177	.4269	.47	.7006
.078	.2474	.128	.3478	.178	.4284	.48	.7063
.079	.2496	.129	.3495	.179	.4298	.49	.7117
.080	.2518	.130	.3513	.180	.4313	.50	.7170
.081	.2541	.131	.3531	.181	.4327	.51	.7222
.082	.2563	.132	.3528	.182	.4342	.52	.7272
.083	.2585	.133	.3566	.183	.4356	.53	.7321
.084	.2607	.134	.3583	.184	.4370	.54	.7368
.085	.2629	.135	.3600	.185	.4385	.55	.7413
.086	.2651	.136	.3618	.186	.4400	.56	.7458
.087	.2673	.137	.3635	.187	.4414	.57	.7501
.088	.2694	.138	.3652	.188	.4428	.58	.7543
.089	.2716	.139	.3669	.189	.4442	.59	.7583
.090	.2737	.140	.3686	.190	.4456	.60	.7623
.091	.2759	.141	.3703	.191	.4470	.61	.7661
.092	.2780	.142	.3719	.192	.4484	.62	.7698
.093	.2801	.143	.3736	.193	.4498	.63	.7734
.094	.2822	.144	.3753	.194	.4512	.64	.7769
.095	.2843	.145	.3770	.195	.4525	.65	.7803
.096	.2863	.146	.3787	.196	.4539	.66	.7835
.097	.2884	.147	.3803	.197	.4553	.67	.7867
.098	.2904	.148	.3820	.198	.4566	.68	.7898
.099	.2925	.149	.3836	.199	.4580	.69	.7928
.100	.2945	.150	.3852	.200	.4593	.70	.7957
.101	.2965	.151	.3868	.21	.4725	.71	.7985
.102	.2985	.152	.3885	.22	.4853	.72	.8013
.103	.3005	.153	.3901	.23	.4976	.73	.8039
.104	.3025	.154	.3917	.24	.5095	.74	.8065
.105	.3045	.155	.3933	.25	.5210	.75	.8090
.106	.3065	.156	.3949	.26	.5322	.76	.8114
.107	.3085	.157	.3965	.27	.5429	.77	.8138
.108	.3104	.158	.3981	.28	.5534	.78	.8160
.109	.3124	.159	.3996	.29	.5635	.79	.8182
.110	.3143	.160	.4012	.30	.5732	.80	.8204
.111	.3162	.161	.4028	.31	.5827	.81	.8224
.112	.3181	.162	.4043	.32	.5919	.82	.8244
.113	.3200	.163	.4058	.33	.6008	.83	.8264
.114	.3219	.164	.4074	.34	.6094	.84	.8283
.115	.3238	.165	.4089	.35	.6177	.85	.8301
.116	.3257	.166	.4104	.36	.6258	.86	.8318
.117	.3276	.167	.4120	.37	.6337	.87	.8336
.118	.3295	.168	.4135	.38	.6413	.88	.8352

TABLE 3 (*Continued*)

z	$P_o(z)$	z	$P_o(z)$	z	$P_o(z)$	z	$P_o(z)$
.89	.8368	1.27	.8707	1.65	.8791	2.03	.8808
.90	.8384	1.28	.8711	1.66	.8792	2.04	.8808
.91	.8399	1.29	.8715	1.67	.8793	2.05	.8808
.92	.8413	1.30	.8719	1.68	.8794	2.06	.8808
.93	.8427	1.31	.8722	1.69	.8795	2.07	.8809
.94	.8441	1.32	.8726	1.70	.8795	2.08	.8809
.95	.8454	1.33	.8729	1.71	.8796	2.09	.8809
.96	.8467	1.34	.8733	1.72	.8797	2.10	.8809
.97	.8479	1.35	.8736	1.73	.8797	2.11	.8809
.98	.8491	1.36	.8739	1.74	.8798	2.12	.8809
.99	.8503	1.37	.8742	1.75	.8799	2.13	.8809
1.00	.8514	1.38	.8745	1.76	.8799	2.14	.8809
1.01	.8525	1.39	.8748	1.77	.8800	2.15	.8809
1.02	.8535	1.40	.8750	1.78	.8800	2.16	.8810
1.03	.8545	1.41	.8753	1.79	.8801	2.17	.8810
1.04	.8555	1.42	.8755	1.80	.8801	2.18	.8810
1.05	.8565	1.43	.8758	1.81	.8802	2.19	.8810
1.06	.8574	1.44	.8760	1.82	.8802	2.20	.8810
1.07	.8582	1.45	.8762	1.83	.8803	2.21	.8810
1.08	.8591	1.46	.8764	1.84	.8803	2.22	.8810
1.09	.8599	1.47	.8766	1.85	.8803	2.23	.8810
1.10	.8607	1.48	.8768	1.86	.8804	2.24	.8810
1.11	.8615	1.49	.8770	1.87	.8804	2.25	.8810
1.12	.8622	1.50	.8772	1.88	.8805	2.26	.8810
1.13	.8629	1.51	.8773	1.89	.8805	2.27	.8810
1.14	.8636	1.52	.8775	1.90	.8805	2.28	.8810
1.15	.8643	1.53	.8777	1.91	.8805	2.29	.8810
1.16	.8649	1.54	.8778	1.92	.8806	2.30	.8810
1.17	.8655	1.55	.8780	1.93	.8806	2.31	.8810
1.18	.8661	1.56	.8781	1.94	.8806	2.32	.8810
1.19	.8667	1.57	.8782	1.95	.8806	2.33	.8810
1.20	.8673	1.58	.8783	1.96	.8807	2.34	.8811
1.21	.8678	1.59	.8785	1.97	.8807	2.35	.8811
1.22	.8683	1.60	.8786	1.98	.8807	2.36	.8811
1.23	.8688	1.61	.8787	1.99	.8807	2.37	.8811
1.24	.8693	1.62	.8788	2.00	.8808	2.38	.8811
1.25	.8698	1.63	.8789	2.01	.8808	2.39	.8811
1.26	.8702	1.64	.8790	2.02	.8808	2.40	.8811

Table 4. $\int_{a_1}^{\infty} \frac{e^{-x^2}}{x(x-z)} dx$

The expression $\int_{a_1}^{\infty} \frac{e^{-x^2}}{x(x-z)} dx$ is tabulated to aid in evaluating the function $P_2(z)$ in equation (34).

For values of (a_1) and (z) greater than about (.15), simple interpolation in table 4 is sufficient. Special methods explained in connection with table 6 are required for smaller values of (a_1) and (z).

Table 4

Tabulation of $\int_{a_1}^{\infty} \frac{e^{-x^2}}{x(x-z)} dx$

a_1 \ z	0	.05	.10	.15	.20	.25	.30	.35	.40	.45
.05	18.276									
.10	8.325	11.957								
.15	5.042	6.297	8.959							
.20		4.040	5.091	7.155						
.25			3.370	4.212	5.889					
.30				2.839	3.541	4.931				
.35					2.418	3.012	4.176			
.40						2.070	2.566	3.592		
.45							1.813	2.261	3.148	
.50								1.542	1.920	2.663
.55									1.332	1.658

a_1 \ z	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90
.50	2.663									
.55	1.658	2.300								
.60	1.155	1.439	1.999							
.65		.996	1.242	1.723						
.70			.861	1.073	1.492					
.75				.741	.926	1.290				
.80					.649	.801	1.081			
.85						.552	.656	.959		
.90							.471	.589	.823	
.95								.403	.505	.706
1.00									.344	.430
1.05										.293

TABLE 4 (*Continued*)

a_1 \ z	.90	.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35
.95	.706									
1.00	.430	.602								
1.05	.293	.367	.516							
1.10		.247	.310	.435						
1.15			.210	.264	.371					
1.20				.176	.222	.313				
1.25					.147	.186	.262			
1.30						.123	.156	.220		
1.35							.103	.130	.184	
1.40								.086	.108	.153
1.45									.071	.089
1.50										.058

a_1 \ z	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80
1.40	.153									
1.45	.089	.127								
1.50	.058	.074	.105							
1.55		.047	.060	.085						
1.60			.039	.050	.072					
1.65				.032	.040	.056				
1.70					.026	.032	.047			
1.75						.019	.026	.038		
1.80							.017	.022	.032	
1.85								.013	.016	.023
1.90									.010	.013
1.95										.008

a_1 \ z	1.80	1.85	1.90	1.95	2.00	2.05	2.10	2.15	2.20	2.25
1.85	.023									
1.90	.013	.019								
1.95	.008	.011	.016							
2.00		.006	.008	.012						
2.05			.005	.006	.009					
2.10				.004	.005	.007				
2.15					.002	.003	.004			
2.20						.002	.002	.003		
2.25							.001	.002	.003	
2.30								.001	.001	.002
2.35									.001	.001
2.40										

a_1 \ z	2.25	2.30	2.35	2.40	2.45	2.50	2.55	2.60	2.65	2.70
2.30	.002									
2.35	.001	.001								

TABLE 5. $\int_{a_1}^{\infty} \frac{e^{-x^2}}{(x-z)} dx$

The expression $\int_{a_1}^{\infty} \frac{e^{-x^2}}{x-z} dx$ is tabulated to aid in evaluating the function $P_3(z)$ in equation (35). For values of (a_1) and (z) greater than about (.15), simple interpolation in table 5 is sufficient. Special methods explained in connection with table 6 are required for smaller values of (a_1) and (z).

TABLE 5

Tabulation of $\int_{a_1}^{\infty} \frac{e^{-x^2}}{(x-z)} dx$

a_1 \ z	0	.05	.10	.15	.20	.25	.30	.35	.40	.45
.05	2.709									
.10	2.020	2.621								
.15	1.620	1.938	2.575							
.20		1.543	1.898	2.485						
.25			1.511	1.823	2.386					
.30				1.445	1.740	2.277				
.35					1.375	1.653	2.156			
.40						1.300	1.552	2.051		
.45							1.232	1.485	1.956	
.50								1.143	1.373	1.803
.55									1.064	1.275
.60										.986

a_1 \ z	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90
.50	1.803									
.55	1.275	1.679								
.60	.986	1.184	1.561							
.65		.907	1.088	1.438						
.70			.831	.999	1.319					
.75				.758	.909	1.204				
.80					.687	.825	1.077			
.85						.620	.726	.994		
.90							.564	.670	.894	
.95								.498	.600	.801
1.00									.443	.532
1.05										.392

TABLE 5 (*Continued*)

a_1 \ z	.90	.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35
.95	.801									
1.00	.532	.714								
1.05	.392	.473	.636							
1.10		.344	.415	.559						
1.15			.302	.367	.496					
1.20				.264	.321	.434				
1.25					.229	.278	.375			
1.30						.198	.240	.326		
1.35							.170	.207	.282	
1.40								.146	.177	.242
1.45									.124	.151
1.50										.105

a_1 \ z	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80
1.40	.242									
1.45	.151	.208								
1.50	.105	.130	.177							
1.55		.089	.108	.147						
1.60			.075	.092	.127					
1.65				.063	.075	.104				
1.70					.052	.064	.086			
1.75						.042	.050	.071		
1.80							.034	.043	.063	
1.85								.027	.034	.049
1.90									.023	.030
1.95										.020

a_1 \ z	1.80	1.85	1.90	1.95	2.00	2.05	2.10	2.15	2.20	2.25
1.85	.049									
1.90	.030	.041								
1.95	.020	.026	.034							
2.00		.016	.018	.026						
2.05			.012	.014	.021					
2.10				.010	.013	.017				
2.15					.007	.009	.012			
2.20						.007	.008	.009		
2.25							.006	.007	.008	
2.30								.005	.004	.003
2.35									.004	.003
2.40										.003

TABLE 6. $\left(\frac{1}{b_1-H}\right)\log\left(\frac{b_1}{H}\right)$

The integrals $\int_{a_1}^{\infty}\frac{e^{-x^2}}{x(x-z)}dx$ and $\int_{a_1}^{\infty}\frac{e^{-x^2}}{(x-z)}\,dx$ are entered in tables 4 and 5 for the following values of (a_1) and (z):

$z=$	0	.05	.10	.15	.20	.25	.30	.35	.40	etc.
$a_1=$	.05	.10	.15	.20	.25	.30	.35	.40	.45	etc.
	.10	.15	.20	.25	.30	.35	.40	.45	.50	etc.
	.15	.20	.25	.30	.35	.40	.45	.50	.55	etc.

If the lower limit $(z+H)$ in $P_2(z)$ and $P_3(z)$ does not equal (a_1), find the smallest value of (a_1) that exceeds this lower limit, and find the difference $a_1-(z+H)=k>0$. Then for each of the above integrals

$$\int_{z+H}^{\infty}=\int_{a_1}^{\infty}+\int_{x_1}^{x_2} \text{ where } x_1=z+H;\ x_2=z+H+k=z+b_1=a_1,$$

$$\int_{x_1}^{x_2}\frac{e^{-x^2}}{(x-z)x}dx=\left[\int_{x_1}^{x_2}\frac{e^{-x^2}}{x}dx\right]\left(\frac{1}{b_1-H}\log_e\frac{b_1}{H}\right) \tag{14}$$

$$\int_{x_1}^{x_2}\frac{e^{-x^2}}{(x-z)}dx=z\int_{x_1}^{x_2}\frac{e^{-x^2}}{(x-z)x}dx+\int_{x_1}^{x_2}\frac{e^{-x^2}}{x}dx, \tag{15}$$

approximately.

Because of the values of (a_1) and (z) in the tabulated integrals, the expression $\left(\frac{1}{b_1-H}\log_e\frac{b_1}{H}\right)$ is tabulated only for the values of $b_1=H+k=a_1-z$ equal to .05, .10, and .15. To compute the functions $P_2(z)$ and $P_3(z)$ by the above method use form (3) and the tabulated (z) nearest to the one desired. After computing (k), (b_1), (x_1) and (x_2) proceed as follows: where the bracketed numbers designating the various quantities correspond to the columns of form (3):

$$(6)=\int_{x_1}^{\infty}\frac{e^{-x^2}}{x}dx,\ (7)=\int_{x_2}^{\infty}\frac{e^{-x^2}}{x}dx,\ (8)=(6)-(7)=\int_{x_1}^{x_2}\frac{e^{-x^2}}{x}dx$$

$$(9)=\left(\frac{1}{b_1-H}\log_e\frac{b_1}{H}\right),\ (10)=(8)\times(9)=\int_{x_1}^{x_2}\frac{e^{-x^2}}{(x-z)x}dx$$

$$(17)=\frac{e^{-x_1^2}}{x_1},\quad P_2(z)=\frac{(10)}{B}+\int_{a_1}^{\infty}\frac{e^{-x^2}}{B'(x-z)x}dx+\frac{(17)}{B}=(18)$$

$$(13)=z(10),\text{ and }(15)=\int_{a_1}^{\infty}\frac{e^{-x^2}}{(x-z)}$$

$$(14)=(13)+(8),\text{ and }(14)+(15)=(16)=\int_{z+H}^{\infty}\frac{e^{-x^2}}{(x-z)}dx$$

Finally $(19)=e^{-z^2}$, $P_3(z)=(16)+(19)=(20)$, and interpolation with respect to (z) can be computed if necessary.

Schedule for computing the integrals $\int_{z+H}^{a_1} \frac{e^{-x^2}}{(x-z)x} dx$ and $\int_{z+H}^{a_1} \frac{e^{-x^2}}{(x-z)} dx$ and other expressions involved in the int

Depth unit =

Time unit =

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(1
z	H	$a_1 = z+H+k$ $= z+b_1 = x_2$	b_1	$z+H = x_1$	$\int_{x_1}^{\infty} \frac{e^{-x^2}}{x} dx$	$\int_{x_2}^{\infty} \frac{e^{-x^2}}{x} dx$	(6) − (7)	$\left(\frac{1}{b_1 - H}\right) \times \log_e \left(\frac{b_1}{H}\right)$	(8) × (9)	$\frac{1}{B} \int_{a_1}^{\infty} \frac{e^{-x^2}}{(x-z)x} dx$	$(10) \div B + (11)$ $= \int_{z+H}^{\infty} \frac{e^{-x^2}}{B^1 x(x-z)}$	$z(10)$	

TABLE 6

Tabulation of $\left(\frac{1}{b_1-H}\right)\log_e\frac{b_1}{H}=\frac{2.3026}{b_1-H}\log_{10}\left(\frac{b_1}{H}\right)$

H	$b_1=.05$	.10	.15
.001	79.85	46.54	33.64
.002	67.10	39.93	29.18
.003	59.86	36.15	26.62
.004	54.93	33.54	24.83
.005	51.20	31.54	23.46
.006	48.20	29.94	22.35
.007	45.73	28.60	21.44
.008	43.65	27.46	20.65
.009	41.83	26.48	19.95
.010	40.23	25.58	19.35
.011	38.84	24.81	18.80
.012	37.57	24.10	18.31
.013	36.41	23.46	17.85
.014	35.39	22.87	17.44
.015	34.41	22.33	17.06
.016	33.53	21.82	16.70
.017	32.71	21.36	16.38
.018	31.95	20.92	16.06
.019	31.25	20.51	15.78
.020	30.56	20.12	15.51
.021	29.93	19.77	15.24
.022	29.33	19.42	15.00
.023	28.75	19.10	14.76
.024	28.24	18.78	14.55
.025	27.73	18.49	14.34
.026	27.26	18.21	14.14
.027	26.82	17.95	13.94
.028	26.37	17.69	13.76
.029	25.95	17.43	13.58
.030	25.57	17.21	13.42
.031	25.17	16.99	13.26
.032	24.83	16.76	13.10
.033	24.45	16.56	12.94
.034	24.11	16.36	12.80
.035	23.81	16.16	12.66
.036	23.47	15.97	12.53
.037	23.20	15.79	12.39
.038	22.94	15.62	12.27
.039	22.59	15.43	12.13
.040	22.31	15.27	12.01
.041	22.06	15.12	11.91
.042	21.80	14.98	11.79
.043	21.51	14.82	11.68
.044	21.32	14.66	11.57
.045	21.05	14.53	11.47
.046	20.85	14.38	11.37
.047	20.62	14.25	11.27
.048	20.42	14.12	11.17
.049	20.19	14.01	11.09
.050	(20.00)	13.87	10.99

H	$b_1=.10$	.15
.050	13.87	10.99
.051	13.75	10.91
.052	13.63	10.82
.053	13.51	10.73
.054	13.40	10.65
.055	13.29	10.57
.056	13.19	10.49
.057	13.08	10.42
.058	12.97	10.33
.059	12.87	10.26
.060	12.78	10.18
.061	12.68	10.11
.062	12.58	10.04
.063	12.48	9.973
.064	12.41	9.908
.065	12.31	9.844
.066	12.22	9.773
.067	12.15	9.717
.068	12.07	9.652
.069	11.97	9.583
.070	11.91	9.530
.071	11.83	9.474
.072	11.74	9.411
.073	11.66	9.362
.074	11.60	9.298
.075	11.50	9.245
.076	11.48	9.187
.077	11.35	9.139
.078	11.30	9.085
.079	11.27	9.032
.080	11.16	8.984
.081	11.11	8.936
.082	11.01	8.881
.083	10.97	8.837
.084	10.92	8.791
.085	10.87	8.736
.086	10.79	8.696
.087	10.75	8.650
.088	10.70	8.611
.089	10.63	8.569
.090	10.53	8.518
.091	10.48	8.470
.092	10.43	8.439
.093	10.37	8.390
.094	10.31	8.362
.095	10.26	8.299
.096	10.21	8.276
.097	10.15	8.235
.098	10.10	8.195
.099	10.04	8.149
.100	(10.00)	

H	$b_1=.15$
.100	8.111
.101	8.073
.102	8.040
.103	7.999
.104	7.966
.105	7.931
.106	7.894
.107	7.858
.108	7.825
.109	7.788
.110	7.754
.111	7.726
.112	7.689
.113	7.654
.114	7.628
.115	7.594
.116	7.563
.117	7.533
.118	7.504
.119	7.473
.120	7.443
.121	7.409
.122	7.385
.123	7.351
.124	7.325
.125	7.295
.126	7.266
.127	7.243
.128	7.213
.129	7.186
.130	7.157
.131	7.129
.132	7.108
.133	7.077
.134	7.050
.135	7.027
.136	6.998
.137	6.974
.138	6.949
.139	6.923
.140	6.896
.141	6.876
.142	6.844
.143	6.835
.144	6.809
.145	6.783
.146	6.758
.147	6.731
.148	6.701
.149	6.678
.150	(6.667)

TABLE 7

x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$
.0010	1000.00	.0250	39.975	.058	17.183	.106	9.3286
.0015	666.67	.0255	39.190	.059	16.890	.107	9.2394
.0020	500.00	.0260	38.436	.060	16.607	.108	9.1509
.0025	400.00	.0265	37.709	.061	16.332	.109	9.0659
.0030	333.33	.0270	37.010	.062	16.067	.110	8.9816
.0035	285.71	.0275	36.336	.063	15.810	.111	8.8987
.0040	250.00	.0280	35.686	.064	15.561	.112	8.8173
.0045	222.22	.0285	35.059	.065	15.320	.113	8.7373
.0050	200.00	.0290	34.454	.066	15.086	.114	8.6586
.0055	181.81	.0295	33.868	.067	14.858	.115	8.5815
.0060	166.66	.0300	33.303	.068	14.638	.116	8.5055
.0065	153.84	.0305	32.756	.069	14.424	.117	8.4308
.0070	142.85	.0310	32.227	.070	14.216	.118	8.3574
.0075	133.33	.0315	31.714	.071	14.014	.119	8.2852
.0080	124.99	.0320	31.218	.072	13.817	.120	8.2142
.0085	117.64	.0325	30.736	.073	13.626	.121	8.1444
.0090	111.10	.0330	30.270	.074	13.440	.122	8.0756
.0095	105.25	.0335	29.817	.075	13.258	.123	8.0080
.0100	99.990	.0340	29.378	.076	13.082	.124	7.9415
.0105	95.227	.0345	28.951	.077	12.910	.125	7.8760
.0110	90.898	.0350	28.536	.078	12.743	.126	7.8115
.0115	86.945	.0355	28.133	.079	12.579	.127	7.7480
.0120	83.321	.0360	27.742	.080	12.420	.128	7.6855
.0125	79.987	.0365	27.360	.081	12.265	.129	7.6240
.0130	76.910	.0370	26.990	.082	12.113	.130	7.5634
.0135	75.060	.0375	26.629	.083	11.965	.131	7.5037
.0140	71.415	.0380	26.278	.084	11.821	.132	7.4449
.0145	68.951	.0385	25.935	.085	11.680	.133	7.3870
.0150	66.652	.0390	25.602	.086	11.542	.134	7.3299
.0155	64.500	.0395	25.276	.087	11.407	.135	7.2736
.0160	62.484	.0400	24.960	.088	11.276	.136	7.2182
.0165	60.589	.041	24.349	.089	11.147	.137	7.1636
.0170	58.807	.042	23.768	.090	11.021	.138	7.1097
.0175	57.125	.043	23.213	.091	10.898	.139	7.0565
.0180	55.538	.044	22.683	.092	10.778	.140	7.0043
.0185	54.035	.045	22.177	.093	10.660	.141	6.9526
.0190	52.613	.046	21.693	.094	10.544	.142	6.9017
.0195	51.262	.047	21.230	.095	10.431	.143	6.8515
.0200	49.980	.048	20.785	.096	10.321	.144	6.8019
.0205	48.759	.049	20.359	.097	10.213	.145	6.7531
.0210	47.598	.050	19.950	.098	10.107	.146	6.7049
.0215	46.490	.051	19.557	.099	10.003	.147	6.6573
.0220	45.433	.052	19.179	.100	9.9010	.148	6.6104
.0225	44.421	.053	18.815	.101	9.8005	.149	6.5641
.0230	43.455	.054	18.465	.102	9.7024	.150	6.5184
.0235	42.529	.055	18.127	.103	9.6062	.151	6.4732
.0240	41.643	.056	17.801	.104	9.5120	.152	6.4287
.0245	40.791	.057	17.487	.105	9.4194	.153	6.3847

TABLE 7 (*Continued*)

x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$
.154	6.3413	.202	4.7525	.250	3.7576	.73	.80391
.155	6.2985	.203	4.7272	.26	3.5948	.74	.78124
.156	6.2562	.204	4.7021	.27	3.4433	.75	.75931
.157	6.2143	.205	4.6772	.28	3.3021	.76	.73819
.158	6.1731	.206	4.6527	.29	3.1702	.77	.71775
.159	6.1323	.207	4.6283	.30	3.0464	.78	.69803
.160	6.0920	.208	4.6041	.31	2.9302	.79	.67822
.161	6.0523	.209	4.5801	.32	2.8220	.80	.65911
.162	6.0129	.210	4.5564	.33	2.7174	.81	.64066
.163	5.9742	.211	4.5329	.34	2.6191	.82	.62279
.164	5.9358	.212	4.5097	.35	2.5264	.83	.60491
.165	5.8978	.213	4.4865	.36	2.4392	.84	.58764
.166	5.8604	.214	4.4637	.37	2.3567	.85	.57094
.167	5.8233	.215	4.4411	.38	2.2787	.86	.55474
.168	5.7868	.216	4.4185	.39	2.2025	.87	.53915
.169	5.7506	.217	4.3963	.40	2.1304	.88	.52406
.170	5.7149	.218	4.3744	.41	2.0618	.89	.50892
.171	5.6795	.219	4.3522	.42	1.9968	.90	.49428
.172	5.6445	.220	4.3307	.43	1.9328	.91	.48013
.173	5.6099	.221	4.3094	.44	1.8719	.92	.46646
.174	5.5757	.222	4.2878	.45	1.8139	.93	.45276
.175	5.5420	.223	4.2669	.46	1.7586	.94	.43949
.176	5.5085	.224	4.2457	.47	1.7058	.95	.42667
.177	5.4755	.225	4.2251	.48	1.6552	.96	.41430
.178	5.4428	.226	4.2044	.49	1.6054	.97	.40230
.179	5.4105	.227	4.1842	.50	1.5576	.98	.39070
.180	5.3785	.228	4.1638	.51	1.5119	.99	.37910
.181	5.3469	.229	4.1439	.52	1.4681	1.00	.36788
.182	5.3155	.230	4.1238	.53	1.4246	1.01	.35702
.183	5.2846	.231	4.1039	.54	1.3829	1.02	.34652
.184	5.2539	.232	4.0845	.55	1.3429	1.03	.33603
.185	5.2236	.233	4.0650	.56	1.3045	1.04	.32589
.186	5.1935	.234	4.0456	.57	1.2676	1.05	.31607
.187	5.1639	.235	4.0268	.58	1.2321	1.06	.30659
.188	5.1344	.236	4.0077	.59	1.1968	1.07	.29740
.189	5.1054	.237	3.9888	.60	1.1628	1.08	.28853
.190	5.0766	.238	3.9705	.61	1.1301	1.09	.28883
.191	5.0481	.239	3.9519	.62	1.0986	1.10	.27109
.192	5.0198	.240	3.9335	.63	1.0672	1.11	.26280
.193	4.9919	.241	3.9152	.64	1.0370	1.12	.25318
.194	4.9643	.242	3.8970	.65	1.0078	1.13	.24679
.195	4.9369	.243	3.8790	.66	.97991	1.14	.23906
.196	4.9098	.244	3.8617	.67	.95262	1.15	.23160
.197	4.8829	.245	3.8447	.68	.92651	1.16	.22438
.198	4.8564	.246	3.8263	.69	.90039	1.17	.21740
.199	4.8300	.247	3.8090	.70	.87520	1.18	.21066
.200	4.8039	.248	3.7918	.71	.85089	1.19	.20393
.201	4.7781	.249	3.7747	.72	.82738	1.20	.19744

TABLE 7 (*Continued*)

x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$	x	$\frac{e^{-x^2}}{x}$
1.21	.19117	1.38	.10795	1.55	.058352	1.72	.030188
1.22	.18510	1.39	.10422	1.56	.056209	1.73	.028981
1.23	.17907	1.40	.10061	1.57	.054146	1.74	.027768
1.24	.17323	1.41	.097142	1.58	.052161	1.75	.026793
1.25	.16761	1.42	.093796	1.59	.050200	1.80	.021758
1.26	.16217	1.43	.090475	1.60	.048316	1.85	.017682
1.27	.15692	1.44	.087277	1.61	.046503	1.90	.014238
1.28	.15185	1.45	.084201	1.62	.044760	1.95	.011472
1.29	.14681	1.46	.081233	1.63	.043042	2.00	.0091580
1.30	.14194	1.47	.078374	1.64	.041391	2.05	.0073150
1.31	.13724	1.48	.075622	1.65	.039804	2.10	.0057881
1.32	.13271	1.49	.072893	1.66	.038280	2.15	.0045828
1.33	.12820	1.50	.070267	1.67	.036816	2.20	.0035941
1.34	.12385	1.51	.067735	1.68	.035409	2.25	.0028204
1.35	.11966	1.52	.065303	1.69	.034023	2.30	.0021922
1.36	.11562	1.53	.062896	1.70	.032692	2.35	.0017047
1.37	.11172	1.54	.060580	1.71	.031414		

$\frac{e^{-x^2}}{x}$
.030188
.028981
.027768
.026793
.021758
.017682
.014238
.011472
.0091580
.0073150
.0057881
.0045828
.0035941
.0028204
.0021922
.0017047

Table 8. $f(y)=1-e^{-ay}\cos ay$

Relative vertical velocities in an upwelling region are given (McEwen, 1919, pages 402–403) by the expression

$$f(y)=1-e^{-ay}\cos ay \tag{16}$$

derived from Ekman's theory where $a=\frac{\pi}{D}$ and the coefficient of frictional resistance is proportional to (D), the "depth of frictional resistance." For convenience the function $f(y)$ is tabulated with respect to the depth (y) for appropriate values of (D).

Table 8

Vertical velocity equals $W_1 f(y)$

Tabulation of $f(y)=1-e^{-ay}\cos ay$

Where $a=\frac{\pi}{D}$ (coefficient of friction is proportional to D)

$D=$	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
unit is 10 meters								
$y=0$	0	0	0	0	0	0	0	0
.5	.568	.305	.207	.156	.125	.104	.090	.079
1.0	.912	.568	.399	.305	.247	.207	.178	.156
1.5	1.0	.771	.568	.444	.362	.305	.264	.232
2.0	1.0	.912	.710	.568	.470	.399	.347	.305
2.5	1.0	1.0	.825	.678	.568	.488	.425	.376
3.0	1.0	1.0	.912	.771	.657	.568	.500	.444
3.5	1.0	1.0	.977	.849	.736	.643	.568	.508
4.0	1.0	1.0	1.0	.912	.805	.710	.633	.568
4.5	1.0	1.0	1.0	.962	.857	.771	.692	.626
5.0	1.0	1.0	1.0	1.0	.912	.825	.746	.678
6.0	1.0	1.0	1.0	1.0	.987	.912	.883	.771
7.0	1.0	1.0	1.0	1.0	1.0	.976	.912	.849
8.0	1.0	1.0	1.0	1.0	1.0	1.0	.968	.912
9.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.962
10.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE 9. PARAMETERS OF EXPONENTIAL EQUATION TABULATED WITH RESPECT TO SLOPE OF GRAPH ON CERTAIN KINDS OF SEMILOGARITHMIC PAPER

Certain kinds of "Codex paper" having a uniform horizontal scale and a logarithmic vertical scale are convenient for numerical applications of the equations presented in this paper. In particular it is necessary to compute the coefficient of (y) in the exponential expression (Ae^{-ay}) from a straight line graph determined by plotted points. In table 9, (b) is the unit of length used in plotting the abscissa (y) on the scale of equal parts, that corresponds to the unit of depth. Let (nb) equal the distance between (1) and (10) on the logarithmic scale measured in terms of the unit (b), then (nb) corresponds to $\log_{10} 10 = 1$.

Multiply the slope of the line by the factors $\left(-\frac{1}{n \log_{10} e}\right)$ to obtain (a), and the intercept for $y=0$, by $\frac{1}{n}$ to obtain $\log_{10} A$. The value of (A) can also be read off directly from the graph. The slope can be determined by the ratio of the change in ordinate to change in abscissa, or from the measured angle of the line with the horizontal. The value of (a) can be read off from table 9 for several kinds of Codex paper and lengths of the smallest division on the horizontal scale.

The time derivative $\frac{\partial \theta}{\partial t}$ corresponding to the slope of a temperature time curve plotted on paper No. 4117 (with plain ruling) is entered in the last column of table 9. The ordinate unit is ten times the smallest vertical division, the abscissa unit, one month is thirty times the smallest horizontal division.

TABLE 9

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-ay}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals....		3138	3115	3115	4115	4115	4115	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Length of smallest division equals...........		b	$\frac{b}{5}$	$\frac{b}{10}$	$\frac{b}{5}$	$\frac{b}{10}$	b	
Angle = A	Slope = tan A	.0962 tan A	.1531 tan A	.3062 tan A	.1919 tan A	.3837 tan A	.1284 tan A	.2475 tan A
0	0	0	0	0	0	0	0	0
0.5	.0087	.0008	.0013	.0027	.0017	.0033	.0011	.022
1.0	.0175	.0017	.0027	.0054	.0034	.0067	.0022	.043
1.5	.0262	.0025	.0040	.0080	.0050	.0101	.0034	.065
2.0	.0349	.0034	.0053	.0107	.0067	.0134	.0045	.086
2.5	.0437	.0042	.0067	.0134	.0084	.0168	.0056	.108
3.0	.0524	.0050	.0080	.0160	.0101	.0201	.0067	.130
3.5	.0612	.0059	.0094	.0187	.0117	.0235	.0079	.151
4.0	.0699	.0067	.0107	.0214	.0134	.0268	.0090	.173
4.5	.0787	.0076	.0120	.0241	.0151	.0302	.0101	.195
5.0	.0875	.0084	.0134	.0268	.0168	.0336	.0112	.217
5.5	.0963	.0093	.0147	.0295	.0185	.0369	.0124	.238
6.0	.1051	.0101	.0161	.0321	.0202	.0403	.0135	.260
6.5	.1139	.0110	.0174	.0349	.0219	.0437	.0146	.282
7.0	.1228	.0118	.0188	.0376	.0236	.0471	.0158	.304
7.5	.1317	.0127	.0202	.0403	.0253	.0505	.0169	.326
8.0	.1405	.0135	.0215	.0430	.0270	.0539	.0180	.348
8.5	.1495	.0144	.0229	.0458	.0287	.0574	.0192	.370
9.0	.1584	.0152	.0243	.0485	.0304	.0608	.0203	.392
9.5	.1673	.016	.026	.051	.032	.064	.021	.414
10.0	.1763	.017	.027	.054	.034	.068	.023	.436

TABLE 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-av}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals.... / Length of smallest division equals............		3138 b	3115 $\frac{b}{5}$	3115 $\frac{b}{10}$	4115 $\frac{b}{5}$	4115 $\frac{b}{10}$	4115 b	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Angle = A	Slope = tan A	.0962 tan A	.1531 tan A	.3062 tan A	.1919 tan A	.3837 tan A	.1284 tan A	.2475 tan A
10.5	.1853	.018	.028	.057	.036	.071	.024	.459
11.0	.1944	.019	.030	.060	.037	.075	.025	.481
11.5	.2035	.020	.031	.062	.039	.078	.026	.504
12.0	.2126	.020	.032	.065	.041	.082	.027	.526
12.5	.2217	.021	.034	.068	.042	.085	.028	.548
13.0	.2309	.022	.035	.071	.044	.089	.030	.572
13.5	.2401	.023	.037	.073	.046	.092	.031	.594
14.0	.2493	.024	.038	.076	.048	.096	.032	.617
14.5	.2586	.025	.040	.079	.050	.099	.033	.639
15.0	.2679	.026	.041	.082	.052	.103	.034	.663
15.5	.2773	.027	.042	.085	.053	.106	.036	.686
16.0	.2867	.028	.044	.088	.055	.110	.037	.709
16.5	.2962	.028	.045	.091	.057	.114	.038	.733
17.0	.3057	.029	.047	.093	.059	.117	.039	.756
17.5	.3153	.030	.048	.096	.060	.121	.040	.781
18.0	.3249	.031	.050	.099	.062	.125	.042	.804
18.5	.3346	.032	.051	.102	.064	.128	.043	.824
19.0	.3443	.033	.0537	.105	.066	.132	.044	.851
19.5	.3541	.034	.054	.108	.068	.136	.045	.876
20.0	.3640	.035	.056	.111	.070	.140	.047	.900
20.5	.3739	.036	.057	.114	.072	.143	.048	.924

19.5	.3541	.034	.054	.108	.068	.136	.045	.876
20.0	.3640	.035	.056	.11'	.070	.140	.047	.900
20.5	.3739	.036	.057	.11	.072	.143	.048	.924

TABLE 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-ay}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals.... Length of smallest division equals..........		3138 b	3115 $\frac{b}{5}$	3115 $\frac{b}{10}$	4115 $\frac{b}{5}$	4115 $\frac{b}{10}$	4115 b	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Angle = A	Slope = tan A	.0962 tan A	.1531 tan A	.3062 tan A	.1919 tan A	.3837 tan A	.1284 tan A	.2475 tan A
21.0	.3839	.037	.059	.118	.074	.147	.049	.949
21.5	.3939	.038	.060	.121	.076	.151	.051	.974
22.0	.4040	.039	.062	.124	.077	.155	.052	.999
22.5	.4142	.040	.063	.127	.079	.159	.053	1.025
23.0	.4245	.041	.065	.130	.081	.163	.055	1.050
23.5	.4348	.042	.067	.133	.083	.167	.056	1.076
24.0	.4452	.043	.068	.136	.085	.171	.057	1.10
24.5	.4557	.044	.070	.139	.087	.175	.059	1.13
25.0	.4663	.045	.071	.143	.089	.179	.060	1.15
25.5	.4770	.046	.073	.146	.092	.183	.061	1.18
26.0	.4887	.047	.075	.149	.094	.187	.063	1.21
26.5	.4986	.048	.076	.153	.096	.191	.064	1.23
27.0	.5095	.049	.078	.156	.098	.195	.065	1.26
27.5	.5206	.050	.080	.159	.100	.200	.067	1.29
28.0	.5317	.051	.081	.163	.102	.204	.068	1.32
28.5	.5430	.052	.083	.166	.104	.208	.070	1.34
29.0	.5543	.053	.085	.170	.106	.212	.071	1.37
29.5	.5658	.054	.087	.173	.109	.217	.073	1.40
30.0	.5774	.056	.088	.177	.111	.221	.074	1.43
30.5	.5891	.057	.090	.180	.113	.226	.076	1.46
31.0	.6009	.058	.092	.184	.115	.230	.077	1.49

TABLE 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-ay}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals.... / Length of smallest division equals..........		3138	3115	3115	4115	4115	4115	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
		b	$\frac{b}{5}$	$\frac{b}{10}$	$\frac{b}{5}$	$\frac{b}{10}$	b	
Angle $= A$	Slope $= \tan A$	.0962 tan A	.1531 tan A	.3062 tan A	.1919 tan A	.3837 tan A	.1284 tan A	.2475 tan A
31.5	.6128	.059	.094	.188	.118	.235	.079	1.52
32.0	.6249	.060	.096	.191	.120	.240	.080	1.55
32.5	.6371	.061	.098	.195	.122	.244	.082	1.58
33.0	.6494	.062	.099	.199	.125	.249	.083	1.61
33.5	.6619	.064	.101	.203	.127	.254	.085	1.64
34.0	.6745	.065	.103	.207	.129	.259	.087	1.67
34.5	.6873	.066	.105	.210	.132	.264	.088	1.70
35.0	.7002	.067	.107	.214	.134	.269	.090	1.73
35.5	.7133	.069	.109	.218	.137	.274	.092	1.76
36.0	.7265	.070	.111	.222	.139	.279	.093	1.80
36.5	.7400	.071	.113	.226	.142	.284	.095	1.83
37.0	.7536	.072	.115	.331	.145	.289	.097	1.86
37.5	.7673	.074	.117	.335	.147	.294	.099	1.90
38.0	.7813	.075	.120	.239	.150	.300	.100	1.93
38.5	.7954	.077	.122	.243	.153	.305	.102	1.97
39.0	.8098	.078	.124	.248	.155	.311	.104	2.00
39.5	.8243	.079	.126	.252	.158	.316	.106	2.04
40.0	.8391	.081	.128	.257	.161	.322	.108	2.08
40.5	.8541	.082	.131	.262	.164	.328	.110	2.11
41.0	.8693	.084	.133	.266	.167	.333	.112	2.15
41.5	.8847	.085	.135	.271	.170	.339	.114	2.19

TABLE 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-ay}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals.... Length of smallest division equals..........		3138 b	3115 $\frac{b}{5}$	3115 $\frac{b}{10}$	4115 $\frac{b}{5}$	4115 $\frac{b}{10}$	4115 b	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Angle $= A$	Slope $= \tan A$	.0962 tan A	.1531 tan A	.3062 tan A	.1919 tan A	.3837 tan A	.1284 tan A	.2475 tan A
42.0	.9004	.087	.138	.276	.173	.345	.116	2.23
42.5	.9163	.088	.140	.280	.176	.351	.118	2.27
43.0	.9325	.090	.143	.285	.179	.358	.120	2.31
43.5	.9490	.091	.145	.290	.182	.364	.122	2.35
44.0	.9657	.093	.148	.296	.185	.370	.124	2.39
44.5	.9827	.094	.150	.301	.189	.377	.126	2.43
45.0	1.0000	.096	.153	.306	.192	.384	.128	2.47
45.5	1.018	.098	.156	.312	.195	.390	.131	2.52
46.0	1.036	.100	.158	.317	.199	.397	.133	2.56
46.5	1.054	.101	.161	.323	.202	.404	.135	2.61
47.0	1.072	.103	.164	.328	.206	.411	.138	2.65
47.5	1.091	.105	.167	.334	.209	.418	.140	2.70
48.0	1.111	.107	.170	.340	.213	.426	.143	2.75
48.5	1.130	.109	.173	.346	.217	.433	.145	2.80
49.0	1.150	.111	.176	.352	.221	.441	.147	2.85
49.5	1.171	.113	.179	.358	.225	.449	.150	2.90
50.0	1.192	.115	.182	.365	.229	.457	.153	2.95
50.5	1.213	.117	.186	.371	.232	.465	.156	3.00
51.0	1.235	.119	.189	.378	.237	.474	.159	3.05
51.5	1.257	.121	.192	.385	.241	.482	.161	3.11
52.0	1.280	.123	.196	.392	.246	.491	.164	3.17
52.5	1.303	.125	.199	.399	.250	.500	.167	3.22

TABLE 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-av}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals.... / Length of smallest division equals....		3138 b	3115 $\frac{b}{5}$	3115 $\frac{b}{10}$	4115 $\frac{b}{5}$	4115 $\frac{b}{10}$	4115 b	Value of $\frac{\partial \theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Angle = A	Slope = tan A	.0962 tan A	.1531 tan A	.3062 tan A	.1919 tan A	.3837 tan A	.1284 tan A	.2475 tan A
53.0	1.327	.128	.203	.406	.255	.509	.170	3.28
53.5	1.351	.130	.207	.414	.259	.518	.173	3.34
54.0	1.376	.132	.211	.421	.264	.528	.177	3.40
54.5	1.402	.135	.215	.429	.269	.538	.180	3.47
55.0	1.428	.137	.219	.437	.274	.548	.183	3.53
55.5	1.455	.140	.223	.445	.279	.558	.187	3.60
56.0	1.483	.143	.227	.454	.285	.569	.190	3.67
56.5	1.511	.145	.231	.463	.290	.580	.194	3.74
57.0	1.540	.148	.236	.472	.296	.591	.198	3.81
57.5	1.570	.151	.240	.481	.301	.602	.202	3.88
58.0	1.600	.154	.245	.490	.307	.614	.206	3.96
58.5	1.632	.157	.250	.499	.313	.626	.210	3.44
59.0	1.1664	.160	.255	.509	.319	.638	.213	4.12
59.5	1.698	.163	.260	.519	.326	.651	.218	4.20
60.0	1.732	.167	.265	.530	.332	.664	.222	4.28
60.5	1.767	.170	.270	.541	.339	.678	.227	4.37
61.0	1.804	.173	.276	.552	.346	.692	.232	4.46
61.5	1.842	.177	.282	.564	.353	.706	.236	4.55
62.0	1.881	.181	.288	.576	.361	.721	.241	4.65
62.5	1.921	.185	.294	.588	.369	.736	.246	4.75
63.0	1.963	.189	.300	.601	.377	.752	.252	4.85
63.5	2.006	.193	.307	.614	.385	.769	.257	4.96

TABLE 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-av}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals.... Length of smallest division equals...........		3138 b	3115 $\frac{b}{5}$	3115 $\frac{b}{10}$	4115 $\frac{b}{5}$	4115 $\frac{b}{10}$	4115 b	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Angle $= A$	Slope $= \tan A$	.0962 $\tan A$	.1531 $\tan A$	.3062 $\tan A$	.1919 $\tan A$	.3837 $\tan A$	.1284 $\tan A$	.2475 $\tan A$
64.0	2.050	.197	.314	.628	.393	.786	.263	5.07
64.5	2.096	.201	.321	.642	.402	.804	.269	5.18
65.0	2.144	.206	.328	.656	.411	.822	.275	5.30
65.5	2.194	.211	.336	.672	.421	.891	.281	5.43
66.0	2.246	.216	.344	.688	.431	.861	.288	5.56
66.5	2.300	.221	.352	.704	.441	.882	.295	5.69
67.0	2.356	.226	.361	.722	.452	.903	.303	5.82
67.5	2.414	.232	.370	.739	.463	.926	.310	5.96
68.0	2.475	.238	.379	.758	.475	.949	.318	6.11
68.5	2.539	.244	.389	.778	.487	.974	.326	6.27
69.0	2.605	.250	.399	.798	.500	.999	.335	6.44
69.5	2.675	.257	.410	.819	.513	1.026	.344	6.61
70.0	2.747	.265	.420	.841	.527	1.054	.353	6.79
70.5	2.824	.272	.432	.865	.542	1.084	.362	6.99
71.0	2.904	.279	.444	.889	.558	1.114	.373	7.18
71.5	2.989	.287	.457	.915	.574	1.147	.384	7.40
72.0	3.078	.296	.471	.942	.591	1.180	.395	7.61
72.5	3.172	.305	.485	.971	.609	1.216	.407	7.85
73.0	3.271	.315	.501	1.002	.628	1.255	.420	8.09
73.5	3.376	.325	.517	1.034	.648	1.295	.434	8.35
74.0	3.487	.336	.534	1.067	.669	1.338	.448	8.63
74.5	3.606	.347	.552	1.103	.692	1.384	.463	8.92

Table 9 (*Continued*)

Values of (a) corresponding to the graph of $(\theta - C) = Ae^{-ay}$ on several different kinds of "Codex" semilogarithmic paper

Semilogarithmic paper number equals....		3138	3115	3115	4115	4115	4115	Value of $\frac{\partial\theta}{\partial t}$ from graph on Codex paper number 4117 for ordinate, in units of ten times the smallest division against abscissa (t) in units 30 times the smallest division
Length of smallest division equals..........		b	$\frac{b}{5}$	$\frac{b}{10}$	$\frac{b}{5}$	$\frac{b}{10}$	b	
Angle $=A$	Slope $=\tan A$	.0962 $\tan A$	.1531 $\tan A$	.3062 $\tan A$	.1919 $\tan A$	.3837 $\tan A$	.1284 $\tan A$	.2475 $\tan A$
75.0	3.732	.359	.571	1.142	.716	1.432	.479	9.24
75.5	3.867	.372	.592	1.183	.742	1.483	.496	9.57
76.0	4.011	.386	.614	1.227	.769	1.539	.515	9.92
76.5	4.165	.401	.638	1.275	.799	1.598	.535	10.3
77.0	4.331	.417	.663	1.325	.831	1.662	.556	10.7
77.5	4.511	.434	.690	1.381	.866	1.730	.579	11.1
78.0	4.705	.453	.720	1.440	.903	1.805	.604	11.6
78.5	4.915	.473	.752	1.505	.943	1.886	.631	12.2
79.0	5.145	.495	.787	1.575	.988	1.974	.661	12.7
79.5	5.396	.519	.826	1.652	1.035	2.070	.692	13.3
80.0	5.671	.546	.868	1.736	1.088	2.175	.728	14.0
80.5	5.976	.575	.914	1.829	1.146	2.292	.767	14.8
81.0	6.314	.606	.966	1.931	1.211	2.422	.810	15.6
81.5	6.691	.643	1.024	2.048	1.284	2.567	.859	16.6
82.0	7.115	.684	1.089	2.178	1.365	2.730	.913	17.6
82.5	7.596	.730	1.150	2.327	1.456	2.912	.962	18.8
83.0	8.144	.784	1.246	2.493	1.564	3.126	1.043	20.1
83.5	8.777	.844	1.344	2.867	1.685	3.369	1.127	21.7
84.0	9.514	.915	1.456	2.912	1.826	3.651	1.220	23.5
84.5	10.385	.999	1.590	3.180	1.992	3.987	1.330	25.7
85.0	11.430	1.100	1.750	3.500	2.194	4.387	1.468	28.3
85.5	12.706	1.222	1.946	3.890	2.439	4.875	1.631	31.4

TABLE 10. COMPUTATION OF $(\theta_o - v)$ AND THE RATIO $\frac{10^5}{B}$ FOR SALT WATER FROM THE SURFACE TEMPERATURE AND SERIAL VALUES OF $10^5(\sigma - \sigma_o)$ DETERMINED FROM OBSERVED SERIAL TEMPERATURES AND SALINITIES

As explained on page 212 the temperature (θ') at any depth (y) in sea water does not in general have the same value as the decreased temperature (v) of a surface element whose corresponding increase in specific gravity $(\sigma - \sigma_o)$ equals that at the depth (y). Accordingly it is necessary to compute the value of (v) and $\left(\frac{10^5}{B}\right)$ in the equation

$$\frac{10^5(\sigma - \sigma_o)}{(\theta_o - v)} = \frac{10^5}{B}$$

subject to the condition that

$$(S - S_o) = \lambda\left(\frac{S_o}{L}\right)(\theta_o - v) \tag{77}$$

which holds for surface elements where a drop of $(\theta_o - v)$ below the surface temperature is accompanied by the increase $(S - S_o)$ in salinity shown by equation (77). From hydrographic tables (Knudsen, 1901) the change in specific gravity caused by a change of $(S - S_o)$ in salinity equals approximately .00076 $(S - S_o)$ for the usual values of temperature and salinity in the sea. Therefore, if we denote by $f(\theta_o - v)$ the change in $10^5(\sigma - \sigma_o)$ due solely to the temperature change we get

$$10^5(\sigma - \sigma_o) = f(\theta_o - v) + 76\left(\frac{\lambda S_o}{L}\right)(\theta_o - v) \tag{17}$$

when the salinity changes according to equation (77). The corresponding value of (v) can be found by trial from tables of specific gravity of sea water and an assumed value of (λ) which need be only a rough approximation to the true value owing to the relatively small influence of salinity changes as compared to the accompanying temperature changes. As explained on page 222, (λ) is approximately unity.

To facilitate this computation table (10) has been prepared. Values of $10^5(\sigma - \sigma_o)$ and $\frac{10^5}{B} = \frac{10^5(\sigma - \sigma_o)}{(\theta_o - v)}$ are entered in the first part of the table for a series of values of θ_o and $(\theta_o - v)$ assuming a constant salinity 33.70‰. Even a very large departure from this average salinity causes but an insignificant change in the entries which are differences and ratios of differences First compute $10^5(\sigma - \sigma_o)$ corresponding to a value of (θ_o) to the nearest tenth degree by means of the supplementary table of increments of $10^5(\sigma - \sigma_o)$ and add to these the values of $(\theta_o - v)\left(\frac{C_s}{76}\right)D$ from part 3 of the supplementary table, thus obtaining corrected values of $10^5(\sigma-\sigma_o)$. The coefficient $\left(\frac{C_s}{76}\right)$ departs less than four per cent from unity within the indicated temperature range and (D) has the value $\left(76\frac{\lambda S_o}{L}\right)$. Next determine the corrected value of $\left(\frac{10^5}{B}\right)$ by adding entries read from part 2 of the supplementary tables of the increments of $\left(\frac{10^5}{B}\right)$ for the assumed value of (λ) and corresponding values of (D) and $\frac{C_s}{76}D$. Plot these corrected values of $\left(\frac{10^5}{B}\right)$ against the corrected values of $10^5(\sigma-\sigma_o)$ and read off from the graph the values of $\left(\frac{10^5}{B}\right)$ corresponding to the values of $10^5(\sigma-\sigma_o)$ estimated from the observed serial temperatures and salinities. These results may be checked as explained on page 00 where the coefficient (76) should be multiplied by one-tenth of the entry under $D=10$ in the supplementary table of $\left(\frac{C_s}{76}\right)D$.

departs less than
erature range and
ted value of $\left(\frac{10^5}{B}\right)$
tary tables of the
nd corresponding
$\left(\frac{10^5}{B}\right)$ against the
aph the values of
mated from the
results may be
(76) should be
supplementary

Table 10, Part 2

Increment of value of $\frac{10^5}{B}$ for tempera tabulated with respect to the temperature increment $(\Delta\theta_o)$ and $(\theta_o - v)$

$(\theta_o - v)$ \ $\Delta\theta_o = .1$	.2	.3	.4	.5	.6	.7	.8	.9	1.0
12 1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
13 1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
14 1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
15 1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
16 1:0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
17 1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3

Table 10, Part 1

Tabulation of $10^5(\sigma - \sigma_o)$ and $\frac{10^5}{B} = \frac{10^5(\sigma - \sigma_o)}{\theta_o - v}$ with respect t difference $(\theta_o - v)$ for constant sa

$\theta_o =$	10		11		12		13		14		15		16		17		18
$10^5(\sigma_o - 1) =$	2595.5		2577.8		2559.5		2540.3		2520.0		2498.8		2476.9		2453.7		2430.
$(\theta_o - v)$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma - \sigma_o)$
0	0	16.9	0	18.2	0	18.7	0	19.5	0	20.7	0	21.6	0	22.3	0	23.6	0
1	16.4	16.4	17.7	17.7	18.3	18.3	19.2	19.2	20.3	20.3	21.2	21.2	21.9	21.9	23.2	23.2	23.6
2	31.8	15.9	34.1	17.0	36.0	18.0	37.5	18.7	39.5	19.7	41.5	20.8	43.1	21.5	45.1	22.6	46.8
3			49.5	16.5	52.4	17.4	55.2	18.4	57.8	19.3	60.7	20.2	63.4	21.1	66.3	22.1	68.7
4					67.8	16.9	71.6	17.9	75.5	18.9	79.0	19.8	82.6	20.7	86.6	21.7	89.9
5							87.0	17.4	91.9	18.4	96.7	19.3	100.9	20.2	105.8	21.1	110.2
6									107.3	17.9	113.1	18.9	118.6	19.8	124.1	20.7	129.4
7											128.5	18.4	135.0	19.3	141.8	20.2	147.7
8													150.4	18.8	158.2	19.8	165.4
9															173.6	19.3	181.8
10																	197.2
11																	
12																	
13																	
14																	
15																	
16																	
17																	

TABLE 10, PART 1

Tabulation of $10^5(\sigma-\sigma_o)$ and $\frac{10^5}{B}=\frac{10^5(\sigma-\sigma_o)}{\theta_o-v)}$ with respect to the surface temperature θ_o and the difference (θ_o-v) for constant salinity=33.70

15 2498.8		16 2476.9		17 2453.7		18 2430.1		19 2405.0		20 2379.2		21 2352.6		22 2325.1		23 2296.8		24 2267.5		25 2237.5		
$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	$10^5(\sigma-\sigma_o)$	$\frac{10^5}{B}$	(θ_o-v)
0	21.6	0	22.3	0	23.6	0	23.9	0	25.5	0	26.3	0	27.1	0	27.9	0	28.8	0	29.8	0	30.5	0
21.2	21.2	21.9	21.9	23.2	23.2	23.6	23.6	25.1	25.1	25.8	25.8	26.6	26.6	27.5	27.5	28.3	28.3	29.3	29.3	30.0	30.0	1
41.5	20.8	43.1	21.5	45.1	22.6	46.8	23.4	48.7	24.3	50.9	25.4	52.4	26.2	54.1	27.0	55.8	27.9	57.6	28.8	59.3	29.6	2
60.7	20.2	63.4	21.1	66.3	22.1	68.7	22.9	71.9	23.9	74.5	24.8	77.5	25.8	79.9	26.6	82.4	27.5	85.1	28.4	87.6	29.2	3
79.0	19.8	82.6	20.7	86.6	21.7	89.9	22.5	93.8	23.4	97.7	24.4	101.1	25.3	105.0	26.2	108.2	27.0	111.7	27.9	115.1	28.8	4
96.7	19.3	100.9	20.2	105.8	21.1	110.2	22.0	115.0	23.0	119.6	23.9	124.3	24.8	128.6	25.7	133.3	26.6	137.5	27.5	141.7	28.3	5
113.1	18.9	118.6	19.8	124.1	20.7	129.4	21.6	135.3	22.6	140.8	23.4	146.2	24.4	151.8	25.3	156.9	26.1	162.6	27.1	167.5	27.9	6
128.5	18.4	135.0	19.3	141.8	20.2	147.7	21.1	154.5	22.1	161.1	23.0	167.4	23.9	173.7	24.8	180.1	25.8	186.2	26.6	192.6	27.5	7
		150.4	18.8	158.2	19.8	165.4	20.7	172.8	21.6	180.3	22.5	187.7	23.4	194.9	24.4	202.0	25.2	209.4	26.2	216.2	27.0	8
				173.6	19.3	181.8	20.2	190.5	21.2	198.6	22.0	206.9	23.0	215.2	23.9	223.2	24.8	231.3	25.7	239.4	26.6	9
						197.2	19.7	206.9	20.7	216.3	21.6	225.2	22.5	234.4	23.4	243.5	24.3	252.5	25.2	261.3	26.1	10
								222.3	20.4	232.7	21.2	242.9	22.1	252.7	23.0	262.7	23.9	272.8	24.8	282.5	25.7	11
										248.1	20.7	259.3	21.6	270.4	22.5	281.0	23.4	292.0	24.3	302.8	25.2	12
												274.7	21.1	286.8	22.0	298.7	23.0	310.3	23.9	322.0	24.8	13
														302.2	21.6	315.1	22.5	328.0	23.4	340.3	24.3	14
																330.5	22.0	344.4	23.0	358.0	23.9	15
																		359.8	22.5	374.4	23.4	16
																				389.8	22.9	17

TABLE 10, PART 2

Increment of value of $\frac{10^5}{B}$ for temperature tabulated with respect to the temperature increment ($\Delta\theta_o$) and ($\theta_o - v$)

$(\theta_o - v)$ \ $\Delta\theta_o =$ .1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0 .1	.2	.3	.4	.5	.6	.7	.8	.9	1.00
1 .1	.2	.3	.4	.5	.6	.7	.8	.9	1.00
2 .1	.2	.3	.4	.5	.6	.7	.8	.9	1.00
3 .1	.2	.3	.4	.5	.6	.7	.7	.8	.93
4 .1	.2	.3	.3	.4	.5	.6	.7	.8	.87
5 .1	.2	.3	.4	.5	.6	.7	.8	.9	.96
6 .1	.2	.3	.4	.5	.6	.6	.7	.8	.92
7 .1	.2	.3	.4	.5	.5	.6	.7	.8	.90
8 .1	.2	.3	.4	.5	.6	.6	.7	.8	.92
9 .1	.2	.3	.4	.5	.5	.6	.7	.8	.91
10 .1	.2	.3	.4	.5	.6	.6	.7	.8	.92
11 .1	.2	.3	.4	.5	.5	.6	.7	.8	.91
12 .1	.2	.3	.4	.5	.6	.6	.7	.8	.92
13 .1	.2	.3	.4	.5	.5	.6	.7	.8	.91
14 .1	.2	.3	.4	.5	.5	.6	.7	.8	.91
15 .1	.2	.3	.4	.5	.5	.6	.7	.8	.91
16 .1	.2	.3	.4	.5	.5	.6	.7	.8	.91
17									

TABLE 10, PART 2 (*Continued*)

Value of $\left(\frac{C_s}{76}\right)D$ = increment of $\frac{10^5}{B}$ tabulated with respect to (D) and ($\theta_o - v$)

$(\theta_o - v)$	$D=$ 1	2	3	4	5	6	7	8	9	10
0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
1	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
2	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
3	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
4	1.0	2.0	3.0	4.0	5.0	6.1	7.1	8.1	9.1	10.1
5	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
6	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
7	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
8	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
9	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
10	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
11	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
12	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
13	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
14	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
15	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
16	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3
17	1.0	2.1	3.1	4.1	5.1	6.2	7.2	8.2	9.3	10.3

TABLE 10, PART 3

Increment of value of $10^5(\sigma-\sigma_o)$ for temperature, tabulated with respect to the temperature increment $(\Delta\theta_o)$ and (θ_o-v)

$(\theta_o=v)$ \ $\Delta\theta_o=.1$	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0	0	0	0	0	0	0	0	0	0
1	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
2	.2	.4	.6	.8	1.0	1.2	1.4	1.6	1.8	2.0
3	.3	.6	.8	1.1	1.4	1.7	2.0	2.2	2.5	2.8
4	.4	.7	1.0	1.4	1.7	2.1	2.5	2.8	3.1	3.5
5	.5	1.0	1.4	1.9	2.4	2.9	3.4	3.8	4.3	4.8
6	.5	1.1	1.6	2.2	2.7	3.3	3.9	4.4	4.9	5.5
7	.6	1.3	1.9	2.5	3.1	3.8	4.4	5.0	5.7	6.3
8	.7	1.5	2.3	3.0	3.7	4.4	5.2	5.9	6.6	7.4
9	.8	1.6	2.5	3.3	4.1	4.9	5.7	6.6	7.4	8.2
10	.9	1.8	2.8	3.7	4.6	5.5	6.4	7.3	8.3	9.2
11	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
12	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9	11.0
13	1.2	2.4	3.5	4.7	5.9	7.1	8.3	9.4	10.6	11.8
14	1.3	2.5	3.8	5.1	6.3	7.6	8.9	10.2	11.4	12.7
15	1.4	2.7	4.1	5.5	6.8	8.2	9.6	11.0	12.3	13.7
16	1.5	2.9	4.4	5.8	7.3	8.7	10.2	11.7	13.1	14.6
17										

TABLE 10, PART 3 (*Continued*)

Values of $(\theta_o-v)\left(\frac{C_s}{76}\right)D$ = increment of $10^5(\sigma-\sigma_o)$ tabulated with respect to (D)

(θ_o-v) \ $D=$ 1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
2	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0	20.0
3	3.0	6.0	9.3	12.3	15.3	18.3	21.3	24.6	27.6	30.6
4	4.0	8.0	12.0	16.0	20.0	24.4	28.4	32.4	36.4	40.4
5	5.0	10.0	15.5	20.5	25.5	30.5	35.5	41.0	46.0	51.0
6	6.0	12.0	18.6	24.6	30.6	36.6	42.6	49.2	55.2	61.2
7	7.0	14.0	21.7	28.7	35.7	42.7	49.7	57.4	64.4	71.4
8	8.0	16.8	24.8	32.8	40.8	49.6	57.6	65.6	74.7	82.4
9	9.0	18.0	27.9	36.9	45.9	54.9	63.9	73.8	82.8	91.8
10	10.0	20.0	31.0	41.0	51.0	61.0	71.0	82.0	92.0	102.0
11	11.0	23.1	34.1	45.1	56.1	67.1	79.2	90.4	102.3	113.4
12	12.0	24.0	37.2	49.1	61.1	73.1	85.2	97.5	110.5	122.4
13	13.0	27.3	40.3	53.3	66.4	80.6	93.5	106.6	121.0	133.9
14	14.0	29.4	43.4	57.4	71.4	86.7	100.8	114.6	130.0	144.0
15	15.0	31.5	46.5	61.5	76.5	93.0	108.0	123.0	139.4	154.4
16	16.0	33.6	49.6	65.6	81.6	99.1	115.1	131.0	148.8	164.8
17	17.0	35.7	52.7	69.7	86.7	105.3	122.4	139.2	158.0	175.0

LITERATURE CITED

Bowen, I. S.
1926. The ratio of heat losses by conduction and by evaporation from any water surface. Physical Review, **27**:779-787, 1 fig. in text.

Cummings, N. W., and Richardson, Burt.
1927. Evaporation from lakes. Physical Review, 30, 527–534, 4 figs. in text.

Ekman, V. W.
1905–06. On the influence of the earth's rotation on ocean currents. Arkiv. für Matematik, Astronomi, och Fysik, 2, 1–53, 1 pl., 10 figs. in text.
1927. Meeresströmungen, in Handbuch der Physikalischen und Technischen Mechanik, **5**:177–206, 10 figs. in text.

Fowle, Frederick E.
1914. Smithsonian Physical Tables, Smithsonian Miscellaneous Collections, **63**:xxxvi+355.

Freeman, John R.
1926. Regulation of elevation and discharge of the Great Lakes. xxiii+548 pp., 110 figs. in text. Published by the Board of Trustees of the Sanitary District of Chicago.

Glaisher, T. W. L.
1870. Tables of the numerical values of the sine-integral, cosine-integral, and exponential-integral. Philos. Trans. Roy. Soc. London, **160**:367–388, 3 figs. in text.

Hayashi, Keiichi.
1926. Sieben-und mehrstellige Tafeln der Kreis-und Hyperbelfunctionen und deren Produkte sowie der Gamma-function (Berlin, Julius Springer), v+283 pp., 1 fig. in text.

Horton, Robert E., and Grunsky, C. E.
1927. Report of the Engineering Board of Review of the Sanitary District of Chicago on the Lake Lowering Controversy and a Program of Remedial Measures, xvii+432 pp., 7 figs. in text. Published by the Board of Trustees of the Sanitary District of Chicago.

Jeffreys, Harold.
1920. On turbulence in the ocean. Philos. Mag., **39**:578–586.

Knudsen, M.
1901. Hydrographische Tabellen (Copenhagen, Gad), v+63 pp.

McEwen, G. F.
1912. The distribution of ocean temperatures along the west coast of North America, deduced from Ekman's theory of the upwelling of cold water from adjacent ocean depths. Internat. Review ges. Hydrobiol. u. Hydrog., **5**:234–286, 21 figs. in text.
1914. Peculiarities of the California climate, explained on the basis of general principles of atmospheric and oceanic circulation. Monthly Weather Review, **42**:14–23, 13 figs. in text.
1915. Oceanic circulation and temperature off the Pacific Coast, *in* Nature and Science on the Pacific Coast (San Francisco, Elder), pp. 132–140, 4 figs. in text.
1916. Summary and interpretation of the hydrographic observations made by the Scripps Institution for Biological Research of the University of California, 1908–15. Univ. Calif. Publ. Zool., **15**:255–356, pls. 1–38.

1919. Ocean temperatures, their relation to solar radiation and oceanic circulation. Semicentennial Publications of the University of California, Miscellaneous Studie , pp. 334–421, 19 figs. in text.

1920. "Physical Constants" pertaining to the ocean. Science, n.s., **52**:62–63.

1922. Suggestions relative to the application of mathematical methods to certain basic problems of dynamical oceanography. Bulletin of the National Research Council, **3**, pt. 2, 78–81.

1924. A mathematical theory of the temperature distribution in water due to solar radiation, evaporation, and convection. *Ibid.*, **7**, pt. 5:139–140.

1927. Dynamical oceanography and certain elements of hydrodynamics upon which it is based. Bulletin no. 14 (non-technical) of the Scripps Institution of Oceanography of the University of California, pp. 1–20.

Michael, E. L., and McEwen, G. F.

1915. Hydrographic, plankton and dredging records of the Scripps Institution for Biological Research of the University of California, 1901–1912. Univ. Calif. Publ. Zool., **15**:1–206, 4 figs., 1 map in text.

1916. Continuation of hydrographic, plankton, and dredging records of the Scripps Institution for Biological Research of the University of California, 1913–1915. *Ibid.*, **15**:207–254, 4 figs. in text.

Miller, W. Lash, and Rosebrugh, T. R.

1903. Numerical values of certain functions involving e^{-x}. Proc. and Trans. Roy. Soc. Canada, ser. 2, **9**:73–107.

Pearson, K.

1914. Tables for statisticians and biometricians (Cambridge, University Press), lxxxiii+143 pp., 11 figs. in text.

1922. Tables of the incomplete Γ-function computed by the staff of the department of applied statistics, University of London, University College (London, His Majesty's Stationery Office), xxxi+164 pp.

Richardson, Burt, and Montgomery, Carol.

1929. The measurement of insolation by means of a pan. Transactions of the American Geophysical Union Ninth Annual Meeting, April 26 and 27, 1929, in Bulletin of the National Research Council, No. 68, pp. 56–61.

Richardson, Burt.

1929. Evaporation as a function of insolation. Proceedings of the American Society of Civil Engineers, (In press, June, 1929)

Schmidt, Wilhelm.

1915. Strahlung und Verdunstung an freien Wasser Flächen. Ein Beitrage zum Wärmehaushalt des Weltmeers und um Wasserhaushalt der Erde. Ann. Hydr. u. Marit. Meteor., **43**:111–124.

Taylor, G. I.

1915. Eddy motion in the atmosphere. Philos. Trans. Roy. Soc. London, **215**, ser. A: 1–26, 5 figs. in text.

Thorade, H.

1923. Ungeordnete Bewegung und Mischung im Meere. Die Naturwissenschaften, **1**:1001–1006.

Von Sanden, H.

1923. Practical mathematical analysis (London, Methuen and Co.), xv+194 pp., 27 figs. in text.

Wüst, G.

1920. Die Verdunstung auf dem Meere. Veröffentlichungen des Instituts für Meereskunde an der Universität Berlin. A. Geographisch-naturwissenschaftliche Reihe, No. 6:1–95, 11 figs. in text.