

A Newton-Imbedding Procedure for Solutions of Semilinear Boundary Value Problems in Sobolev Spaces

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*Dedicated to Professor Guo Chun Wen
on the occasion of his 70th birthday*

Abstract

This paper is concerned with the application of the Newton-imbedding iteration procedure to nonlinear boundary value problems in Sobolev spaces. A simple model problem for the second-order semilinear elliptic equations is considered to illustrate the main idea. The essence of the method hinges on the a priori estimates of solutions of the associated linear problem in appropriate Sobolev spaces. It is to our surprise that $H^1(\Omega)$ -solution is not smooth enough to guarantee the convergence of the sequence generated by the procedure. Existence and uniqueness of solution to the original nonlinear problem are established constructively. An application of this approach to the Lamé system with nonlinear body force and its generalization to contain a nonlinear surface traction in elasticity will also be discussed.

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1 Introduction

In [19], an integral equation method was introduced for solving boundary value problem for generalized analytic functions. By a combination of an imbedding method and the Newton iteration procedure according to [18], this integral equation method then leads to a constructive scheme for solving semilinear first-order problems [19], [21]. Such combinations between imbedding and Newton's iteration procedure have been successfully employed for existence proofs for the solutions to boundary value problems for a class of first-order semilinear elliptic systems with linear and nonlinear boundary conditions (see, e.g., [2], [3] - [6], [8], [20]).

In this paper, we shall adopt a similar approach employed in [19], however, to the construction of variational solutions of semilinear second-order boundary value problems in Sobolev spaces. As will be seen, this approach depends primarily on the nature of the associated linear equation derived from the original semilinear equation by applying the Newton iteration procedure. The basic mathematical ingredients here are the a priori estimates of the solutions of the linear problem in suitable function spaces. The associated linear problem in fact can be reduced to a pair of coupled linear integral equations, one integral equation of the second kind over the domain, and one boundary integral equation of the first kind. The existence and uniqueness of the solutions of this system of integral equations are still remained to be analyzed. Nevertheless, the present approach gives a practical numerical scheme and offers an additional method in the connection of application of boundary element methods to nonlinear problems where the nonlinearity occurs in the partial differential equations of the semilinear type (see e.g., [9],[11],[10], and [7]).

We organize the paper as follows: In Section 2, the Newton-imbedding method is described by using a simple model problem. As will be seen, in each iteration step, it involves a linear boundary value problem for a linear equation which contains a zero-th order term with variable coefficient. We refer to a typical problem of these forms as the associated linear problem. Section 3 contains the relevant a priori estimates of the solution of the associated problem and a reduction of the problem to the coupled integral equations. The a priori estimates will be needed for the convergence proof for the Newton-imbedding iteration procedure in Section 4. We conclude the paper by extending the approach to the case of a semilinear equation in elasticity.

2 The Newton-imbedding procedure

Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary Γ . We consider the simple model problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad (2.1)$$

$$u|_{\Gamma} = \varphi(x) \quad \text{on } \Gamma, \quad (2.2)$$

where as usual, Δ denotes the Laplacian. Here f and φ are given data satisfying certain regularity conditions which will be specified later in order to ensure the existence of the unique solution of the problem. To describe the procedure, our first step is to imbed the problem in a family of problems (P_t) which consists of

$$-\Delta u = tf(u) \quad \text{in } \Omega$$

together with the same boundary condition (2.2) as before. Here $t \in [0, 1]$ is the imbedding parameter. If we denote the solution of (P_t) by $u(x, t)$ which we assume to exist for the time being, then $u(x, 1)$ is the desired solution of the original problem, (2.1), (2.2).

We note that for $t = 0$, $u(x, 0)$ is the solution of the Dirichlet problem for the Laplacian, and can be easily constructed by the boundary element method. Using the solution, $u(x, 0)$ as the initial approximation, we may solve (P_t) for $t = t_1 > 0$ by the Newton iteration procedure. The known solution $u(x, t_1)$ can then be used again as the initial approximation for $u(x, t_2)$ with $t_2 > t_1$ by the Newton iteration procedure. We shall show that after finitely many steps such that

$$0 = t_0 < t_1 < t_2, \dots < t_N = 1,$$

the solution of the problem (P_t) for $t = 1$, the original problem, can be found.

Now for the Newton iteration procedure, let us assume that $u(x, t_{j-1})$ to be the solution of the problem $(P_{t_{j-1}})$ for a fixed t_{j-1} , $0 \leq t_{j-1} < 1$. Then as an approximation to the solution of (P_t) for $t = t_j$, $t_{j-1} < t_j \leq 1$, we define the sequence $u_n(x, t_j)$ by

$$u_0(x, t_j) = u(x, t_{j-1}) \quad (2.3)$$

and by solving the linear boundary value problem below for $u_{n+1}(x, t_j)$, $n = 0, 1, \dots$:

$$-\Delta u_{n+1} = t_j f'(u_n) \{u_{n+1} - u_n\} + t_j f(u_n) \quad \text{in } \Omega \quad (2.4)$$

$$u_{n+1}|_{\Gamma} = \varphi \quad \text{on } \Gamma. \quad (2.5)$$

Such a combination between imbedding and the Newton iteration procedure is generally known as the *Newton-imbedding procedure* (see e.g., [18]). We remark that in the standard Newton's method, the convergence of the approximate sequence depends crucially on the choice of initial guess, which requires to be sufficiently close to the exact solution. However, in the Newton-imbedding scheme, as will be seen, as long as $t_j - t_{j-1}$ is sufficiently small, the initial choice of $u_0(x, t_j)$ given by (2.3) will be a good approximation to $u(x, t_j)$ so that convergence will be ensured.

3 Associated linear problem

We notice that in each of the iterations in the procedure given by (2.4),(2.5), we arrive at a typical linear boundary value problem of the form:

$$-\Delta v + q(x)v = g(x), \quad x \in \Omega, \quad (3.1)$$

$$v|_{\Gamma} = \varphi \quad \text{on } \Gamma. \quad (3.2)$$

Here both the variable coefficient $q(x)$ and the nonhomogeneous term $g(x)$ are known and they are related to f' and f in an obvious manner. In the following, we shall refer to (3.1), (3.2) as the associated linear problem for the nonlinear problem (2.1), (2.2). We are interested in the weak solution of this associated problem in the Sobolev space $H^1(\Omega)$. It is well known that there exists a unique solution $v \in H^1(\Omega)$ for given $g \in L^2(\Omega)$, and $\varphi \in H^{1/2}(\Gamma)$, provided $q \in C^\alpha(\overline{\Omega})$, $\alpha > 0$ with the property that $(qv, v)_0 \geq 0$ for all $v \in H^1(\Omega)$. However, it is to our surprise that $H^1(\Omega)$ -solution is not smooth enough to guarantee the convergence of the sequence $\{u_n\}$ defined by (2.3)-(2.5) in the Newton-imbedding iteration procedure. We need a $H^2(\Omega)$ -solution (or at least a solution $v \in H^s(\Omega)$ with $s - 3/2 \geq \alpha > 0$) in order that the method works. To be more precise, we begin with the following a priori estimate.

Lemma 3.1 *Let $q \in C^\alpha(\overline{\Omega})$ be given such that $\|q\|_{C^\alpha(\overline{\Omega})} \leq M$ for some constant $M > 0$ and let q be positive in the sense that*

$$0 < (qv, v)_0 \quad \text{for all } 0 \neq v \in H^2(\Omega). \quad (3.3)$$

Then for any $v \in H^2(\Omega)$ there holds the a priori estimate:

$$\|v\|_{H^2(\Omega)} \leq c_q \{ \| -\Delta v + qv \|_{L^2(\Omega)} + \|v|_{\Gamma}\|_{H^{3/2}(\Gamma)} \}, \quad (3.4)$$

where c_q is a constant depending on q .

Proof. Let $H_0^1(\Omega)$ denote the Sobolev space defined by

$$H_0^1(\Omega) := \{v \in H^1(\Omega) \mid v|_\Gamma = 0 \text{ on } \Gamma\}.$$

Then by the second Green formula, we have for any $v \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$a(v, w) + (qv, w)_0 = (-\Delta v + qv, w)_0 \quad \text{for all } w \in H_0^1(\Omega). \quad (3.5)$$

Here $a(v, w)$ is the bilinear form

$$a(v, w) := \int_{\Omega} \nabla v \cdot \nabla w dx,$$

and $(\cdot, \cdot)_0$ stands for the $L^2(\Omega)$ inner product. (Without loss of generality, we have assumed here the real $L^2(\Omega)$.) The positivity of q implies that

$$a(v, v) \leq \|-\Delta v + qv\|_0 \|v\|_{H^1(\Omega)}.$$

As a consequence of the Poincaré inequality, we obtain the estimate

$$\|v\|_{H^1(\Omega)} \leq c \|-\Delta v + qv\|_0 \quad \text{for any } v \in H_0^1(\Omega) \cap H^2(\Omega) \quad (3.6)$$

with some constant c independent of q . The later shows that the mapping

$$\mathcal{L} : H^2(\Omega) \longrightarrow L^2(\Omega) \times H^{3/2}(\Gamma)$$

defined by

$$\mathcal{L}v := (-\Delta v + qv, v|_\Gamma)$$

is a continuous isomorphism which gives the desired estimates (3.4). This completes the proof of the lemma.

Lemma 3.2 *Under the same assumption of q in Lemma 3.1, we have the estimate*

$$\|v\|_{H^2(\Omega)} \leq \kappa \|-\Delta v + qv\|_{L^2(\Omega)} \quad \text{for any } v \in H_0^1(\Omega) \cap H^2(\Omega), \quad (3.7)$$

where the constant κ depends on M but not on q .

Proof. Let $H_0^1(\Omega) \cap H^2(\Omega)$. Then we have the estimates

$$\begin{aligned} \|v\|_{H^2(\Omega)} &\leq c_0 \|-\Delta v\|_0 \\ &\leq c_0 (\|-\Delta v + qv\|_0 + \|qv\|_0) \\ &\leq \kappa(M) \|-\Delta v + qv\|_0 \end{aligned}$$

for some constant depending κ depending on the boundness M of $\|q\|_{C^\alpha(\bar{\Omega})}$. The last step in the estimate follows from (3.6).

We remark that from the proof of Lemma 3.1, it is clear that the estimate also holds for $(qv, v)_0 \geq 0$. However, our integral equation method works only for either q is positive or $q \equiv 0$. In the latter, the function f in (2.1) is no longer depending on u . Hence in what follows we assume that q is always positive.

As a consequence of Lemma 3.1, we have the following existence and uniqueness results for the solution of the associated problem.

Lemma 3.3 *For given $g \in L^2(\Omega)$ and $\varphi \in H^{3/2}(\Gamma)$, under the assumptions of q in Lemma 3.1, the problem (3.1), (3.2) has a unique solution $v \in H^2(\Omega)$.*

The constructive solutions of the associated linear problem can be obtained by using finite element or integral equation methods for the numerical approximations. We now give a brief description of the later. We seek a solution in the form

$$\begin{aligned} v(x) &= \int_{\Gamma} E(x, y)\sigma(y)ds_y + \int_{\Gamma} \frac{\partial}{\partial n_y} E(x, y)\varphi(y)ds_y + \int_{\Omega} E(x, y)g(y)dy \\ &- \int_{\Omega} E(x, y)v(y)dy, \quad x \in \Omega, \end{aligned} \quad (3.8)$$

where

$$E(x, y) = \frac{1}{4\pi} \frac{1}{|x - y|}$$

is the fundamental solution of $-\Delta$ and

$$\sigma = \gamma_1 v \frac{\partial}{\partial n} v|_{\Gamma} \quad \text{and} \quad \varphi = \gamma_0 = v|_{\Gamma}$$

are the Cauchy data of v . In operator form this becomes

$$v + \mathcal{T}v = \mathcal{V}\sigma - \mathcal{W}\varphi + \mathcal{N}g, x \in \Omega,$$

where \mathcal{V} , and \mathcal{W} are the simple and double -layer surface potential operators, where \mathcal{N} is the Newtonian potential operator and $\mathcal{T}v := \mathcal{N}(qv)$ is a modified Newtonian operator. In this representation, $v \in H^2(\Omega)$ and $\sigma \in H^{1/2}(\Gamma)$ are the unknowns. By using the boundary condition (2.2), and from the

properties of these potentials, the associated linear problem is reduced to the following system of integral equations

$$(I + \mathcal{T})v - \mathcal{V}\sigma = \mathcal{N}g - \mathcal{W}\varphi \quad \text{in } \Omega, \quad (3.9)$$

$$-\gamma_0 \mathcal{T}v + V\sigma = -\gamma_0 \mathcal{N}g + (1/2I + K)\varphi \quad \text{on } \Gamma, \quad (3.10)$$

where V and K are the familiar simple- and double-layer boundary integral operators such that $V = \gamma_0 \mathcal{V}$ and $K = I + \gamma_0 \mathcal{W}$. These is a coupling of domain and boundary integral equations. In particular, we know that the mapping defined by

$$\mathcal{T} : H^2(\Omega) \rightarrow H^2(\Omega)$$

is compact while the mapping defined by

$$V : H^{1/2}(\Gamma) \rightarrow H^{3/2}(\Gamma)$$

is an isomorphism. Equation (3.9) corresponds a domain integral equation of the second kind for the unknown v and (3.10) a boundary integral equation of the first kind for the unknown σ . The existence and uniqueness results of these equations can be best analyzed by using the theory of pseudo-differential operators (see, e.g., [15]) and will not be pursued here.

Lemmas 3.2 and 3.3 form the basis for our scheme. The a priori estimate (3.7) provides the tool for establishing the convergence of the sequence in the Newton-imbedding iterations.

4 Convergence results

In this section, we now devote to the convergence proof of the scheme (2.3)-(2.5), and their related questions. Let us first begin formally with the problem satisfied by the difference $(u_{n+1} - u_n)$,

$$\begin{aligned} -\Delta(u_{n+1} - u_n) &= t_j \{ f'(u_n)(u_{n+1} - u_n) \\ &\quad + \frac{1}{2} \tilde{f}_{uu}(u_{n-1})(u_n - u_{n-1})^2 \} \quad \text{in } \Omega, \end{aligned} \quad (4.1)$$

$$(u_{n+1} - u_n)|_{\Gamma} = 0 \quad \text{on } \Gamma,$$

where \tilde{f}_{uu} is defined by the mean value formula

$$\tilde{f}_{uu}(u_{n-1}) := \int_0^1 f''(\tau u_n + (1 - \tau)u_{n-1})\tau d\tau. \quad (4.2)$$

In order to justify the procedure defined by (2.3)-(2.5) rigorously, and to make use of the a priori estimate (3.7) for the convergence proof, we now make the following assumptions on the nonlinear term f :

(A1) The functions f is continuous from $H^2(\Omega)$ to $L^2(\Omega)$, while its derivatives f' and f'' are continuous from $H^1(\Omega)$ to $C^\alpha(\bar{\Omega})$. Moreover, f and its derivatives f' and f'' are all bounded: There is a constant $M > 0$ such that

$$\|f\|_{L^2(\Omega)} \leq M, \quad \|f'\|_{C^\alpha(\bar{\Omega})} \leq M, \quad \|f''\|_{C^\alpha(\bar{\Omega})} \leq M.$$

(A2) The derivative $-f'$ is positive in the sense that for any $u \in H^2(\Omega)$,

$$(-f'(u)v, v)_0 > 0 \quad \text{for all } 0 \neq v \in H^2(\Omega).$$

Under these assumptions, we first note that the linear problems defined by (2.4),(2.5) are uniquely solvable and each of the solutions $u_{n+1}(x)$ is again in $H^2(\Omega)$, if $u_n(x)$ belongs to $H^1(\Omega)$. By applying the a priori estimate (3.7) to (4.1), we obtain the estimate

$$\begin{aligned} \|u_{n+1} - u_n\|_{H^2(\Omega)} &\leq \kappa \left\| \frac{1}{2} t_j \tilde{f}_{uu}(u_{n-1})(u_n - u_{n-1})^2 \right\|_{L^2(\Omega)} \\ &\leq \left(\frac{\kappa}{2} t_j M \|u_n - u_{n-1}\|_{C^\alpha(\bar{\Omega})} \right) \|u_n - u_{n-1}\|_{L^2(\Omega)} \\ &\leq \left(\frac{\kappa}{2} t_j c_0 M \|u_n - u_{n-1}\|_{H^2(\Omega)} \right) \|u_n - u_{n-1}\|_{H^2(\Omega)} \end{aligned} \quad (4.3)$$

for $n = 0, 1, \dots$, where c_0 is a constant due to the Sobolev imbedding theorem (see, e.g., [1]). Then from (4.2) it is not difficult to show that the sequence $\{u_n\}$ converges in $H^1(\Omega)$, if

$$\frac{\kappa}{2} t_j c_0 M \|u_1 - u_0\|_{H^2(\Omega)} < 1. \quad (4.4)$$

Assume that (4.3) holds and denote by u_* the limit of the sequence $\{u_n\}$ in $H^1(\Omega)$. One can easily verify that $u_* \in H^2(\Omega)$ satisfies an equation of the form

$$-\Delta u_* t_j f(u_*) = R_n(u_*) \quad \text{in } \Omega.$$

Here $R_n(u_*)$ is given by

$$R_n(u_*) = -\Delta(u_* - u_{n+1}) - t_j(f(u_*) - f(u_n)) + t_j(f'(u_n)(u_{n+1} - u_n)).$$

Thus by the continuity of f and the boundness of f' , it can be shown that

$$\|R_n(u_*)\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Also we see that

$$\begin{aligned} \|u_*|_\Gamma - \varphi\|_{H^{3/2}(\Gamma)} &= \|(u_* - u_{n+1})|_\Gamma\|_{H^{3/2}(\Gamma)} \\ &\leq c\|u_* - u_{n+1}\|_{H^2(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence by the uniqueness of the solution of (2.1), (2.2) below, we conclude that the sequence $\{u_n\}$ converges to $u_* = u(x, t_j)$, the solution of (2.1),(2.2) for $t = t_j$.

To fulfill the condition (4.4), it suffices to require t_j be chosen near t_{j-1} enough so that

$$t_j - t_{j-1} < \left(\frac{1}{2}c_0\kappa^2M^2\right)^{-1} \quad (4.5)$$

for $j = 1, 2, \dots$. Indeed, from the definition of u_0 and u_1 , we arrive at the estimate

$$\begin{aligned} \|u_1 - u_0\|_{H^2(\Omega)} &\leq \kappa\|(t_j - t_{j-1})f(u_0)\|_{L^2(\Omega)} \\ &\leq \kappa M(t_j - t_{j-1}). \end{aligned} \quad (4.6)$$

According to (4.6), if we choose t_j such that

$$\frac{\kappa^2}{2}c_0 t_j(t_j - t_{j-1})M^2 < 1$$

or

$$t_j(t_j - t_{j-1}) < \left(\frac{1}{2}c_0\kappa^2M^2\right)^{-1},$$

$j = 1, 2, \dots$, then the condition (4.4) will be satisfied. Since $t_j \leq 1$, the condition (4.5) will be sufficient for this purpose. We emphasize that the right hand side of (4.5) provides a uniform bound for the difference $(t_j - t_{j-1})$, the imbedding method may be extended to $t = 1$ in finitely many steps.

We now turn our attention to the uniqueness of the solution of problem defined by (2.1), (2.2). Let u and v be any two solutions of the problem. Then the difference must satisfy the boundary value problem of the form

$$\begin{aligned} -\Delta(u - v) &= \tilde{f}_u(u - v) \quad \text{in } \Omega, \\ (u - v)|_\Gamma &= 0 \quad \text{on } \Gamma. \end{aligned}$$

The function \tilde{f}_u is defined similarly as in (4.2) and is negative by the assumption of (A2). Hence it follows from Lemma 3.3, $u \equiv v$.

We summarize all these results up to now in the following theorem.

Theorem 4.1 *For given $\varphi \in H^{3/2}(\Omega)$, under the assumptions of (A1)-(A2), the boundary value problem, (??), (2.2) has a unique solution $u \in H^2(\Omega)$ which can be constructed by the Newton-embedding method, provided the condition (4.5) is satisfied.*

We remark that from the computational point of view, it is important to know the rate of convergence as well as the accuracy of the approximate solutions, if one uses $u_n(x, t_j)$ to approximate the exact solution $u(x) = u(x, 1)$. Indeed, this information is revealed in the theorem below.

Theorem 4.2 *Let α be any given real number such that $0 < \lambda < 1$. For any fixed $t_j, 0 \leq t_j \leq 1$, suppose that the t -step size of the imbedding satisfies the uniform bound*

$$t_k - t_{k-1} \leq \frac{2\lambda}{c_0 \kappa^2 M^2}$$

for $k = 1, 2, \dots, j$. Then the following error estimate holds

$$\|u - u_n(\cdot, t_j)\|_{H^2(\Omega)} \leq \sqrt{c_0} \kappa M (1 - t_j) + 2(\lambda t_j)^{2(n-1)}.$$

The proof of this theorem is lengthy but straightforward. We omit the details. We comment that in the estimate, the first term on the right is due to the contribution of $\|u - u(\cdot, t_j)\|_{H^2(\Omega)}$, while the second term is from the estimate of $\|u(\cdot, t_j) - u_n(\cdot, t_j)\|_{H^2(\Omega)}$. Since $\lambda < 1$, the error will be eventually dominated by the first term for n sufficiently large, say n_0 , such that the iteration solution $u_{n_0}(x, t_j)$ will give just a good approximation to u as all the higher order terms $u_n(x, t_j)$, for $n > n_0$. Hence one can stop the iteration there.

5 Concluding Remarks

In this concluding remarks, we wish first to exploit the possibility of extending the approach for the model problem in Section 2 to nonlinear problems in elasticity. As a natural extension, we consider the Dirichlet problem for the semilinear Lamé system

$$-\Delta^* \mathbf{u} := -\mu \Delta \mathbf{u} - (\lambda + \mu) \text{grad div } \mathbf{u} = \mathbf{f}(\mathbf{u}) \quad \text{in } \Omega, \quad (5.1)$$

$$\mathbf{u}|_{\Gamma} = \varphi \quad \text{on } \Gamma. \quad (5.2)$$

Again $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ , and we are interested in the weak solution for the displacement field $\mathbf{u} \in (H^1(\Omega))^3$. Here μ and λ as usual denote the Lamé constants with $\mu > 0$, and $3\lambda + 2\mu > 0$; $\boldsymbol{\varphi} \in (H^{1/2}(\Gamma))^3$ is the prescribed surface displacement vector, while \mathbf{f} is the nonlinear body force. From the model problem of (2.1), (2.2), it suggests that we should modify the assumptions $A(1) - A(2)$ properly in order to accommodate all the necessary changes in the Newton-imbedding iteration procedure due to the vector-valued nonlinear term in (5.1). This leads us to the following assumptions:

(A1) The vector-valued function \mathbf{f} and its derivatives $\nabla_{\mathbf{u}}\mathbf{f}$ and $\nabla_{\mathbf{u}}^2\mathbf{f}$ are continuous as the mappings defined below:

$$\begin{aligned}\mathbf{f} &: (H^2(\Omega))^3 \longrightarrow (L^2(\Omega))^3 \\ \nabla_{\mathbf{u}}\mathbf{f} &: (H^2(\Omega))^3 \longrightarrow (C^\alpha(\bar{\Omega}))^{3 \times 3} \\ \nabla_{\mathbf{u}}^2\mathbf{f} &: (H^2(\Omega))^3 \longrightarrow (C^\alpha(\bar{\Omega}))^{3 \times 3 \times 3}.\end{aligned}$$

Moreover, \mathbf{f} and its derivatives $\nabla_{\mathbf{u}}\mathbf{f}$ and $\nabla_{\mathbf{u}}^2\mathbf{f}$ are all bounded: There exists a constant $M > 0$ such that for all $\mathbf{u} \in (H^2(\Omega))^3$,

$$\|\mathbf{f}\|_{(L^2(\Omega))^3} \leq M, \quad \left\| \frac{\partial f^i}{\partial u_j} \right\|_{C^\alpha(\bar{\Omega})} \leq \frac{M}{3}, \quad \left\| \frac{\partial^2 f^i}{\partial u_j \partial u_k} \right\|_{C^\alpha(\bar{\Omega})} \leq \frac{M}{\gamma(c_0, \Omega)},$$

where $\gamma(c_0, \Omega)$ is a constant depending on the imbedding constant c_0 and the measure of Ω .

(A2) The derivative $-\nabla_{\mathbf{u}}\mathbf{f}$ is positive in the sense that for any $\mathbf{u} \in (H_2(\Omega))^3$

$$(-\nabla_{\mathbf{u}}\mathbf{f}(\mathbf{u})\mathbf{v}, \mathbf{v})_0 > 0$$

for all $\mathbf{0} \neq \mathbf{v} \in (H^2(\Omega))^3$.

We remark that the derivatives $\nabla_{\mathbf{u}}\mathbf{f}$ and $\nabla_{\mathbf{u}}^2\mathbf{f}$ are 2nd- and 3rd- order tensors respectively. The entries of these derivatives are, respectively $\partial f^i / \partial u_j$ and $\partial^2 f^i / \partial u_j \partial u_k$ for indices $i, j, k = 0, \dots, 3$. These terms will appear in the Newton-imbedding procedure for (5.1). It is not difficult to see that under the assumptions **(A1)**-**(A2)** together with the restriction of the t -step size, our results for the model problem (2.1), (2.2), Theorems 4.1 and 4.2, will remain valid for the nonlinear problem, (5.1) and (5.2), in elasticity.

Indeed, if we denote by $\mathbf{u}(x, t)$ the solution of the family of the imbedding systems corresponding to (5.1), (5.2), then for the imbedding parameter $t = 0$, we will arrive at the Dirichlet problem for the homogeneous Lamé system

$$-\Delta^* \mathbf{u}(\cdot, 0) = \mathbf{0} \quad \text{in } \Omega, \quad \mathbf{u}(\cdot, 0)|_{\Gamma} = \boldsymbol{\varphi} \quad \text{on } \Gamma. \quad (5.3)$$

The solution of (5.3) can be represented again as in the case for the model problem in the form of a simple-layer potential

$$\mathbf{u}(x, 0) = \int_{\Gamma} \mathbf{E}(x, y) \boldsymbol{\sigma}_0(y) ds_y, \quad x \in \Omega, \quad (5.4)$$

where $\mathbf{E}(x, y)$ is the fundamental tensor

$$\mathbf{E}(x, y) = \frac{\lambda + 3\mu}{8\pi\mu(\lambda + 2\mu)} \left\{ \frac{1}{|x - y|} \mathbf{I} + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{1}{|x - y|^3} (x - y)(x - y)^T \right\}.$$

Here the unknown density $\boldsymbol{\sigma}_0$ satisfies the boundary integral equation of the first kind

$$\mathbf{V} \boldsymbol{\sigma}_0 := \int_{\Gamma} \mathbf{E}(\cdot, y) \boldsymbol{\sigma}_0(y) ds_y = \boldsymbol{\varphi} \quad \text{on } \Gamma,$$

which has a unique solution $\boldsymbol{\sigma}_0 \in (H^{1/2}(\Gamma))^3$, since

$$\mathbf{V} : (H^{1/2}(\Gamma))^3 \rightarrow (H^{3/2}(\Gamma))^3$$

is a Fredholm operator of index zero, and moreover \mathbf{V} is $(H^{-1/2}(\Gamma))^3$ -elliptic, that is, there exists a constant $c_0 > 0$ such that

$$\langle \mathbf{V} \boldsymbol{\sigma}_0, \boldsymbol{\sigma}_0 \rangle \geq c_0 \|\boldsymbol{\sigma}_0\|_{(H^{-1/2}(\Gamma))^3}^2 \quad \forall \boldsymbol{\sigma}_0 \in (H^{-1/2}(\Gamma))^3.$$

For $t > 0$ in the iteration procedure, the prototype of the corresponding associated problem is now given by

$$-\Delta^* \mathbf{v} + \mathbf{Q}(x) \mathbf{v} = \mathbf{g}(x) \quad \text{in } \Omega, \quad (5.5)$$

$$\mathbf{v}|_{\Gamma} = \boldsymbol{\varphi} \quad \text{on } \Gamma, \quad (5.6)$$

for given $\mathbf{g} \in (L^2(\Omega))^3$ and $\boldsymbol{\varphi} \in (H^{3/2}(\Gamma))^3$. Here \mathbf{Q} is a 2nd-order tensor which can be identify with

$$\mathbf{Q} := -\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}_n).$$

Hence in analogy to the condition in Lemmas 3.1, 3.2, we require that $\mathbf{Q} \in (C^\alpha(\overline{\Omega}))^{3 \times 3}$ and is bounded above and positive as in (A.1) and (A.2). Then under the positivity and boundness of \mathbf{Q} , the a priori estimate (3.4) now reads: *For any $\mathbf{v} \in (H_0^1(\Omega))^3 \cap (H^2(\Omega))^3$ the following estimate holds:*

$$\|\mathbf{v}\|_{(H^2(\Omega))^3} \leq \kappa \|-\Delta^* \mathbf{v} + \mathbf{Q}\mathbf{v}\|_{(L^2(\Omega))^3},$$

where κ is a constant depending on M but not on \mathbf{Q} .

This estimate can be derived in the same manner as in the scalar case and will be employed for establishing the convergence of sequences in the Newton-imbedding procedure. Details will be omitted.

An attempt has also been made to apply the Newton- imbedding method to the nonlinear problem consists of (5.1) together with a nonlinear Robin condition of the form

$$T[\mathbf{u}]|_\Gamma + \Phi(x, \mathbf{u}) = \mathbf{0} \quad \text{on } \Gamma, \quad (5.7)$$

where \mathbf{T} stands for the traction operator, and Φ is a nonlinear function which satisfies a Caratheodory, Lipschitz as well as a strong monotonicity condition (see [11]). In the case when the body force $\mathbf{f} = \mathbf{f}(x)$ is independent of \mathbf{u} , based on the theory of monotone operators, this problem has been treated in [11] via boundary integral equation methods and was motivated by a similar problem for the Laplacian in [17], [16]. For the nonlinear body force $\mathbf{f}(\mathbf{u})$, in principle, by following [4], we may imbed both (5.1) and (5.7) into a family of nonlinear Robin problems. However, we note that for $t = 0$, we have a pure traction problem for the homogeneous Lamé system whose solution is unique only up to rigid motions. In spite of this, a more serious technical difficulty that we seem to have for the time being is the lack of an appropriate a priori estimate for the corresponding associated problem. We hope to return to this investigation in the near future.

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