REPRESENTING MARKETS USING THE BOLTZMANN DISTRIBUTION

by

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ACKNOWLEDGEMENTS

Modeling economic markets is a problem so vast and demanding, that in no way could this attempted monograph give it the treatment all economists or 'econophysicists' strive to accomplish. I did not realize how extensive the existing literature on the application of statistical mechanics and related techniques to economics was until the very end of my inquiry. This thesis represents the developing knowledge, and perhaps naive, thoughts of a physics student attempting to contribute something meaningful to a problem outside of his discipline. Moreover, it is also a memento dedicated to great minds whose investigations preceded my own. Certainly, without the guidance and support of faculty and mentors, this cursory review could not even be possible.

I would like to thank my mentor Dr. Adam Fleischhacker for his support and guidance, especially when I was lost in the rabbit-hole of direction-less inquiry and second reader, Dr. Karl M. Unruh, for his support and thought-provoking critique, and Dr. Edward Lyman for introducing me to the rich world of statistical mechanics. Also, I would like to give a special thanks to the McNair program at the University of Delaware for their continued support, Dr. Yakovenko from the University of Maryland at College Park for his pioneering work in the field of 'econophysics' and for allowing me to sit in on his talk, and the Santa Fe Institute for the innovative spirit of inquiry they helped foster in me.

ABSTRACT

The applicability of using the Boltzmann distribution for modeling economic markets is reviewed. While remaining accessible to business students and professionals, this review provides a clear, yet rigorous account of using a Boltzmann-inspired statistical formulation of market behavior. The formulation's flexibility for modeling market behavior in two types of markets is shown and the distribution's use for practical analysis encouraged. Suggestions for future study are also given.

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Chapter

§ 1.0. Introduction

As long as a good or product is both desired and scarce, one may always value that good at a particular price, allowing one to form a market of that good or service. Labor, currency and stocks, consumer products, and commodities all lend themselves to price valuation and therefore each correspond to a particular market that allows their exchange. In this paper, the term 'market' does not represent the physical or virtual space which allows this exchange, but rather the collective purchasing and selling behavior of buyers and sellers respectively.

These *goods and services* are finite, just as the *number of people* that desire them are. And, it is the interaction between these finite quantities that yields the price dynamics of a given market. Most markets, regardless of the variety of product or service exchanged, involve a large number of economic agents whose collective behavior affects the state of the entire market. This is the first of several general principles concerning markets which will allow us to eventually create a common model invoking the Boltzmann distribution.

Let us also consider the notions of supply and demand. For, while supply is easy to quantify, demand is inherently uncertain; although the number of people willing to purchase a product or service is a finite number, the value which each person assigns to a product or service varies across buyers and time, and is therefore not subject to straightforward quantification. Consequently, the model we create must take into account this underlying uncertainty in demand across buyers while preserving the notions of a finite supply and set of buyers, and the inverse law of supply and demand. It is from these considerations that the notion of price comes about, which is the critical parameter upon which a Boltzmann distribution may be built. We therefore seek a simple relationship between price and demand, or returns, that agrees with empirical truth. First, I will review the definition of a market as presented in a traditional introductory economics course. Second, I will then introduce the varieties of markets which will be discussed. Lastly, I will review and comment on the shortcomings of the standard methods involving the maximization (or minimization) of objective functions.

§ 1.1. Markets

Markets may be classified into one of two broad categories i.e., perfect or imperfect competition. Markets subsumed under the category of perfect competition have a large number of buyers and sellers of one particular product. In this case, price emerges only from the collective interests of buyers and sellers and is therefore *insensitive* to the whims of an individual buyer and seller. For markets which reside in the category of imperfect competition, the most basic and fundamental of these is the monopoly, a market for which there are many interested buyers but only one seller of a particular product. In this case, it is clear that the only limitation that the market imposes upon the price at which the seller finds to be optimal is the demand of the buyers or their willingness to purchase the product at a given price. Because the primary dynamic which governs price for monopolistic markets is demand, we focus on the development of a simple model of demand for a monopolistic market in section §3.1 to establish the fundamental importance of the Boltzmann distribution. In conclusion, although these two varieties of markets are highly idealized and is therefore hardly veridical, they are important building blocks in the effort to understand the relationship between price, demand, and profit.

§ 1.2. A Review of The Standard Description of Economic Behavior

Full-Knowledge Methodologies: Deterministic and Probabilistic Outcomes

The standard 'full-knowledge' model of economic behavior as presented by Neumann, and later Hadar, relies upon assumptions about the motivations of decision-makers and nature of their behavior.[1] These assumptions are incorporated into the model ab initio, providing an explicit description of consumer behavior. However reasonable these assumptions may be, they are nevertheless presumptions which lead to a loss of generality, the primary presumption being that all decision-makers have the

[1] See chapter 10 of Hadar[3], and Neumann[2], §3. Also, see §A.1, A.2 in the addendum for a detailed elaboration on the critical differences between these two accounts regarding the construction of a utility function and fundamental assumptions.

ability, constitution, and full knowledge to act in the same (rational) way. The statistical description described herein, however, yields a robust model that is free of this assumption. This suggests that the Boltzmann-inspired statistical formulation is a promising alternative to the standard.

Specifically, these assumptions are needed to form the notion of the utility function for the case of deterministic and probabilistic outcomes. While the deterministic case is much more limited in its applicability, the development for uncertain outcomes given by the probabilistic case is general enough so that it may be applied to a host of economic scenarios. However, even for the probabilistic case, a great deal of assumptions regarding the utility function must be made. While this development might yield mathematically rigorous descriptions, this rigor is not always helpful for the average decision maker. Additionally, the notions (See (1),(2),(3*),(4*) of §A.1, A.2) of preferences, transitivity, continuity, and independence seem to be useful for further analysis, but will not be needed in the general derivation of the Boltzmann distribution, which forms the core of this work. Finally, there is an additional matter regarding the objective function: its description, designed to be a quantitative representation of decision-maker preference, contains unavoidable arbitrariness;[2] this is quite unlike the statistical approach,[3] which embraces such uncertainties in our knowledge and therefore yields results that are more generally valid.

The conventional methodology that allows for decision making involving uncertain prospects, relies upon the principle that all decision makers aim to maximize (or minimize) an objective function (as is the case for a consumer who wishes to maximize their utility). This method requires one to exactly state how each individual behaves in

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[2] The utility function can only be determined up to a linear transformation. This would suggest that direct calculation of utility is necessarily unexact, rendering exact valuations somewhat meaningless. See the technical development of the utility function found in Neumann [2], §3.4.4 - 3.4.5.

[3] The theory of distributions and their properties, is an ideal mathematical tool-set for handling such arbitrarily defined functions. For example, one can choose to employ normalized probability distributions to subdue unknown scaling factors.

the economy. While this information might indeed be useful if such knowledge is fully acquired, a proper analysis of all the possible interactions between each individual for a large market as a whole becomes unrealistic. Even if such an analysis could be carried out for a large number of individuals, it would require a wealth of computational resources and time. To demonstrate the validity of this claim, suppose that each agent does in fact have complete information regarding the decisions of all other agents and the total number of agents is 100 . Therefore, each agent would have at their disposal an appropriate matrix which describes the set of optimal decisions given the state of all other 99 agents. However, creating an array of all potential strategies of 99 agents with the associated optimal decision is an intractable task! If most markets have much greater than 100 agents then, a fortiori, the task of formulating one's strategy in this way is exorbitant at best. Given the considerable computational complexity of this problem, an unassuming statistical formulation is ideal.

Thankfully, one may approach this problem from a 'holisitic' view, i.e. by focusing our attention on the net behavior of economic agents, or descriptions of net quanitites, we are able to avoid this problem while making great strides towards a better understanding. In light of this systems-oriented view, the Boltzmann-inspired formulation will be discussed.

§ 2.0. The General Statistical Derivation for the Boltzmann Distribution

Suppose we observe the characteristics of a market for which there are a finite set of unique choices (or, equivalently, alternatives) $\{x_1, x_2, x_3\}$ each participating agent may choose from. Without knowing anything about the preferences of each agent, we must assume all agents prefer no one choice over another, in accordance with the "principle of sufficient reason". Although this is an obvious oversimplification in this economic context, it allows us to start with absolutely no assumptions while allowing us to make powerful statistical conclusions.[4]

Firstly, if there are three choices an agent may be assigned to, then the total number of ways we may group two agents, while treating them as *indistinguishable* from one another, is six - there are three ways both may be placed into a group corresponding to one of the three choices, and three ways each one may be grouped into a unique choice.[5] In general, because of our ignorance, we must treat each category as equally likely to be chosen by any given agent. Hence, our prediction of the total number of ways N people will choose from the set of n choices, may be written as Eq. 1.

$$\Omega = \frac{(N+n-1)!}{(n-1)!(N)!}. \tag{1}$$

Moreover, if there is a total of N people that may choose among n alternatives, where there are N_i people who choose alternative i ($i = 1, 2, ...\ n$), then the total number of ways the state $w = \{N_i, N_{i+1}, N_{i+2}, ...N_n\}$ may be realized is given by

$$\Omega(w, n) = \frac{N!}{\prod_i^n N_i!} \tag{2}$$

What we find is that the number of configurations, where the alternatives are each chosen by an equal number of people, exceed the number of configurations which favor any one choice over another and becomes more pronounced as we increase the number of individuals participating. Therefore, if no one choice is favored initially over

[4] See development of Boltzmann distribution in Greiner[5].

[5] The reader should prove this himself.

another, the likelihood that the total population will evenly divide amongst each choice is much greater than any other distribution of choices.

Thus, the probability of obtaining the state w, with N_i individuals choosing option x_i out of the the set of n potential choices $\{x_1, x_2, ...x_n\}$, is the multiplicity of that state i.e., the number of configurations if we were to treat each agent as distinct, of the system that reside in state w, divided by the total number of possible configurations. Hence, for N individuals and a set of n choices,

$$P(w) = \frac{\Omega(w, n)}{n^N} \tag{3}$$

We now wish to find the state w for which the probability $P(w)$ is at a maximum i.e., the most likely state. To this end, we first consider the natural log of the multiplicity

$$\ln \Omega(w, n) = \ln N! - \sum_{i=1}^{n} \ln N_i! \tag{4}$$

. We now attempt to maximize Eq. 4.

At this point, one might be tempted take the derivative and set it equal to zero to find this maximum; this is not exactly correct. It is important to consider some natural constraints, like the finiteness of the population before doing so. Specifically, the natural constraints are (1) $N = \sum_{i=1}^{n} N_i$, and (2) the 'average' choice $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} x_i$ supposing x_i represents a monetary or positive numerical value between 0 and a maximum (positive) number x_{max}.

Moreover, if we allow $N \to \infty$, we may make use of Stirling's approximation[6] so that Eq. 4 reduces to

$$\ln \Omega(w, n) \approx N \ln N - \sum_{i=1}^{n} N_i \ln N_i \tag{5}$$

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[6] See the classic text: Schroeder, Daniel V. An introduction to thermal physics. Vol. 60. New York: Addison Wesley, 2000.

Letting $\Omega(N_i, N_{i+1}, N_{i+2}, ...N_n, n) \equiv \Omega$ and maximizing Eq. 5 with respect to N_i subject to the constraints, we take the derivative of Ω with respect to the number of individuals N_i to obtain

$$\sum_{k}^{n}(\lambda_1 + \lambda_2 x_i + \ln N_i)\, dN_i = 0$$

$$\lambda_1 + \lambda_2 x_i + \ln N_i = 0$$

$$\boxed{N_i = e^{-\lambda_1} e^{-\lambda_2 x_i}} \tag{6}$$

where λ_1, λ_2 are Lagrangian multipliers. Furthermore, by considering the ratio $\frac{N_i}{N}$, we find that the probability of having N_i people associated with alternative x_i is given by

$$P_i := \frac{N_i}{N} = \frac{e^{-\lambda_2 x_i}}{\sum_{i=1}^{n} e^{-\lambda_2 x_i}}$$

where $\sum_{i=1}^{\infty} e^{-\lambda_2 x_i}$ is defined to be the partition function Z.[7] Hence, by replacing λ_2 with β, we obtain the familiar result.

$$P_i = \frac{e^{-\beta x_i}}{Z} \to P(x) = Ce^{-\beta x} \tag{7}$$

where $\frac{1}{Z}$ or C is a constant required for normalization and $P(x)$ is the continuous form of the former, as displayed in the above expression (An example of this distribution for $C = 1$ and $\beta = \frac{1}{20}$ is given in Figure 1.). In the next section, we apply this distribution to a simple revenue management problem for a monopolistic market and briefly review and explore additional examples .

[7] Note that by changing n to ∞ for the sum $\sum_{i=1}^{n} e^{-\lambda_2 x_i}$, the sum is unchanged since $e^{-\lambda_2 x_i}$ for $i > n$ is simply zero, or does not exist. By considering the continuous case where $n \to \infty$, this feature remains true since we assume that $N(x)$ is a positive quantity on the finite interval $(0, x_{max})$ and zero everywhere else.

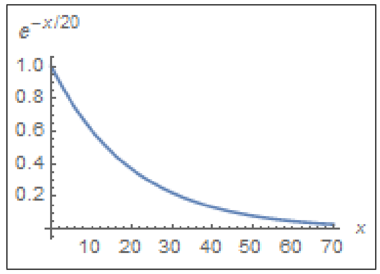

Figure 1: A Boltzmann Distribution

3.0. Application of the Boltzmann Distribution

§3.1. The Simple Revenue Management Problem for Monopolistic Markets

Problem Description and Application of Boltzmann Distribution

Practically, business management must often make decisions based upon forecasts of demand for goods or services they offer. Given information about the market and other relevant parameters, management relies on mathematical technique to aid in the process of deciding the best course of action. However, it is often the case that these demand distributions are unknown, i.e. the company has little to no information about the purchasing behavior of the consumers. For the case of a monopoly[8], we wish to provide an answer to the revenue management problem - what is the ideal selling price for a product if little to no information is known about the purchasing behavior of the consumers or state of the market?

Suppose Company A wishes to determine the optimal selling price given its limited knowledge of the demand in the form of past revenue and other reasonable constraints. Here, we aim to determine the optimal decisions available given these constraints. We introduce a simple expression for revenue R_k obtained for a particular price of m_k set by the company measured in the recent past, where we assume that all individuals with some reservation price $m_j \geq m_k$ purchases the product at the *offered price* m_k, where N_j is the number of people with the corresponding reservation price

[8] A comparative development of the monopolistic market as an analogue to the thermodynamic model of an ideal gas is given in §A and §B.

m_j and the index j represents the lowest price for which the reservation price is greater or equal to the offered price.

$$R_k = m_k \sum_{i=j}^{n} N_i. \tag{8}$$

Therefore, the goal is to find the choice of m_k which will produce the maximal value of R_k a moment after the revenue is measured for a set of offered prices.[9]

We begin our analysis of this problem by invoking a derivation of the Boltzmann distribution parallel to that presented in the previous section, by maximizing Eq.5 given the modified natural constraints[10] that (1) $N = \sum_{i=1}^{n} N_i$, and (2) $\bar{m} = \frac{1}{N}\sum_{i=1}^{n} N_i m_i$. Hence, if we replace x_i with the reservation price m_i i.e., the maximum price a buyer is willing and able to spend, of a buyer, we obtain

$$P_i = \frac{e^{-\beta m_i}}{Z}. \tag{9}$$

In an effort to determine the exact meaning and value of β and Z (as defined in the last section), we revisit Eq. 5.

The Market Temperature T

The natural log of the multiplicity of states denoted as $ln\,\Omega$ is proportional to the entropy S of the state. Hence, letting $n \to \infty$,

$$S = k[N \ln N - \sum_{i=1}^{n} N_i \ln N_i] = k\left[N \ln N - \sum_{i=1}^{\infty} N_i \ln\left(N\frac{e^{-\beta m_i}}{Z}\right)\right]$$

$$S = k\left[N \ln N - \sum_{i=1}^{\infty} N_i \ln N + \sum_{i=1}^{\infty} N_i \ln Z + \sum_{i=1}^{\infty} \beta N_i m_i\right]$$

[9] It seems reasonable to assume that the validity of this method depends on, and is inversely proportional to, the size of the time separation between measurement and decision.

[10] A discussion of these constraints is given in §B.

which reduces to

$$\boxed{S = k\, N\, [ln\, Z + \beta\, \bar{m}]} \tag{10}$$

By considering the change in multiplicity or entropy as a function of changing average reservation price $\bar{m}$ (holding N constant), in accordance with well-known results, we obtain

$$\frac{\partial S}{\partial \bar{m}} = k\, N \left[\beta + \frac{\partial}{\partial \beta}\left(ln \sum_{i=1}^{\infty} e^{-\beta m_i} + \beta \bar{m}\right)\frac{\partial \beta}{\partial \bar{m}}\right] \tag{11}$$

But, since

$$\frac{\partial}{\partial \beta}\left(ln\left(\sum_{i=1}^{\infty} e^{-\beta m_i}\right) + \beta \bar{m}\right) = -\frac{\sum_{i=1}^{\infty} e^{-\alpha} m_i\, e^{-\beta m_i}}{\sum_{i=1}^{\infty} e^{-\alpha} e^{-\beta m_i}} + \bar{m} = 0 \tag{12}$$

(after multiplying the first term by $\frac{e^{-\alpha}}{e^{-\alpha}}$, with $\lambda_1 := \alpha$)

Therefore, Eq. 12 reduces to

$$\boxed{\frac{\partial S}{\partial \bar{m}} = Nk\, \beta}, \tag{13}$$

where β is commonly written as $\frac{1}{T}$, where T is a parameter analogous to the temperature of a thermodynamic system. The applicability or relevance of the market temperature T, for a special case of the revenue management problem, will be explored in an upcoming discussion.

The Distribution

The distribution N_i may now be written as

$$\left[N_i = e^{-\alpha} e^{-\beta m_i}\right] \equiv \left[N(m) = e^{-\alpha} e^{-\beta m}\right]$$

Setting $N(m_{max}) = 1$, with $m \in (0, m_{max}]$[11], we find that

$$e^{-\alpha} = e^{\beta m_{max}}$$

yielding our final result for the (continuous or discrete) distribution of people with reservation price m:

$$\boxed{N(m) = e^{\beta(m_{max}-m)}} \tag{14}$$

Result

By writing the continuous form of Eq. 8 ($n \to \infty$), the revenue is

$$R(m^*) = m^* e^{\beta m_{max}} \int_{m^*}^{m_{max}} e^{-\beta m} dm \tag{15}$$

so that

$$R(m^*) = \frac{m*}{\beta} \left[e^{\beta(m_{max}-m^*)} - 1 \right] \tag{16}$$

where m^* is the offering price. If we now set $N(m_{max}) = 1$ and $m \in (0, M_{max}))$

$$R(m^*) = \frac{m^*}{\beta} \left[e^{\beta(m_{max}-m^*)} - 1 \right] + \frac{m^*}{m_{max}}. \tag{17}$$

Or,

$$\boxed{R(m) = m\tilde{Z}(m) + C} \tag{18}$$

where $C = \frac{m}{m_{max}}$ is a correction term to give $R(m_{max}) = m_{max}$, and the partition function

$$Z = Z(m) + \tilde{Z}(m) = \int_0^m N(m)\, dm + \int_m^{m_{max}} N(m)\, dm = \frac{e^{\beta m_{max}} - 1}{\beta}$$

[11] $N(0)$ has no practical meaning and shall be neglected.

Summary

We have shown that given the constraints of a finite population and a fixed market size, we obtain a Boltzmann distribution of reservation prices. Therefore, the distribution of reservation prices is given by the following:

$$N(m) = e^{(m_{max}-m)/T}$$

Additionally, we have obtained the expression for revenue below:

$$R(m) = m\,T\ \left[e^{(m_{max}-m)/T} - 1\right] + \frac{m}{m_{max}}$$

In the high market temperature (or low β) regime we find the following relation:

$$N(m) \approx 1$$

.

Which implies that for high T, i.e. a large value of T, the reservation prices of the buyers are uniformly spread over the entire range of prices such that each individual has a unique reservation price as depicted in Figure 2. Finally, for low T, one may readily see that the distribution diverges for small values of m and has a uniform distribution of $N(m) \approx 1$ for larger values of m.

We may also determine the answer to the revenue maximization problem by considering the zeros of the derivative of $R(m)$; that is, we consider the following expression:

$$\frac{dR(m)}{dm} = \frac{e^{\beta\ (m_{max}-m)}}{\beta}\left[1 - \beta m - e^{-\beta(m_{max}-m)}\right] + \frac{1}{m_{max}} = 0 \tag{19}$$

In the limit of large m_{max} (allowing us to neglect the term $1/m_{max}$) and low T, the term $e^{-\beta(m_{max}-m)}$ is negligible, we therefore obtain (Since $1/\beta \equiv T$)

$$\boxed{m \approx T} \tag{20}$$

Equation 20 demonstrates that for low T, the ideal selling price for the revenue management problem explored in §3.1 is approximately the value of the market temperature T.

For the high market temperature limit, we may use the approximation:

$$e^{-\beta(m_{max}-m)} \approx 1 - \beta(m_{max} - m)$$

so that

$$\beta \left[m_{max} - 2m\right] = 0$$

$$\boxed{m \approx \frac{m_{max}}{2}} \tag{21}$$

Hence, the ideal selling price in the high market temperature limit is about half of the maximum reservation price.

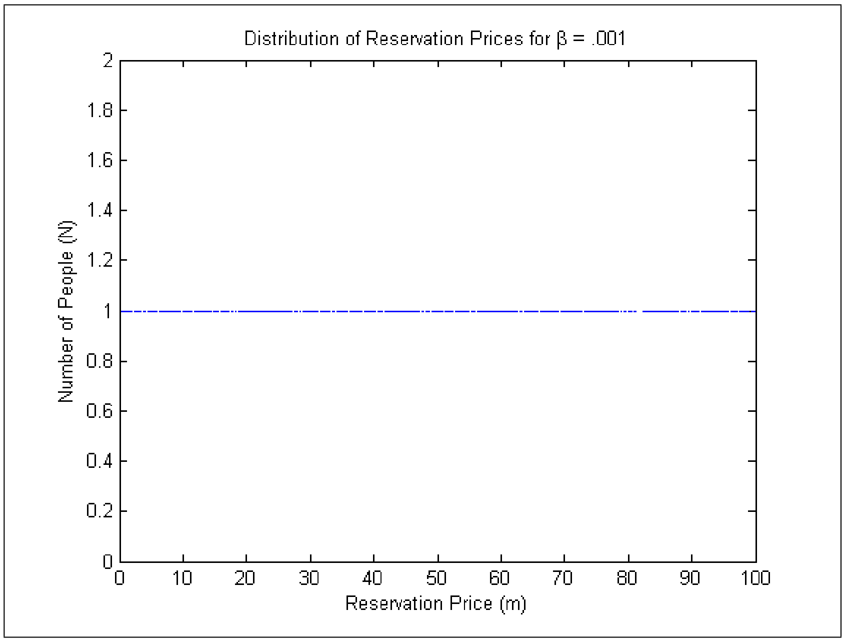

Figure 2: Distribution of Reservation Prices for High T
$(T = 1000,\ M_{max} = 100,\ N = 1000)$

Discussion

We now have an expression, á la Boltzmann, that represents the distribution of reservation prices given a minimum of assumptions. Additionally, the Boltzmann distribution yields the fundamental law of diminishing demand i.e., that demand decreases with increasing price. The following graphs illustrate what the unnormalized distribution of reservation prices might look like, given an arbitrary set of data points, for several values of the parameter β or T and the corresponding normalized revenue with the analytically derived curves.[12] The result clearly demonstrates the validity of the approximations of Eq. 20 and Eq. 21. [13]

[12] These graphs were generated by Matlab, with data points being arbitrarily selected through the use of a random seed generator. See Addendum for examples of Matlab c ode.

[13] The empirically obtained (piecewise continuous) curves will necessarily be bounded within the (continuous) analytical curves for revenue.

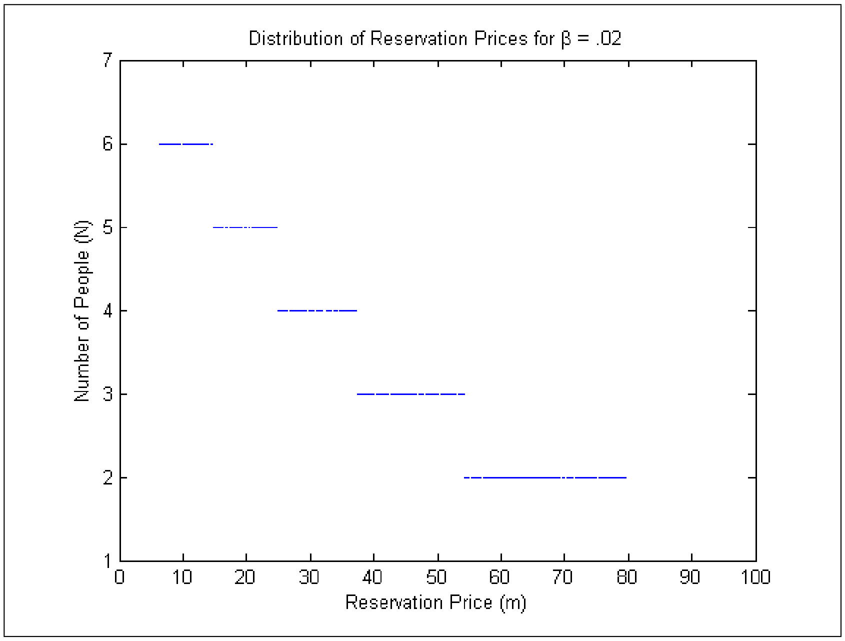

Figure 3: Distribution of Reservation Prices
$(T = 50,\ M_{max} = 100,\ N = 3224)$

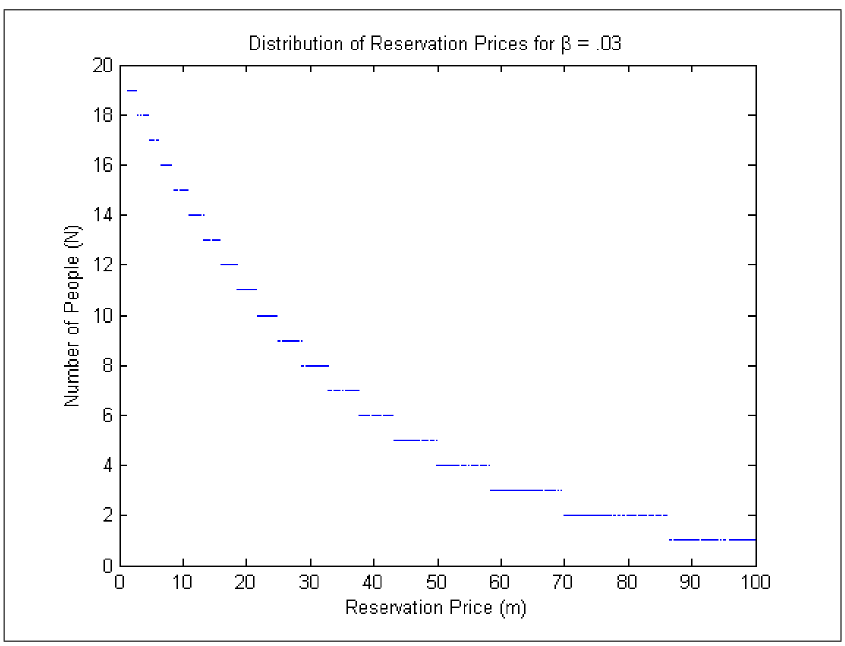

Figure 4: Distribution of Reservation Prices
$(T \approx 33,\ M_{max} = 100,\ N = 6065)$

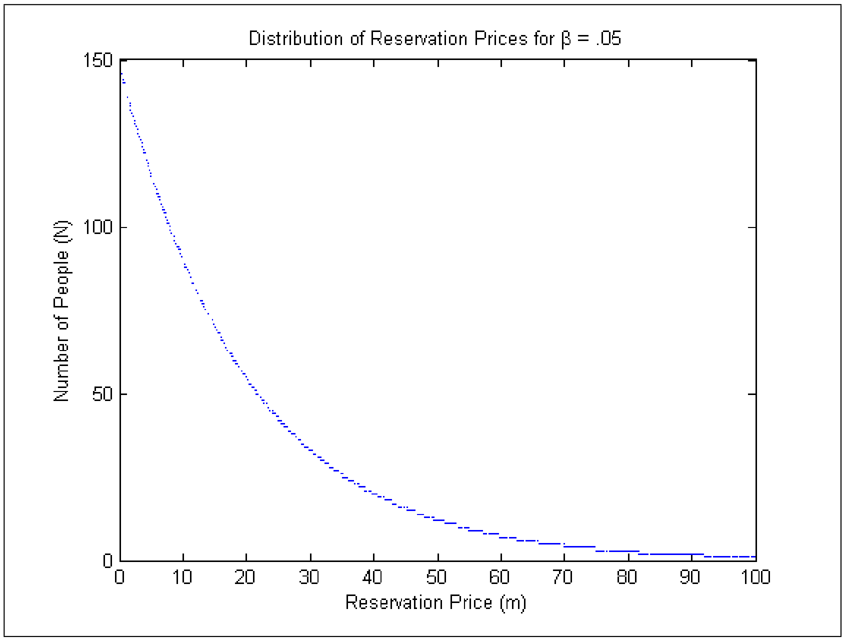

Figure 5: Distribution of Reservation Prices
($T = 20,\ M_{max} = 100,\ N = 29262$)

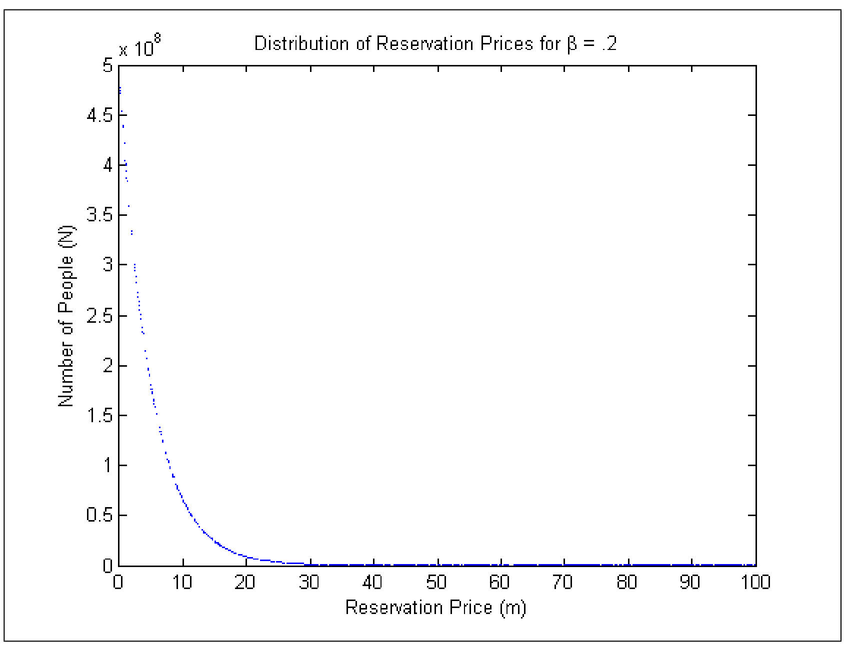

Figure 6: Distribution of Reservation Prices for Low T
($T = 5,\ M_{max} = 100,\ N = 2.236 \times 10^{10}$)

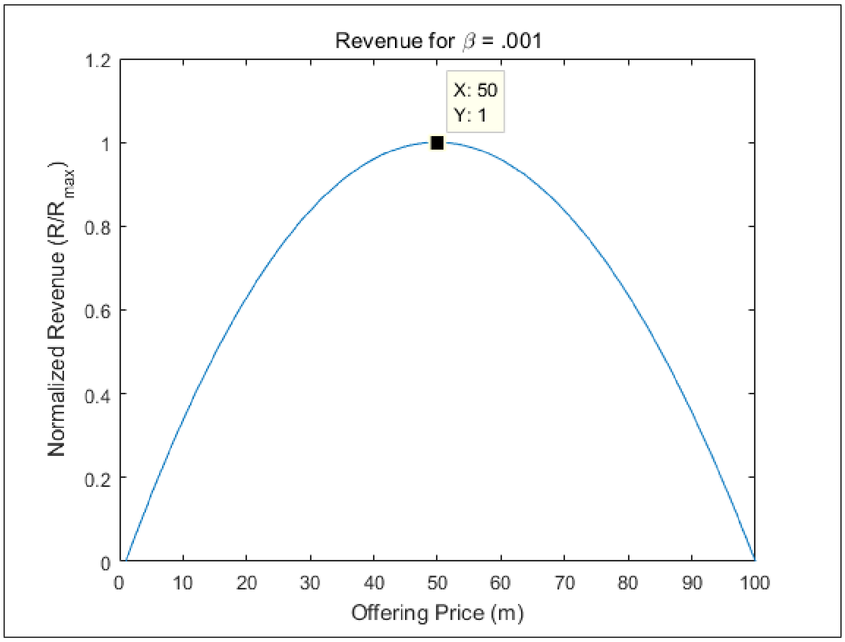

Figure 7: Expected Revenue for $T = 1000$

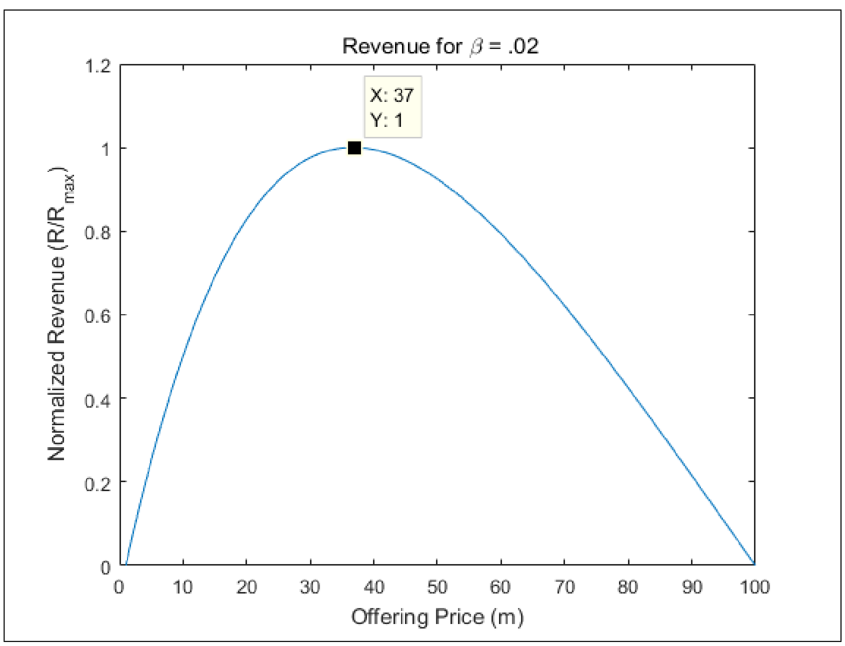

Figure 8: Expected Revenue for $T = 50$

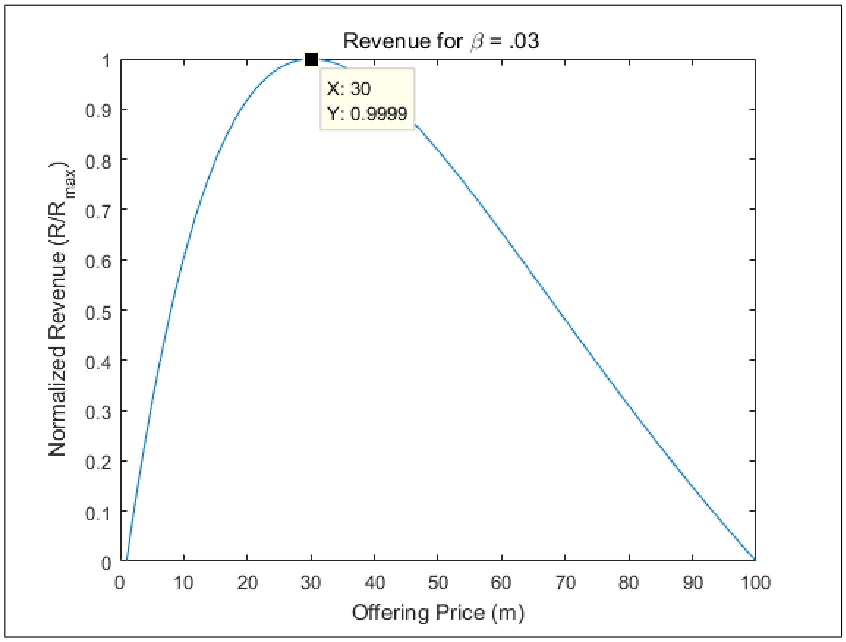

Figure 9: Expected Revenue for $T \approx 33$

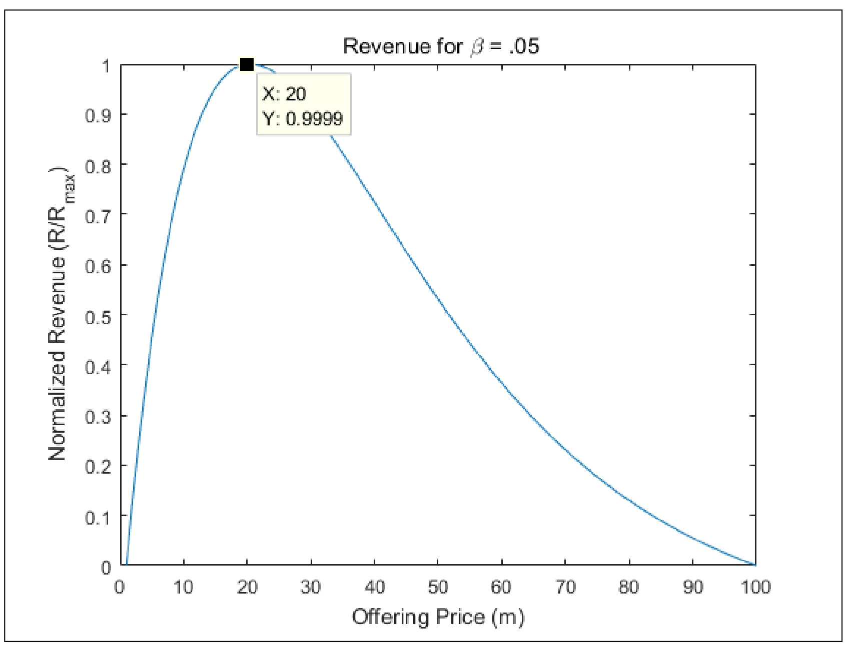

Figure 10: Expected Revenue for $T = 20$

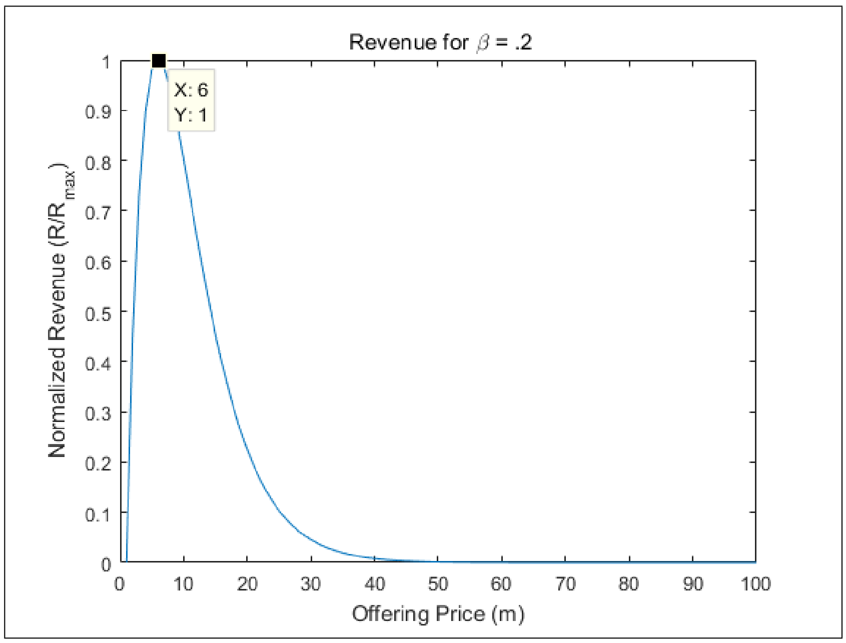

Figure 11: Expected Revenue for $T = 5$

§ 3.2. The Distribution of Minute Stock Returns

Kleinert and Chen [6] observed that the absolute value of stock returns, for small time increments, has a Boltzman distribution $\tilde{B}(z)$ for $z(t) = \Delta x(t)$, where $x(t)$ is the log of the stock price at time t.

$$\tilde{B}(z) = \frac{e^{\frac{-|z|}{T}}}{2T}.$$

I will show how this empirically derived expression for the distribution of returns (for small increments of time) may easily be derived from the general Boltzmann result previously outlined. While it seems there is no *prima facie* agent model which is responsible for the empirically observed Boltzmann distribution, a mathematical comparison between the empirical and analytical expressions is presumably appropriate and elucidative.

Recall that the Boltzmann distribution is given by

$$P(x) = \frac{e^{\frac{-x}{T}}}{Z}$$

by introducing a factor of $e^{\frac{x'}{T}}$, with $x - x' = \Delta x$,[14] the distribution becomes

$$P(\Delta x) = \frac{e^{\frac{-\Delta x}{T}}}{\sum e^{\frac{-\Delta x}{T}}} \tag{22}$$

where the sum is over the entire space of continuous values of Δx. For Eq. 22, it is assumed that $\Delta x \in (0, \infty)$. If we now extend the space to include negative intervals, then

$$P(|\Delta x|) = \frac{e^{\frac{-|\Delta x|}{T}}}{\sum_{-\infty}^{\infty} e^{\frac{-|\Delta x|}{T}}} = \frac{e^{\frac{-|z|}{T}}}{\int_{-\infty}^{\infty} e^{\frac{-|z|}{T}}} \tag{23}$$

where the interval for a given time period τ is $\Delta x = z(\tau) = z$.

Noting that

[14] See §A.2 for conceptual justification

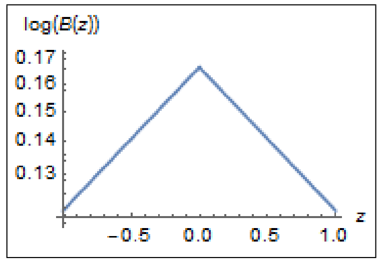

Figure 12: The Distribution of Minute Returns

$$\int_{-\infty}^{\infty} e^{\frac{-|z|}{T}} dz \;=\; 2\int_{0}^{\infty} e^{\frac{-z}{T}} \;=\; -2T$$

we obtain

$$\boxed{\tilde{B}(z) = |P(z)| = \frac{e^{-\frac{|z|}{T}}}{2T}}$$

.

A Log plot of this result is given with $T = 3$ in Figure 12. [15]

§ 3.3. Perfect Competition Markets: An Additional Application

Markets with many buyers and one 'seller' were considered in the previous applications. Now, let us consider a market with many buyers and sellers to demonstrate the flexibility of the Boltzmann-inspired formulation. Since this section is necessarily a cursory review, the reader is encouraged to consult the appropriate resources for further study.

Dragulescu[8], Yakovenko[7], and Wannier[11] derived the Boltzmann distribution for a closed economic system with conserved money. For example, Wannier derived the distribution $P(m) = ce^{-\frac{m}{T}}$, where T is the average amount of money per agent for the 'simple economy' model, where each agent has an amount m in dollars and the total money in the market is M. To simulate the exchange of money or interaction between agents, we randomly choose a buyer and seller (each has equal probability of

[15] See Kleinert[6]

being selected) and choose a random amount to be exchanged between them, within the limits of the amounts each buyer is able to exchange. Wannier et al. showed that by iterating this process many times, the distribution of money becomes a Boltzmann distribution, which models a critical feature of a capitalistic economy.[16]

[16] It is important to note the critical difference between the development given in §3.1 and the closed economic model. For the revenue management problem, money is not conserved nor is it claimed to be: the average is simply taken to be fixed at some arbitrary time t.

4.0. Discussion and Closing Remarks

§ 4.1. Necessary Features of a Boltzmann-inspired Model

Although the developments in sections §3.1, and §3.2 are for two markets that clearly differ in several respects, the above presentation leads one to consider the possibility that there are similarities which allow the expression of a mathematical description that is common to both. Although a definitive answer to this inquiry has not yet been found, we are able to review the following critical features of the models explored in §3.1 and §3.3: (1) A population of agents exists. (2) Each agent may choose any value of a relevant quantity from a continuum of values bounded appropriately. (3) We assume that there is an *equal-a-priori* probability that an agent will choose any one value over any other value because of our ignorance.

§ 4.2. Suggestions for Advancement

To provide an account of interesting and useful features of this formulation as a topic for future study and investigation, I mention other mathematical toolkits and developments which, I suspect, hold the keys to providing a greater understanding of the proper role and scope of the Boltzmann distribution in its application to economic problems of interest. First, one may easily incorporate additional constraints on the system through the use of Lagrange multipliers, as done for Eq.5. This view is endorsed by Fleischhacker and Fok.[17]

Second, the techniques used in time series analysis seem to be an invaluable tool for extracting average values of relevant parameters and other descriptions of the time-evolution of a distribution, like variance. This has in fact been demonstrated by Kleinert to show that a high market temperature corresponds to market volatility and downturn.

[17] See Fleischhacker[12], [13].

Finally, a development of this formulation that includes the evolution of the Boltzmann-inspired distribution in time which utilizes path integrals and other toolsets found in the statistical physics or non-equilibrium statistical physics literature remains undiscovered. The unique features of choatic systems also seems to offer insight into this problem. Specifically, many chaotic systems preserve their fractal dimension as the evolve with time. This property, if proved to be true for economic systems, may prove useful in predicting the evolution of a distribution with time. The reader is encouraged to consult the work of Voit[15], Ingber[14], Bouchard[16], and Kleinert[17].

§ 4.2.2 Preliminary Investigations towards a Time-Dependent Formulation

The notion of equilibrium rests upon the constancy of an average in time. For the monopoly developed in §3.1, $\bar{m}$ is the average reservation price given a distribution $N(m)$ and corresponding probability distribution $P(m) = \frac{N(m)}{N}$. Hence, if $\bar{m} \to \bar{m}(t)$, the probability distribution also changes with time such that

$$\bar{m}(t) = \int_0^{m_{max}} m\, P(m,t)\, dm \tag{24}$$

Now, how should the average reservation price fluctuate? If we know the time-varying distribution $P(m,t)$ in full, we may simply compute $\bar{m}$(t) using Eq. 24 for all time. However, there is no straightforward way to determine $P(m,t)$, since there is no reasonable *a priori* deterministic set of laws which govern these fluctuations so that one may deduce the value of $\bar{m}(t)$ for all time. We must therefore [18] rely on our observation of how $\bar{m}$ changes with time, i.e. we must consider the average change (in time) of the average reservation price if we are to describe how it changes in time. This suggests that our best statement of the time variation of $\bar{m}$ is merely the time average of the variation of $\bar{m}$ in time, or

$$\frac{d}{dt}\,\bar{m}(t) \to \left\langle \frac{d}{dt}\,\bar{m}(t) \right\rangle_t$$

[18] Such data could be obtained from past revenue or values of the distribution of reservation prices obtained by survey, for which the distribution of reservation prices would take the form $N(m) = \sum_i N_i\, \delta(m - m_i)$ where (m_i, N_i) are the data points.

where the brackets denote an average over time (as suggested by the subscript).

Still, however, our expression lacks a critical component. Given a minimal understanding of market data, one immediately notices that there are irregularities or noise in such data; that is to say, our grasp of the average is always "fuzzy" or is never fully represented empirically. With this dilemma in mind, we add noise to arrive at

$$\boxed{\frac{d}{dt}\,\bar{m}(t) = \left\langle \frac{d}{dt}\,\bar{m}(t) \right\rangle_t + \eta(t)} \tag{25}$$

which denotes the fact that $\frac{d\bar{m}}{dt}$ fluctuates with white noise (That is, $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\,\eta(t') \rangle = \sigma^2\,\delta(t - t')$) about its average with a variance of σ^2.[19]

Given the validity of Eq. 25, there are three observations worthy of mention.

Temperature as a measure for the scale of fluctuations

Revisiting Eq. 13, we find that the change in the multiplicity or entropy of the distribution of reservation prices with respect to time is - via the chain rule - given by

$$\frac{\partial S}{\partial t} = \frac{Nk}{T}\frac{d}{dt}\,(\bar{m}(t))$$

Or, equivalently

$$\boxed{\partial_t\, S(\bar{m}(t)) = \frac{Nk}{T}\,\partial_t\,\bar{m}(t)} \tag{26}$$

Equation 26 implies that the time-dependent response of the market is proportional to the fluctuations of the average reservation price, where the constant of proportionality is inversely related to temperature. For example, given a small value

[19] See Kleinert [17], for a full development of stochastic technique.

of T, we expect that the fluctuations in the average reservation price will yield great changes in the temporal evolution of the system.

The Lyapunov Exponent

In an effort to find a more tractable form of Eq. 25, we note that

$$\left\langle \frac{d}{dt}\,\bar{m}(t) \right\rangle_\tau = \left[\frac{1}{\tau} \int_t^{t+\tau} \left(\frac{d}{dt'}\,\bar{m}(t') \right)\, dt' \right]$$

which reduces to

$$\left[\frac{\bar{m}(t')\, dt'}{\tau} \right]_t^{t+\tau} = \frac{\bar{m}(t+\tau) - \bar{m}(t)}{\tau}$$

Taking the limit as $\tau \to 0$

$$\lim_{\tau \to 0} \left[\frac{\bar{m}(t+\tau) - \bar{m}(t)}{\tau} \right] = \delta\bar{m}_t$$

$\delta\bar{m}(t)$ represents an infinitesimal deviation of the average value of the reservation price.[20] If we think of this infinitesimal variation as the difference between two trajectories at a particular time, we may write:

$$\delta\bar{m}_t = \delta\bar{m}(t) - \delta\bar{m}(t)_0 \tag{27}$$

letting $\delta\bar{m}_0$ be defined in a similar way, we note that the Lyapunov exponent λ characterizes the evolution of the system if

$$|\delta\bar{m}_t| \approx e^{\lambda t}\, |\delta\bar{m}_0| \tag{28}$$

[20] To avoid confusion, it is important to note the differences between $\bar{m}$ and $\langle \bar{m} \rangle$: The former is the average reservation price given a distribution $N(m)$ at a particular time t; the latter is the time average of this average reservation price. Also note that $\langle m \rangle = m$, since m clearly cannot change with time.

Several methods for the determination of the Lyapunov exponent exist and the reader is encouraged to consult the literature, since it is beyond the scope of this paper. In the end, however, we find the final expression:

$$\boxed{\left|\left\langle \frac{d}{dt}\,\bar{m}(t)\right\rangle_\tau\right| \approx e^{\lambda t}\,|\delta\bar{m}_0|} \tag{29}$$

The Fokker-Planck Equation

In an effort to write an expression which models the diffusion or changes in the probability distribution $\Psi(\bar{m}, t)$ with time, we say that for an interval of time Δt, the Fokker Planck equation may be written in the following form:[21]

$$\boxed{\frac{\partial\Psi}{\partial t} = \frac{\partial}{\partial\bar{m}}\left(\alpha\Psi + \frac{\partial}{\partial\bar{m}}(\beta\Psi)\right)} \tag{30}$$

Where $\alpha = -\left\langle\frac{\Delta\bar{m}}{\Delta t}\right\rangle$ is the drift coefficient and $\beta = \left\langle\frac{\Delta\bar{m}^2}{2\Delta t}\right\rangle$ is the diffusion coefficient. [22] If we recall Ito's rule for a function $f(\bar{m}(t))$ given by:

$$\partial_t f(\bar{m}(t)) = f'(\bar{m}(t))\partial_t\bar{m}(t) + \frac{\sigma^2}{2}f''(\bar{m}(t))$$

We may perform a termwise comparison to find, after rewriting Eq. 30 and letting $f(\bar{m}(t)) = \Psi(\bar{m}(t))$, the following expression:

$$\partial_t\bar{m}(t)\,\frac{\partial\Psi}{\partial\bar{m}} + \frac{\sigma^2}{2}\frac{\partial^2\Psi}{\partial\bar{m}^2} = -\alpha\frac{\partial\Psi}{\partial\bar{m}} + \beta\frac{\partial^2\Psi}{\partial\bar{m}^2} \tag{31}$$

Hence, the drift of the probability distribution for $\bar{m}$, as described by Eq. 25, is the expectation of the change of $\bar{m}$ with time. Additionally, Eq. 31 demonstrates that the coefficient governing the spread of the distribution with time is proportional to the variance of the time series which describes $\bar{m}$.

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[21] The motivation and form for this expression is inspired by a casual talk given by Dr. Victor Yakovenko to undergraduates at the University of Maryland at College Park.

[22] Note that $\frac{\Delta\bar{m}}{\Delta t}$ may be rewritten as $\left\langle\frac{d}{dt}\,\bar{m}(t)\right\rangle_{\Delta t}$.

Concluding Remarks

Although the full formulation for the time-dependent formulation is postponed, this investigation suggests several promising paths that may lead to an unassuming yet rigorous description of consumer behavior. What has been more fully demonstrated, however, is that the Boltzmann derivation is not merely a hopeful theoretical construct, but is a representation that corresponds with actual empirical patterns. Ultimately, we have taken a glimpse of the variety of economic scenarios which it seems apt to describe, with the hope that its generality and scope extends to aid in practical application. The real value in this development is the absence of the need to specify the behavior of each individual i.e., the Boltzmann-inspired formulation depends only on a description of important macroscopic details, rather than the complex 'microscopic' or agent-centered analysis which characterizes a great deal of approaches in economics.

Addendum

Matlab Code Examples

Matlab Code for Distribution of Reservation Prices for $\beta = .2$

```
\\Set Max Reservation Price
Max = 100;
\\ Set value of Beta
c = .2;
\\ Generate 1000 data points
for i = 1:1000
\\ Obtain a randomly generated price on the interval (o,Max)
resprice(i) = (rand)*Max;
\\ Compute N(m) for these generated data points and round to whole number
N(i) = round((exp(c*(Max-resprice(i))))) ;
end
for j = 1:1000
plot(resprice(j),N(j));
hold on;
end
title('Distribution of Reservation Prices for \beta = .2')
xlabel('Reservation Price (m)');
ylabel('Number of People (N)');
\\ Determine the total number of people given the data
sum(N)
```

Matlab Code for Revenue for $\beta = .001$

```
\\ Set Max reservation Price
Max = 100;
\\ Set Beta
beta = .001;
\\ Setup X-axis
m = linspace(0,Max);
\\ Plot Normalized Revenue
\\ (Normalized Numerically with Max at about 2564 for Unnormalized Revenue)
plot((((m/beta).*(exp(beta.*(Max-m))-1) + (m/Max))/(2564)))
title('Revenue for \beta = .001')
xlabel('Reservation Price (m)');
ylabel('Normalized Revenue (R/R_{max})');
```

§ *A* Assumptions for Decision-Maker Behavior: Deterministic Outcomes

The assumptions, upon which the formulation of the utility function is built, are as follows: (1) the decision maker, when presented with two options, must always prefer one option over another or remain indifferent with respect to both (Hadar uses $x_1 \} x_2$ to denote a collection or bundle of goods x_1 is preferred to x_2, and $x_1 \sim x_2$ to denote indifference.), (2) the binary relation '}' is transitive, (3) a consumer always prefers the bundle that contains more of a particular good, *ceteris paribus* and, (4) the preferences of each individual consumer is fully known.

Assumptions (1) and (2) are reasonable requirements for the description of the actions of any decision-maker.[23] However, the generality of these presumptions are limited by assumptions (3) and (4) which follow for the general consumer. Hadar claims, in accordance with (3), that all consumers behave in the same 'greedy' way, which enables a numerical representation for the utility function. This assumption is not required for the construction of such a mapping, and is an undesirable constraint which severely limits the usefulness of the model, which is avoided in the statistical formulation. Finally, my revision of assumption (4) is a corollary of (3) that is, as discussed in §1.2, a gross oversimplification.

It is important to note that there is a fundamental difference between the general development given by Neumann: by ascribing an additional property to the utility function (which Hadar reserves for the case of probabilistic outcomes), Neumann arrives at the same result without the need to rely on assumption (3).[24] Further elaboration is postponed until the next section.

The original phrasing of (4) given by Hadar differs in a significant way from my revision. Firstly, Hadar comments that assumption (3) alone cannot guarantee

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[23] Assumptions (1) and (2) are important in that they may govern the time-evolution of probability distributions, which is a subject of future work.

[24] While Neumann does not explicitly state the greediness assumption, his notation is ambiguous and might suggest it: the meaning of $x_1 > x_2$ sometimes means "x_1 is preferred to x_2", or "x_1 is greater than x_2" depending on the context.

the existence of a functional relationship or rule of correspondence between a bundle of goods and a real positive number, upon which a utility function should be built. Secondly, he asserts that this utility function contains all relevant information regarding consumer preference. While this is true if greediness is the primary motivation, this formulation is one of many for which all relevant information about consumer preference may be fully represented. According to the statistical methodology, one may conceive of probability distributions which represent the net result of unknown laws of preference or motivations; these descriptions contain all relevant information - without the need for an explicit rule for the preferences of each individual. Therefore, 'representability' does not require assumption (3), and (4) should be revised to read: the preferences of consumers is fully representable. However, this revision seems to be obvious, since one would not wield mathematical technique without this belief.

§*A*.2 Assumptions for Decision-Maker Behavior: Probabilistic Outcomes

If we now let x be a set of outcomes along with their respective probabilities, x represents the outcome of a decision an economic agent must make in the face of uncertainty. For the consumer, x is the 'uncertain bundle'[25] or set of income levels (or intervals of income) and their respective probability of attainment (or probability that their initial income will be incremented by the respective interval)[26] after the actions corresponding to x is chosen.

Along with the assumptions (1) and (2) of §4.1, we have: (3*) if $x_1 \succ x_2 \succ x_3$, then for $0 < \alpha < 1$ (α represents a probabilistic weighting),

$$x_2 \sim \alpha x_1 + (1 - \alpha) x_3$$

[25] See chapter 12 of Hadar[3].

[26] The fact that this probability may be formulated for an interval or level of income supports the result that the developments of §3.1. and §3.2. are linked, *mutatis mutandis*, via simple substitution.

for some x_2,[27] and (4*) if $x_1 \} x_2$ for some x_1 and x_2, then for some x_3,

$$(x_1, x_3; \alpha, (1-\alpha)) \} (x_2, x_3; \alpha, (1-\alpha))$$

where (3*) establishes continuity and (4*) establishes independence.

A few comments on the above modifications and additions: these adjustments allow for a probabilistic description of future outcomes and do not *prima facie* impose 'uniformity of behavior' (e.g. the greediness assumption). From this set of modified suppositions, one may again construct a utility function. Neumann demonstrates how this may be done in §3.5 of his classical text.

[27] This relation reads: a decision maker is indifferent between option x_2 and the uncertain outcome described by x_1 with probability α and x_3 with probability $(1-\alpha)$.

§ *B* The Monopoly and the Thermodynamic System

Consider a system with a very large number of particles or objects denoted by N. We may also break up N into n groups (with $1 \leq n \leq N$), each with N_i particles so that

$$N = \sum_{i=1}^{n} N_i \tag{32}$$

for N particles of some kind, contained in a fixed volume V. We observe that this system has *extensive* properties i.e., quantities of interest which are proportional to the size or number of particles N, and *intensive* properties which remain constant even if the size of the system changes. If these particles do not interact with each other, we may write the total energy of the system – an extensive property[28] – as

$$E = \sum_{i=1}^{n} N_i E_i \tag{33}$$

where E_i represents a value in a discrete set of energies and N_i is the number of particles to which the value E_i may be ascribed.

Now suppose that we may envision these N particles as individuals in an economy who are willing and able to participate in the basic actions of commerce i.e., buying and selling. However, let us only consider one product which 'Company A' sells and N individuals purchase. Clearly, N may be considered to be quite large – this is the first similarity to the system hitherto discussed. The second similarity or analogy may be drawn with regard to the energy E_i. Firstly, like energy, our analogous quantity must be proportional to the number of individuals N participating in the act of purchasing 'Product P'. Secondly, we would like a discrete quantity which we may meaningfully sum, as in Eq. 33. With these considerations in mind, the notion of money as the

[28] The total energy E is extensive only in the sense that if we consider uniform multiplication by a scalar λ over a particular distribution $\{N_1, N_2, ...N_n\}$ with each N_i held fixed, then the total energy is λE. We then say that E is extensive in N for a particular distribution or set $\{N_i\}$.

quantity corresponding to energy becomes plausible. To this end, we make a slight adjustment to Eq. 33, leaving most of the notation unchanged.

$$M = \sum_{i=1}^{n} N_i M_i = N\bar{M} \tag{34}$$

where M_i represents the maximum price at least one individual is willing to pay for a product, the *reservation price*, N_i is the number of individuals with reservation price M_i, M is the *market size* of the system, $\bar{M}$ is the average reservation price, and $N = \sum N_i$ still holds. Note that if M is held fixed, the average $\bar{M}$ must vary according to the distribution of market prices (the values of N_i and M_i). However, if we fix this average, then the market size M must now vary. These two options will give rise to two notions of an equilibrium state constrained by the condition of a fixed markets size M or a fixed average reservation price $\bar{M}$. We call these two formulations equilibrium and time-dependent quasi-equilibrium respectively.

It is important to note that both M and E naturally varies according to the distribution of particles or individuals with a given energy or price respectively. If we fix the average of these quantities, we achieve *thermal equilibrium* , which has a clear meaning for the physical N-particle model: the temperature, directly related to the average energy of the particles, is held constant. However, we have yet to understand what this means in an economic context. before this understanding is achieved, we expound upon the notions of equilibrium and time-dependent quasi-equilibrium.

The time-independent equilibrium state is the most probable state which arises from a fixed market size M or M_α *ab initio*.

$$M_\alpha = \sum_{i=1}^{n} N_i M_i \tag{35}$$

Therefore, in this case, there is no well-defined average for a set of identically prepared systems[29]. And, by fixing the market size, only the sets $[N_i, M_i]_{i=1}^{n}$ that satisfy Eq.35 are allowed.

Note that the market size is not an ideal measure. It only describes the total money collected if every person where to purchase a product at the maximum price they themselves are willing to pay. In reality, charging each customer the price they are maximally willing to pay for a product is not a realizable business practice. However, in sales, this measure could represent the revenue a perfect salesman would make. Nevertheless, we simply neglect this measure for the problem at hand and consider an equilibrium state established by a fixed average reservation price.

If we now take an ensemble of distributions for a large set of M_α, thus allowing M to vary, we now have the corresponding canonical ensemble for the system. If we fix the average reservation price at a particular time, while allowing the average to change with time so that $\bar{M}(t) = \bar{M}$, We find that the time-dependent quasi-equilibrium state is constrained by the following condition.

$$\bar{M}(t) = \bar{M} = \frac{\sum_{i=1}^{n} N_i M_i}{N} \tag{36}$$

Note that the time-independent formulation has each price M_i explicitly corresponding to the weighting N_i, which means that the ordering of the set $\{M_i\}$ affects the sum. The sum of the products of these factors can only be preserved in a unique set of cases. Hence, degeneracies for a given M_α are a unique feature rather than a critical one. However, for the time-dependent case, only the sum of one factor is conserved: the ordering of the set of M_α may freely vary without affecting the sum. This in turn creates degeneracies or symmetries which yield Boltzmann-like statistics.

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[29] If we imagine this ensemble of identically prepared systems as instances of one system in time, the notion of time-independent equilibrium is made clear.

Considerations

It is important to note that for the system of particles, the parameters upon which the law governing the assignment of energies to each particle are known; the number of particles and the volume of the space confining them, along with their energies, constitute all we need to know to assign a particular value of energy to each particle. An equally straightforward enumeration of parameters should not be done, however, for the laws governing the threshold price of each individual. Therefore, we accept that the law or function which assigns a threshold price to each individual has unknown parameters, which is summarized in a more succinct way below,

$$E_i = f(V) \Leftrightarrow M_i = g() \tag{37}$$

where $f(V)$ is some function of volume, while $g()$ is the unknown function on which M_i depends.

A second assumption, for the ideal case, which we must analogously derive is the requirement that the particle system does not contain 'interacting' parts. This condition implies that the eventual value of energy of each particle does not change as a result of a specific interaction it has with another particle it comes into contact with. In other words, particles remain distinct and independent, so that the energy of an individual particle does not depend on that of another. We now analogously require that each individual does not exchange money with other individuals, or create a pool of money from which to draw from or save. By stipulating that the only interaction involving the exchange of money is the purchase of product P from company A by an individual,this *condition of monopoly* greatly simplifies our reasoning while theoretically allowing us to make useful statements about the collective purchasing behavior of the group.

§ *C* The N-Individual System: Microstates of a Monopoly

Methods of Evaluation

For many real systems, it is impractical to measure exactly the series of values which it will take for some span of time. For the market demand of a particular product, one will observe that the state of demand, across the domain of consumers, is 'fuzzy' and dynamic: one can never accurately construct a description of the demand with regard to its specific values which it assumes or, with regard to its behavior for any duration of time. For systems that exhibit these two features, one cannot immediately construct a curve or calculate a metric in the usual way if it is to be of any use, for both are naive attempts which undoubtedly neglect the salient features of interest. When we cannot neglect the inherent uncertainty in a phenomenon, we must carefully manipulate the information available to us such that the phenomenon, in its entirety, may be examined with realism. For this purpose, the tools presented in the theory of distributions are utilized in context.

Microstates

With these considerations in mind, recall that at some point in time, each individual or particle is associated with a value of price or energy. The *microstate* of the system is simply the record or configuration of each object and its respective value. For the N-particle system, this is simply the position and velocity of each particle, or

$$\boldsymbol{\omega} = \{(\vec{r}, \vec{v})_i\} \tag{38}$$

For the N-particle system, the number of configurations or microstates $\boldsymbol{\omega}$ for some value of total energy E is also dependent upon the volume V, number of particles N and the value of total energy E, therefore the total number of microstates Ω is given by some function F so that

$$\Omega = F(N, V, E)$$

At this point, it is difficult to construct the corresponding equation for the economic problem under consideration. To do so, a little more is required.

Development

The N particle system and N individual system differs in an important way. $f(V)$ and the deterministic equations which govern the behavior of each particle are bona fide continuous functions, for which a large body of math is designed to handle. On the other hand, there are no such equations for the latter system.

Firstly, the value of price M_i need not be continuously assigned. If the difference in price $M_j - M_{j-1}$ is too small for consumers to perceive a meaningful difference, one should expect that the demand distribution within this interval is essentially constant. On the other hand, suppose that the price difference is meaningful, so that the demand distribution within this particular 'interval' of price could be any number of potential curves. To this end, each meaningful interval between a discrete set of prices is where the unknown demand distributions will lie, separated by known points given by (M_i, N_i). With this in mind, we envision several curves pieced together at the points previously stated. The entire set of these curves, over the entire range of reserve prices M_i or the *support*, (M_{min}, M_{max}), relates to the distribution over this interval, whose properties shall be utilized as much as it is helpful.

Aiming to construct an expression for the microstate (and the total number of microstates) for the N-individual system, we consider the question: what constitutes full knowledge of the N-individual system? A cursory answer might simply be the set of all numbers which represent the respective sizes of each population $\{N_j\}$ choosing reservation price M_j, after all is this not the critical information which we are most concerned about? Yet, just as the energies of each particle arise from their respective positions and velocities, the set of which constitutes a more detailed description of the state of the system, so it is true that the description of the microstate for the set of individuals is the set of parameters which determine the choice of M_i for each member. Although these parameters i.e., the factors that determine if an individual should choose a price M_{j+1} one interval above or M_{j-1} one interval below some given reservation price M_j, are unknown, we may artfully choose our description to incorporate this insight.

To this end, we consider the distribution ${}_k\phi_j$ which lies in the interval (M_j, M_{j+1}), where ${}_k\phi_j$ represents, through some use of it in calculation, the unknown law that governs the incremental decision making of the kth member i.e., the tendency for a member to choose a reservation price of M_j or a price one increment above it of M_{j+1} where $M_j \in (M_{min}, M_{max-1})$, where $M_{min} = M_0$. Therefore, the full description of the system is the entire set of such distributions.

$$\{({}_1\phi_0, {}_1\phi_1, {}_1\phi_2, ...{}_1\phi_n), ({}_2\phi_0, {}_2\phi_1, {}_2\phi_2, ...{}_2\phi_n), ...({}_N\phi_0, {}_N\phi_1, {}_N\phi_2, ...{}_N\phi_n)\}$$

where N represents the total number of individuals and n represents the cardinality of the set of reservation prices. If

$$({}_k\phi_0, {}_k\phi_1, {}_k\phi_2, ...{}_k\phi_n) = {}_k\vec{\phi}$$

then

$$\boldsymbol{\omega} = \{{}_k\vec{\phi}\} \tag{39}$$

is the analog to Eqn. (38).

If we now modify relation (37) to consider the *total* energy E of the N-particle system and market size M of the analogous system, we obtain

$$E = f(\{(\vec{q}, \vec{p})_i\}) \Leftrightarrow M = g(\{{}_k\vec{\phi}\}) \tag{40}$$

which summarizes the notion that a macroscopic property of the system is a function of the set of values which describe the microstate.[30]

Moreover, it seems we may also write, by considering the net distribution or linear combination of distributions for all individuals for some interval (M_j, M_{j+1}). In this way,

$$\phi_j = \sum_{i=1}^{N} {}_i\phi_j \tag{41}$$

Therefore, since the set $\{\phi_j\}$ is an amalgamation of the distributions of the the members, it may also serve as a description of the state of the entire population or system. Thus,

$$\boldsymbol{\omega} = \{\phi_j\} \tag{42}$$

is a valid alternative to Eq.(39).

Additionally, we immediately know that if all states are equally probable, then

$$\Omega = \frac{(N+n-1)!}{(n-1)!(N)!} \tag{43}$$

where n is the cardinality of the set of possible reservation prices M_j. Unfortunately, we cannot accept that this should be true since all individuals often act upon guiding principles or laws which are often shared amongst a significant portion of the population i.e., reasonable people do not select prices at random. Therefore, one should observe preferences during the group selection process which reflect these principles, clearly violating the equal-probability assumption. We note that Ω may also be written as some functional relationship involving relevant parameters of the system.

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[30] $\vec{r}$ and $\vec{p}$ are in fact functions of time, just as ${}_k\vec{\phi}$ should be. However, time dependence is ignored at present. This means that the *conserved* quantities M and E remain constant in time. Moreover, M should also remain constant if ${}_k\vec{\phi}$ is perturbed from some "true" or chosen distribution.

For an N particle system, each state of the system is described by the N sets of position and momenta each particle possesses. A defined volume in which the substance is contained, naturally constrains the possible values of position coordinates the particles can assume. Thus, the volume is nothing more than something which constrains the range of possible values of $\vec{q}$ each particle may independently assume.

To this end, we seek a notion which constrains $\{\phi_j\}$ or $\{_k\vec{\phi}\}$. A straightforward constraint, valid for both formulations, is that for the predefined interval $\Delta M = M_{max} - M_{min}$, $\phi = 0$ outside of this interval.

The Generalized Function

Let the set of curves $\{\phi_j\}$ defined on the interval (M_{min}, M_{max}) be a *generalized function* of the continuous variable of price m or distribution with $n+1$ discontinuities. Thus Eqn. (10) may be represented as $\mathbb{F}(m)$ where

$$\mathbb{F}(m) := \begin{cases} \phi_0 & : M_{min} < m < M_1 \\ \phi_1 & : M_1 < m < M_2 \\ \phi_2 & : M_2 < m < M_3 \\ . & \\ . & \\ . & \\ \phi_n & : M_{n-1} < m < M_{max} \end{cases} \tag{44}$$

Now then, suppose some test function ψ with support (M_{min}, M_{max}) acts upon the distribution which corresponds to $\mathbb{F}(m)$ and $T_{\mathbb{F}}$, yielding M. Using conventional notation,

$$M = \left\langle T_{\mathbb{F}(m)}, \psi \right\rangle = \langle T_{\mathbb{F}}, \psi \rangle \tag{45}$$

where

$$\langle T_{\mathbb{F}}, \psi \rangle = \int_{M_{min}}^{M_{max}} \mathbb{F}(m)\,\psi\,dm = M \tag{46}$$

and ψ is constructed on its support so that

$$\psi(m) = \psi\left(\frac{m - M_{min}}{M_{max} - M_{min}}\right). \tag{47}$$

Letting $M_{min} = 0$,

$$\psi(m) = \psi\left(\frac{m}{M_{max}}\right) = \psi(m'). \tag{48}$$

That is, ψ evaluated at the scaled value of price is equivalent to ψ at the unscaled value.

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