RADON TRANSFORM, SPHERICAL MEANS, AND AN INVERSE PROBLEM FOR THE WAVE EQUATION

by

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A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

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TABLE OF CONTENTS

LIST OF FIGURES . . . . **vii**
ABSTRACT . . . . **viii**

Chapter

1 INTRODUCTION **1**

1.1 The Problem 1
1.2 Background 3
1.3 Main Results 8
1.4 Overview 11

2 DEFINITIONS AND PROPERTIES **14**

2.1 Schwartz Space 14
2.2 Spherical Mean Value 14
2.3 The Solutions of the IVPs 15
2.4 Radon Transform 16
2.5 Spherical Harmonics Expansion 20

3 GEOMETRIC INVERSION **22**

4 INVERSION OF $\mathcal{U}$, $\mathcal{V}$ AND $\mathcal{W}$ WHEN $n = 3$ **27**

4.1 The inversion of $\mathcal{V}$ 27
4.2 The inversion of $\mathcal{U}$ 33
4.3 The isometry and inversion of $\mathcal{W}$ 38

5 INVERSION OF $\mathcal{U}$, $\mathcal{V}$ AND $\mathcal{W}$ WHEN $n \geq 3$ IS ODD **40**

5.1 The inversion of $\mathcal{V}$ 40
5.2 The inversion of $\mathcal{U}$ 51
5.3 The isometry and inverse for $\mathcal{W}$ 56

6 THE RANGES OF $\mathcal{U}$ AND $\mathcal{V}$ WHEN n IS ODD. **58**

6.1 The Range of $\mathcal{V}$ 58
6.2 The Range of $\mathcal{U}$ 65

7 INVERSION OF $\mathcal{V}$ WHEN $n = 2$ **71**

8 SPHERICAL MEANS OVER SPHERES THROUGH THE ORIGIN **78**

BIBLIOGRAPHY **84**

LIST OF FIGURES

1.1 The light cone for the wave equation. 2

1.2 Thermoacoustic tomography. 5

1.3 Spherical mean value on spheres through the origin. 7

2.1 Two-dimensional Radon transform. It maps a function to its integrals along straight lines orthogonal to $(\cos\theta, \sin\theta)$ and a distance ρ from the origin. 17

3.1 The geometric inversion mapping x to X. 22

8.1 Spheres centered at x of radius $|x|$. 78

ABSTRACT

Consider the linear map which sends the initial data to the trace, on the light cone with vertex at O, of the solution of the initial value problem for the wave equation. We show this map is an isometry, construct its inverse, and give a partial characterization of its range. Our results are better in odd space dimensions than in the even dimensional case. Solutions of the wave equation can be expressed in terms of the spherical averages of the initial data and these spherical averages can be related to the Radon transform of a related function. We obtain our results exploiting this relationship and the isometry, the inversion formula and the range characterization of the Radon transform.

Chapter 1

INTRODUCTION

Consider the linear map which sends the initial data to the trace, on the light cone with vertex at O, of the solution of the initial value problem for the wave equation. This thesis is devoted to the study of the injectivity, the range, and the inversion of this map. In this chapter we state the problem, survey earlier work on related problems, state the main results and summarize the contents of each of the chapters.

1.1 The Problem

Suppose $f(x), g(x)$ are smooth functions on $\mathbb{R}^n$, $n > 1$ and $w(x,t)$ is the solution of the following initial value problem (IVP) for the wave equation:

$$w_{tt} - \Delta w = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.1}$$

$$w(x,0) = f(x),\ w_t(x,0) = g(x), \quad x \in \mathbb{R}^n \tag{1.2}$$

where Δ is the Laplacian with respect to the space variable $x = (x_1, x_2, \ldots, x_n)$. When $f(x), g(x)$ are smooth, this is a well posed problem (see [6]) and explicit formulas are known for $w(x,t)$ in terms of f and g. Define the operator $\mathcal{W} : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \mapsto C(\mathbb{R}^n)$ with

$$\mathcal{W}(f,g)(x) = w(x, \pm|x|), \qquad x \in \mathbb{R}^n \tag{1.3}$$

which maps the initial data (f,g) to the values of $w(x,t)$ on the light cone boundary $t = \pm|x|$. We call $\mathcal{W}$ the forward map and the forward problem is the well understood problem of constructing $w(x,|x|)$ given (f,g). Our goal is the inverse problem consisting of the inversion of $\mathcal{W}$, that is the recovery of f,g given $w(x,\pm|x|)$. In particular, we seek an inversion formula for $\mathcal{W}$ and attempt to characterize the range of $\mathcal{W}$. We show

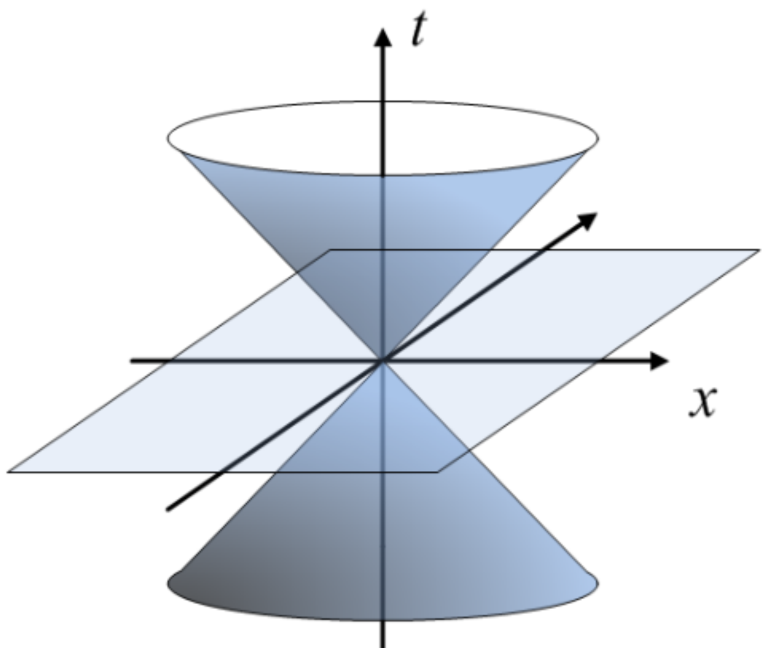

Figure 1.1: The light cone for the wave equation.

that $\mathcal{W}$ is an isometry under appropriate norms, construct the inverse of $\mathcal{W}$ and give a partial characterization of the range of $\mathcal{W}$.

If $w(x,t)$ is the solution of the IVP (1.1) (1.2) and we define the even and the odd (in time) parts of w as

$$u(x,t) = \frac{w(x,t) + w(x,-t)}{2},$$
$$v(x,t) = \frac{w(x,t) - w(x,-t)}{2},$$

then $u(x,t), v(x,t)$ are the unique solutions of the IVPs:

$$u_{tt} - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.4}$$

$$u(x,0) = f(x),\ u_t(x,0) = 0, \quad x \in \mathbb{R}^n \tag{1.5}$$

and

$$v_{tt} - \Delta v = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.6}$$

$$v(x,0) = 0,\ v_t(x,0) = g(x), \quad x \in \mathbb{R}^n \tag{1.7}$$

respectively. Note that $u(x,t)$ is even in t and $v(x,t)$ is odd in t. Define the operators $\mathcal{U} : C^\infty(\mathbb{R}^n) \mapsto C(\mathbb{R}^n)$ and $\mathcal{V} : C^\infty(\mathbb{R}^n) \mapsto C(\mathbb{R}^n)$ with

$$\mathcal{U}: \quad f(x) \quad \longmapsto \quad u(x,|x|), \tag{1.8}$$

$$\mathcal{V}: \quad g(x) \quad \longmapsto \quad v(x,|x|), \tag{1.9}$$

so

$$(\mathcal{U}f)(x) = \frac{w(x,|x|) + w(x,-|x|)}{2},$$
$$(\mathcal{V}g)(x) = \frac{w(x,|x|) - w(x,-|x|)}{2}.$$

Note that knowing $\mathcal{U}$ and $\mathcal{V}$ is equivalent to knowing $\mathcal{W}$. Given $\mathcal{W}(f,g)(x)$ for all $x \in \mathbb{R}^n$, if we are able to invert $\mathcal{U}, \mathcal{V}$, then we will be able to recover $f(x), g(x)$, that is, we will also be able to invert $\mathcal{W}$. So in the following chapters, we will focus on the inversion of $\mathcal{U}, \mathcal{V}$. In particular, when the initial data $f(x), g(x)$ are functions in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, we show that $\mathcal{U}, \mathcal{V}$ are isometries, and inversion formulas are obtained. From these we derive an isometry identity and an inversion formula for $\mathcal{W}$.

1.2 Background

The problem under consideration has a connection with general relativity as well as medical imaging. A fundamental conjecture in general relativity [17], [18] asserts that the domains of outer communication of regular, stationary, four dimensional, vacuum black hole solutions are isometrically diffeomorphic to those of Kerr black holes. So far the conjecture has been resolved by combining results of Hawking [13], Carter [3], and Robinson [29] under additional hypotheses. To extend Hawking's argument, one must solve a boundary value problem (BVP) with data on the horizon for a linear partial differential equation. In the simplest case of the wave equation in the Minkowski space $\mathbb{R}^{3+1}$ with metric $dt^2 - dx_1^2 - dx_2^2 - dx_3^2$, let us consider the boundary value problem in the domain exterior to the light cone

$$E = \{(x,t) \in \mathbb{R}^{3+1} : |x| \geq |t|\}.$$

We seek the solution of the BVP

$$\begin{aligned} &w_{tt} - \Delta w = 0, \quad |x| \geq |t|, \\ &w(x, \pm|x|) = b(x), \qquad x \in \mathbb{R}^3 \end{aligned}$$

for some given function $b(x)$. This problem is known to be ill-posed, that is, solutions may not exist for smooth non-analytic $b(.)$ (as seen in our range characterization result) and it is not clear whether the solution depends continuously on b even when the solution does exist. We show, given b, how one may reconstruct $w(., t=0)$, $w_t(., t=0)$ and hence w on E. We also show that there is an isometry between b and $w(., t = 0)$, $w_t(., t=0)$ providing that w does depend continuously on b. We also make progress towards characterizing for which b there is a solution of this BVP.

Our inverse problem also has a connection with medical imaging. In medical imaging using ultrasound or computerized tomography (CT) one must recover a function from its integrals over a family of lines, hyper-planes or spheres - see [25]. In CT scans, a cross-section of the human body is scanned by a thin X-ray beam and its intensity loss is recorded by a detector and processed by a computer to produce a two-dimensional image. Mathematically this may be modeled as follows.

Let $h(x)$ be the X-ray attenuation coefficient of the tissue at the point x and the X-ray beam is assumed to travel along a line L with initial intensity I_0. When the X-ray beam traverses a small distance δx from x to $x + \delta x$, it suffers a relative intensity loss

$$\delta I / I = h(x) \delta x.$$

After passing through the body, the remaining intensity I_1 is

$$I_1 / I_0 = e^{-\int_L h(x)\, dx},$$

so the scanning process provides us with the line integral of $h(x)$ along L. We need to reconstruct $h(x)$ from $\int_L h(x)\, dx$ along each of the lines L. This transform which maps a function into the set of the its line integrals is called the (two-dimensional) Radon transform and the problem of recovering h from its Radon transform calls for the inversion of the Radon transform. This was done as early as 1917 by Radon who gave an explicit inversion formula. In $\mathbb{R}^n$, for dimension $n \geq 2$, the (n-dimensional) Radon transform is the integral of $h(x)$ on a hyper-plane in $\mathbb{R}^n$.

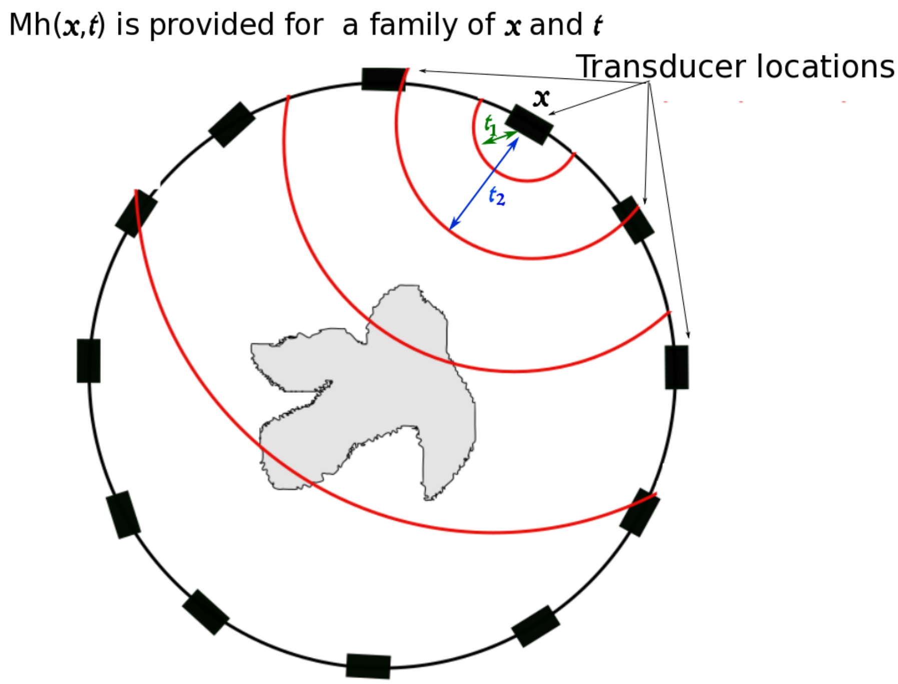

Figure 1.2: Thermoacoustic tomography.

Thermo-acoustic Tomography (TAT), a relatively new hybrid imaging technique, uses radio frequency energy input at time t_0 and measures emitted ultrasound waves (see [21], [22], [23]). In TAT, a short microwave (radio frequency) pulse is sent through an object. At each internal location x, a certain energy $h(x)$ is absorbed, which causes a thermoelastic expansion and creates a pressure wave. This wave can be detected by ultrasound transducers placed on the edges of the object. The cancerous cells often absorb several times more radio frequency energy than the normal cells, and hence show up as a significant contrast to the healthy locations. Assuming constant sound speed c in normal tissue, the sound waves detected at any point x in time $t > t_0$ were generated by inclusions lying on the sphere centered at x and of radius $c(t-t_0)$. Therefore, this imaging technique requires the inversion of a generalized Radon transform that consists of recovering a function from the integrals of $h(x)$ over spheres centered at all available transducers locations x.

For a continuous real valued function $h(x)$ on $\mathbb{R}^n$, x a point in $\mathbb{R}^n$, t a real number, the spherical mean value is defined to be

$$(\mathcal{M}h)(x,t) = \frac{1}{\omega_{n-1}} \int_{|\theta|=1} h(x+t\theta)\, d\theta$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$ and $d\theta$ is the surface area element on the unit sphere. The imaging technique mentioned above requires the recovery of $h(x)$ from a knowledge of $(\mathcal{M}h)(x,t)$ for all centers x on a certain surface S and all radii t in a certain range.

Recovering a function from its mean values over a family of spheres has a long history. The first work to tackle problems of recovering a function from its spherical mean values was John [20] who studied the problem when the centers of the spheres are on a plane. The first reconstruction formulas were derived by Norton [26] who found an inversion method based on harmonic decomposition.

The article [2] has a complete analysis, in the odd dimensional case, of the problem of recovering a function from its spherical mean values over the family of spheres with centers on a hyperplane; additional results are available in [7], [1] and [31]. Goncharov [12] studied the problem of inverting the spherical means restricted to the variety of spheres tangent to a hypersurface. In [10], for odd n, the authors studied the problem of recovering h from the mean values of h on spheres centered on the boundary of a bounded, open, connected subset D of $\mathbb{R}^n$. In [9], the authors studied the same problem when n is even and $(\mathcal{M}h)(x,t)$ is given for all x on the boundary of an open ball B in $\mathbb{R}^n$. A more detailed history of inversion from spherical mean values can be found in [27] and [24].

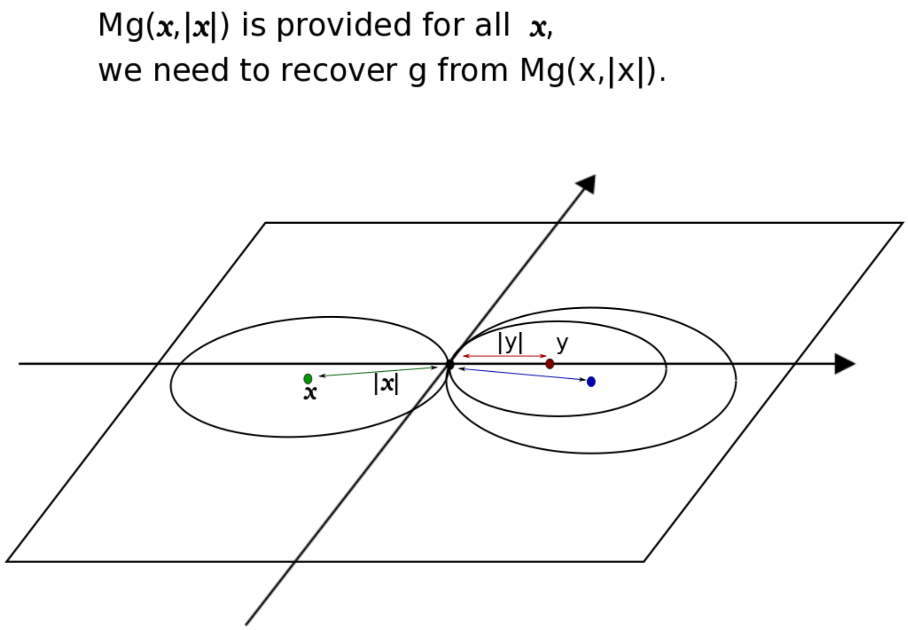

Figure 1.3: Spherical mean value on spheres through the origin.

In Chapter 4, we observe that our goal of inverting $\mathcal{W}$ is closely related to the inversion of $h(x)$ from its spherical mean values $(\mathcal{M}h)(x,|x|)$ (or some derivatives of $(\mathcal{M}h)(x,|x|)$) over all spheres through the origin. When $n=3$ the two problems are equivalent; for other n the connection is complicated because some derivatives and integrals of these spherical averages are also involved. In [5] and [34] the authors studied the recovery of a function from its mean values over spheres passing through a fixed point. In [5], the authors considered the recovery of $h(x)$ from $(\mathcal{M}h)(x,|x|)$. They used the spherical harmonic expansions of $h(x)$ and $(\mathcal{M}h)(x,|x|)$, and gave a formula to recover the spherical harmonic expansion cofficients of $h(x)$ from the spherical harmonic expansion cofficients of $(\mathcal{M}h)(x,|x|)$. In [34], the author gave a recursive method to recover $h(x)$ from $(\mathcal{M}h)(x,|x|)$ when x lies on a straight line and the values of $h(x)$ inside a small ball around the origin are known. In Chapter 8, we study the inversion of $h(x)$ from $(\mathcal{M}h)(x,|x|)$ where we give an isometry and an explicit inversion formula for recovering h from its spherical averages over all spheres through the origin.

An important tool in our work is the observation that the geometric inversion map (inversion across the unit sphere - see Chapter 3) maps spherical mean values of a function over spheres through the origin to the Radon transform of a related function. So the inversion of $g(x)$ from $(\mathcal{M}g)(x, |x|)$ is essentially the inversion of the Radon transform, which is well known (See Theorem 2.1 in [25]). Furthermore, for general n, in Chapters 4, 5, 6 and 7, we see that by geometric inversion, $\mathcal{U}f, \mathcal{V}g$ are related to the derivatives and integrals of the Radon transforms of related functions. Therefore the inversion of $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ is achievable.

1.3 Main Results

We study the inversion of $\mathcal{U}, \mathcal{V}$ in $\mathbb{R}^n$ where $n \geq 2$ is an arbitrary integer. In particular, we find isometry identities and inversion formulas, and partial characterizion of the ranges of $\mathcal{U}, \mathcal{V}$. We answer our questions when n is odd, $n \geq 3$, and when the initial data $f(x), g(x)$ are functions in $\mathscr{S}_0(\mathbb{R}^n)$ which is a subspace of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ (see the definitions of $\mathscr{S}_0(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ in Chapter 2). We also find an isometry and an inversion formula when $n = 2$. In principle, the inverse formula of $\mathcal{V}$ in higher even dimensional space $\mathbb{R}^n$ is achievable, but the formula will be a bit more complicated.

For odd n we have the following isometry identities.

Theorem 1.1 (Isometry identities in $\mathbb{R}^n$ when $n \geq 3$ is odd). *Let $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ where $n \geq 3$ is odd, $(\mathcal{U}f)(x), (\mathcal{V}g)(x)$ be the forward operators defined in (1.8), (1.9) respectively. Then we have*

$$\int_{\mathbb{R}^n} (|x|g(x))^2 \, dx = 8 \int_{\mathbb{R}^n} \frac{|\partial_r(|x|^{\frac{n-1}{2}}(\mathcal{V}g)(x))|^2}{|x|^{n-3}} \, dx$$

and

$$\int_{\mathbb{R}^n} \left(\frac{f(x)}{|x|}\right)^2 \, dx = 2 \int_{\mathbb{R}^n} \left(\frac{(\mathcal{U}f)(x)}{|x|}\right)^2 \, dx.$$

From here, we can obatain an isometry for $\mathcal{W}$.

Theorem 1.2 (**Isometry for $\mathcal{W}$**). *If $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ and $w(x,t)$ is the solution of the IVP (1.1) (1.2), where $n \geq 3$ is odd,then*

$$\int_{\mathbb{R}^n} (|x|g(x))^2 + \left(\frac{f(x)}{|x|}\right)^2 \, dx$$
$$= \int_{\mathbb{R}^n} 2\left(\frac{\partial_r\left(|x|^{\frac{n-1}{2}}(w(x,|x|) - w(x,-|x|))\right)}{|x|^{\frac{n-3}{2}}}\right)^2 + \frac{1}{2}\left(\frac{w(x,|x|) + w(x,-|x|)}{|x|}\right)^2 \, dx.$$

For n odd, we obtain the inverses of $\mathcal{U}$, $\mathcal{V}$ (and hence W) by finding the formal L^2 adjoint of $\mathcal{U}, \mathcal{V}$, and using the isometry obtained above.

Theorem 1.3 (**First inversion formulas for $\mathcal{U}, \mathcal{V}$ when $n \geq 3$ is odd**). *Suppose $n \geq 3$ is odd; for $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ we have*

$$f(x) = 2|x|^2\mathcal{U}^*\left(\frac{(\mathcal{U}f)(x)}{|x|^2}\right)$$

and

$$g(x) = -\frac{8}{|x|^2}\mathcal{V}^*\left(|x|^{\frac{3-n}{2}}\partial_r^2(|x|^{\frac{n+1}{2}}(\mathcal{V}g)(x))\right)$$

where $\mathcal{U}^*, \mathcal{V}^*$ are defined by

$$(\mathcal{U}^*\phi)(x) = \frac{(-1)^{\frac{n-1}{2}}}{2\pi^{\frac{n-1}{2}}|x|^{\frac{n+1}{2}}}\partial_r^{\frac{n-1}{2}}(\mathcal{R}\phi_*)(x/|x|, |x|/2), \quad \forall \phi \in \mathscr{S}_0(\mathbb{R}^n),$$

with $\phi_*(y) = \phi(y)|y|$ and

$$(\mathcal{V}^*\phi)(x) = \frac{(-1)^{\frac{n+1}{2}}}{4\pi^{\frac{n-1}{2}}|x|^{\frac{n-1}{2}}}\partial_r^{\frac{n-3}{2}}(\mathcal{R}\phi)(x/|x|, |x|/2), \quad \forall \phi \in \mathscr{S}_0(\mathbb{R}^n).$$

Using the inversion formula of the Radon transform, we obtain another inversion formulas for $\mathcal{U}, \mathcal{V}$.

Theorem 1.4 (**Second inversion formulas for $\mathcal{U}, \mathcal{V}$ when $n \geq 3$ is odd**). *Let $n = 2m+1$ with $m \geq 1$, then for any $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$, we have*

$$f(x) = \frac{|X|^{2m}}{(4\pi)^m}\int_{S^{n-1}} \partial_s^m\left(\frac{1}{s^m}(\mathcal{U}f)(\frac{\theta}{2s})\right)\Bigg|_{s=X\cdot\theta} d\theta,$$
$$g(x) = -\frac{|X|^{2m+2}}{(4\pi)^m}\int_{S^{n-1}} \partial_s^{m+1}\left(\frac{1}{s^{m-1}|s|}(\mathcal{V}g)(\frac{\theta}{2s})\right)\Bigg|_{s=X\cdot\theta} d\theta,$$

where $X = x/|x|^2$.

The inversion formulas for $\mathcal{U}, \mathcal{V}$ allow us to write the inversion formulas for $\mathcal{W}$.

Theorem 1.5 (**First inversion formula for $\mathcal{W}$ when $n \geq 3$ is odd**). *Suppose $n \geq 3$ is odd; for $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ we have*

$$f(x) = |x|^2 \mathcal{U}^* \left(\frac{w(x, |x| + w(x, -|x|)}{|x|^2} \right),$$
$$g(x) = -\frac{4}{|x|^2} \mathcal{V}^* \left(|x|^{\frac{3-n}{2}} \partial_r^2 \left(|x|^{\frac{n+1}{2}} (w(x, |x|) - w(x, -|x|)) \right) \right),$$

where $\mathcal{U}^*, \mathcal{V}^*$ are the operators defined in Theorem 1.3.

Theorem 1.6 (**Second inverse formula for $\mathcal{W}$ when $n \geq 3$ is odd**). *Suppose $n \geq 3$ is odd; for $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ we have*

$$f(x) = \frac{|X|^{2m}}{2^{2m+1} \pi^m} \int_{S^{n-1}} \partial_s^m \left(\frac{w(\frac{\theta}{2s}, \frac{\theta}{2|s|}) + w(\frac{\theta}{2s}, -\frac{\theta}{2|s|})}{s^m} \right) \Bigg|_{s = X \cdot \theta} d\theta,$$
$$g(x) = -\frac{|X|^{n+1}}{2^{2m+1} \pi^m} \int_{S^{n-1}} \partial_s^{m+1} \left(\frac{w(\frac{\theta}{2s}, \frac{\theta}{2|s|}) - w(\frac{\theta}{2s}, -\frac{\theta}{2|s|})}{s^{m-1}|s|} \right) \Bigg|_{s = X \cdot \theta} d\theta,$$

where $X = x/|x|^2$.

For n even, we are unable to obtain isometries as nice as those for the odd n case. We find more complicated isometries and hence more complicated inversion formulas when n is even. When $n = 2$, we have the following isometry and inverse formula for $\mathcal{V}$.

Theorem 1.7 (**Isometry and inverse formula for $\mathcal{V}$ when $n = 2$**). *Let $n = 2$; then for any $g \in \mathscr{S}_0(\mathbb{R}^2)$ we have*

$$\frac{\pi^2}{4} \int_{S^1} \int_{\mathbb{R}} \left(|s|^{3/2} g(s\theta) \right)^2 \, ds \, d\theta = \int_{S^1} \int_{\mathbb{R}} \left(\frac{\partial}{\partial s} \int_{-\infty}^{s} \frac{\sqrt{|a|}}{\sqrt{s-a}} \frac{\partial}{\partial a} \int_0^{\frac{1}{2a}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2a} - \tau\right|}} \, d\tau \, da \right)^2 ds \, d\theta.$$

The inversion formula is given by

$$g(x) = \frac{-1}{\sqrt{2}\pi^2} \int_{S^1} \int_{\mathbb{R}} \frac{1}{\theta \cdot x|x| - s|x|^3} \frac{\partial}{\partial s} \left(\sqrt{|s|} \frac{\partial}{\partial s} \int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2s} - \tau\right|}} \, d\tau \right) ds \, d\theta.$$

Finally, we give some results for the recovery of a function from its mean values over spheres through the origin. This problem is of independent interest and the results for this problem were incomplete. It is a simple matter for us to obtain the results for this problem, given the work done for $\mathcal{U}$, $\mathcal{V}$. We state the results for completeness. This problem is much easier than the inversion of $\mathcal{U}$ and $\mathcal{V}$ and the inversion formulas can be written down for all $n \geq 2$ - not just odd n or $n = 2$.

Theorem 1.8 (Isometry for spherical means). *Let $h(x) \in \mathscr{S}_0(\mathbb{R}^n)$. For odd n we have*

$$\int_{\mathbb{R}^n} \left(h(x)|x|^{n-2}\right)^2 \, dx = \frac{\pi}{2^{3n-4}\Gamma^2(n/2)} \int_{S^{n-1}} \int_{\mathbb{R}} \left(\partial_s^{\frac{n-1}{2}} \Big(\frac{1}{|s|^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{|2s|})\Big)\right)^2 \, ds \, d\theta.$$

and for even n we have

$$\int_{\mathbb{R}^n} \left(h(x)|x|^{n-2}\right)^2 \, dx =$$
$$\frac{(-1)^{\frac{n}{2}}(n-1)!}{2^{3n-4}\Gamma^2(n/2)} \int_{S^{n-1}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\mathcal{M}h)(\frac{\theta}{2(s+t)}, \frac{1}{2|s+t|})(\mathcal{M}h)(\frac{\theta}{2t}, \frac{1}{|2t|}) \, dt\right) s^{-n} \, ds \, d\theta.$$

Theorem 1.9 (Inversion from spherical means). *If $h \in \mathscr{S}_0(\mathbb{R}^n)$, for odd n we have*

$$h(x) = \frac{(-1)^{\frac{n-1}{2}}}{2^{2n-2}\pi^{n/2-1}\Gamma(n/2)|x|^{2n-2}} \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}}\right)\Big|_{s=X\cdot\theta} \, d\theta,$$

and for even n we have

$$h(x) = \frac{(-1)^{\frac{n}{2}}}{2^{2n-2}\pi^{n/2}\Gamma(n/2)|x|^{2n-2}} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{1}{q} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}}\right)\Big|_{s=X\cdot\theta+q} \, dq \, d\theta$$

where $X = x/|x|^2$.

1.4 Overview

We show that $\mathcal{U}$ and $\mathcal{V}$ are isometries and we characterize the ranges of $\mathcal{U}$ and $\mathcal{V}$. These imply corresponding results for $\mathcal{W}$.

In Chapter 2, we introduce some notation, define some function spaces and operators, and restate some of the known results on the Radon transform and explicit formulas for solutions of the IVP wave equation which will be used often in our thesis.

In Chapter 3, we introduce the geometric inversion map from $\mathbb{R}^n$ to $\mathbb{R}^n$ which maps a point to its reflection across the origin centered unit sphere. Using this we show there is a close relation between spherical mean values on spheres through the origin and the Radon transform, which will be very useful in achieving our goal of inverting $\mathcal{W}$.

In Chapter 4, we study the inversion of $\mathcal{U}, \mathcal{V}$ in $\mathbb{R}^3$. The $n = 3$ case is the simplest case and the analysis here will guide us in the inversion of $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{R}^n$ when $n > 3$ and n is odd and the more complicated case when n is even. When $n = 3$, there is a simple relation between $(\mathcal{V}g)(x)$ and the Radon transform and the isometry of the Radon transform quickly leads to an isometry for $\mathcal{U}$ and $\mathcal{V}$, so a construction of the formal adjoint of $\mathcal{U}$ and $\mathcal{V}$ results in an inversion formula for $\mathcal{U}$ and $\mathcal{V}$.

In Chapter 5, we study the inversion of $\mathcal{U}, \mathcal{V}$ when n is odd and $n > 3$. We find a relation between $(\mathcal{V}g)(x)$ and the derivatives of the Radon transform which helps us find an isometry identity for $\mathcal{V}$. Then we find the adjoint of $\mathcal{V}$ which results in an inversion formula for $\mathcal{V}$. Similar ideas are then applied to $\mathcal{U}$.

In Chapter 6, we attempt to characterize the range of $\mathcal{U}, \mathcal{V}$ when n is odd. Our results are incomplete - we give necessary conditions on the range when the domain of $\mathcal{U}$ and $\mathcal{V}$ consists of $\mathscr{S}(\mathbb{R}^n)$, the space of Scwhartz functions and we characterize a special subspace of the range. A complete characterization of the ranges of $\mathcal{U}$ and $\mathcal{V}$ is a project for the future.

In Chapter 7, we study the inversion of $\mathcal{U}$ and $\mathcal{V}$ when n is even. The even n case is more difficult because the fundamental solution of the wave equation is supported on the surface of the origin based light cone when n is odd but is supported on the solid light cone when n is even. This makes it more difficult to use the relation between spherical mean values on spheres through the origin and the Radon transform. We find an isometry and an inversion formula for $\mathcal{V}$ when $n = 2$. The inversion of $\mathcal{U}$ and $\mathcal{V}$ when n is even and $n > 2$ are projects for the future.

In Chapter 8, we study the problem of recovering a function from its mean values over the family of spheres through the origin. This problem is really an easier version of the problems studied in the earlier chapters. This problem has been studied in the literature but the results in the literature are unsatisfactory and for completeness we state our results and give an outline of the proofs.

Chapter 2

DEFINITIONS AND PROPERTIES

In this chapter, we introduce some notation, define some function spaces and operators, and restate some of the known results on the Radon transform as well as give known explicit formulas for the solution of the IVP for the wave equation.

2.1 Schwartz Space

The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ consists of smooth functions on $\mathbb{R}^n$ all of whose derivatives decay rapidly as $x \to \infty$. More specifically

$$\mathscr{S}(\mathbb{R}^n) = \{h(x) \in C^\infty(\mathbb{R}^n) : \|h\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta h(x)| < \infty,\ \forall \alpha, \beta \in \mathbb{Z}^n_+\},$$

where $\mathbb{Z}^n_+$ consists of non-negative integer tuples and ∂^β is the differential operator $\partial_1^{\beta_1}\partial_2^{\beta_2}\cdots\partial_n^{\beta_n}$ if $\beta = (\beta_1, \beta_2, \dots, \beta_n)$.

Our goal is to study the inversion of $\mathcal{U}$ and $\mathcal{V}$ when their domains are restricted to a special subspace of $\mathscr{S}(\mathbb{R}^n)$ consisting of functions all of whose derivatives decay to zero rapidly as $x \to 0$. Specifically, we define

$$\mathscr{S}_0(\mathbb{R}^n) = \{h(x) \in \mathscr{S}(\mathbb{R}^n) : \lim_{x\to 0} \left|\frac{D^\beta h(x)}{x^\alpha}\right| = 0,\ \forall \alpha, \beta \in \mathbb{Z}^n_+\},$$

We will also work with $\mathscr{S}(S^{n-1}\times\mathbb{R})$ and $\mathscr{S}_0(S^{n-1}\times\mathbb{R})$ which consist of the restrictions of elements of $\mathscr{S}(\mathbb{R}^n\times\mathbb{R})$ and $\mathscr{S}_0(\mathbb{R}^n\times\mathbb{R})$ to $S^{n-1}\times\mathbb{R}$; here S^{n-1} is the origin centered unit sphere in $\mathbb{R}^n$.

2.2 Spherical Mean Value

For the rest of the thesis we define

$$\frac{\partial}{\partial t^2} := \frac{1}{2t}\frac{\partial}{\partial t};$$

one may check that

$$\frac{\partial}{\partial t^2}(h(t^2)) = h'(t^2).$$

For any continuous function $h(x)$ on $\mathbb{R}^n$, define the spherical mean value operator

$$(\mathcal{M}h)(x,t) = \frac{1}{\omega_{n-1}} \int_{|\theta|=1} h(x+t\theta)\, d\theta, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

where $\omega_{n-1} = |S^{n-1}|$ is the surface area of the unit sphere $S^{n-1} = \{\theta \in \mathbb{R}^n : |\theta| = 1\}$ in $\mathbb{R}^n$ and $d\theta$ is the surface area element on S^{n-1}. So $(\mathcal{M}h)(x,t)$ is the spherical average of h on the sphere centered at x and of radius t. When $t > 0$, $(\mathcal{M}h)(x,t)$ has other useful representations, namely

$$\begin{aligned}(\mathcal{M}h)(x,t) &= \frac{1}{\omega_{n-1}t^{n-1}} \int_{|y-x|=t} h(y)\, dS_y \\ &= \frac{2}{\omega_{n-1}t^{n-2}} \int_{\mathbb{R}^n} h(y)\delta(|y-x|^2 - t^2)\, dy,\end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function.

2.3 The Solutions of the IVPs

Considering our goal, it will be important to have explicit expressions for solutions of the IVPs of the wave equation. Consider the IVP

$$v_{tt} - \Delta v = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.1}$$

$$v(x,0) = 0, \; v_t(x,0) = g(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

The solution $v(x,t)$ is given by the following theorem.

Theorem 2.1 (Page 682 in [6]). *If $g \in C^{(n+1)/2}(\mathbb{R}^n)$ when n is odd and $g \in C^{(n+2)/2}(\mathbb{R}^n)$ when n is even then $v(x,t)$, the solution of IVP (2.1) (2.2), is given by*

$$v(x,t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - s^2)^{(n-3)/2} s(\mathcal{M}g)(x,s)\, ds, \quad t \geq 0. \tag{2.3}$$

Furthermore, when n is odd, we also have

$$v(x,t) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n}{2})} \left(\frac{\partial}{\partial t^2}\right)^{(n-3)/2} (t^{n-2}(\mathcal{M}g)(x,t)), \quad t \geq 0, \tag{2.4}$$

and when n is even, we also have

$$v(x,t) = \frac{t}{\Gamma(\frac{n}{2})} \left(\frac{\partial}{\partial t^2} \right)^{(n-2)/2} \left(t^{n-3} \int_0^t \frac{s(\mathcal{M}g)(x,s)}{\sqrt{t^2 - s^2}} \, ds \right), \quad t \geq 0. \tag{2.5}$$

Now consider the IVP:

$$u_{tt} - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.6}$$

$$u(x,0) = f(x), \; u_t(x,0) = 0, \quad x \in \mathbb{R}^n. \tag{2.7}$$

Notice that if $u^*(x,t)$ is the solution of the IVP:

$$u^*_{tt} - \Delta u^* = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

$$u^*(x,0) = 0, \; u^*_t(x,0) = f(x), \quad x \in \mathbb{R}^n,$$

then $u(x,t) = \partial_t u^*(x,t)$ is the solution of the IVP (2.6) (2.7). Since $u^*(x,t)$ is given by (2.3), if $f \in C^\infty(\mathbb{R}^n)$ then

$$u(x,t) = \frac{1}{(n-2)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^t (t^2 - s^2)^{(n-3)/2} (\mathcal{M}f)(x,s) \, ds, \quad t \geq 0. \tag{2.8}$$

2.4 Radon Transform

The n-dimensional Radon transform $\mathcal{R}$ maps a function on $\mathbb{R}^n$ linearly to the set of its integrals on the hyperplanes in $\mathbb{R}^n$. If $h(x) \in \mathscr{S}(\mathbb{R}^n), \theta \in S^{n-1}, s \in \mathbb{R}$, then the Radon transform of $h(x)$ is

$$(\mathcal{R}h)(\theta, s) = \int_{x \cdot \theta = s} h(x) \, dS_x$$

which represents the integral of h on the hyper-plane in $\mathbb{R}^n$, through $s\theta$, and unit normal θ - so this hyper-plane is $|s|$ units away from the origin. Since the hyper-plane corresponding to (θ, s) is the same as the hyper-plane corresponding to $(-\theta, -s)$ we have the useful relation

$$(\mathcal{R}h)(-\theta, -s) = (\mathcal{R}h)(\theta, s), \qquad \theta \in S^{n-1}, \; s \in \mathbb{R}.$$

By Theorem 2.4 (Schwartz theorem), the Radon transform is a linear one-to-one mapping of $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(S^{n-1} \times \mathbb{R})$. The Radon transform is an isomtery.

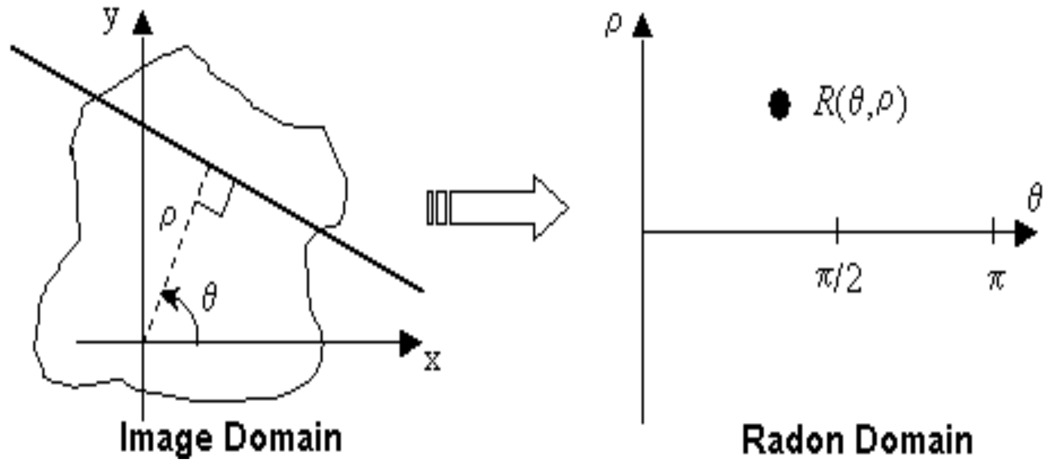

Figure 2.1: Two-dimensional Radon transform. It maps a function to its integrals along straight lines orthogonal to $(\cos\theta, \sin\theta)$ and a distance ρ from the origin.

Theorem 2.2 (Page 12 in [11]). *If $h \in \mathscr{S}(\mathbb{R}^n)$, for odd n we have*

$$\int_{\mathbb{R}^n} |h(x)|^2 \, dx = \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} \int_{\mathbb{R}} \left| \partial_s^{\frac{n-1}{2}} (\mathcal{R}h)(\theta, s) \right|^2 \, ds \, d\theta.$$

and for even n we have

$$\int_{\mathbb{R}^n} |h(x)|^2 \, dx = \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} \int_{\mathbb{R}} \left| {}_{-\infty}D_s^{\frac{n-1}{2}} (\mathcal{R}h)(\theta, s) \right|^2 \, ds \, d\theta,$$

where ${}_{-\infty}D_s^{\frac{n-1}{2}}$ is the $(n-1)/2$-th order Riemann-Liouville fractional derivative with lower limit $-\infty$, defined as

$${}_{-\infty}D_s^{(n-1)/2} (\mathcal{R}h)(\theta, s) = \frac{1}{\sqrt{\pi}} \partial_s^{n/2} \int_{-\infty}^{s} \frac{(\mathcal{R}h)(\theta, t)}{\sqrt{s-t}} \, dt.$$

For even n we also have

$$\int_{\mathbb{R}^n} |h(x)|^2 \, dx = \frac{(-1)^{\frac{n}{2}} (n-1)!}{(2\pi)^n} \int_{S^{n-1}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\mathcal{R}h)(\theta, s+t)(\mathcal{R}h)(\theta, t) \, dt \right) s^{-n} ds \, d\theta.$$

The second identity in Theorem 2.2 is not explicitly mentioned in [11] but follows quickly from some results in [16], [28] and [33]. We give its short proof here.

Proof. Write ${}_{-\infty}D_s^{(n-1)/2} (\mathcal{R}h)(\theta, s)$ in terms of convolution:

$${}_{-\infty}D_s^{(n-1)/2} (\mathcal{R}h)(\theta, s) = \frac{1}{\sqrt{\pi}} \partial_s^{n/2} \left((\mathcal{R}h)(\theta, s) * s_+^{-1/2} \right)$$

where

$$s_+^{-1/2} = s^{-1/2} \quad \text{if } s > 0, \quad s_+^{-1/2} = 0 \quad \text{if } s \leq 0,$$

and $(\mathcal{R}h)(\theta, s) * s_+^{-1/2}$ means the convolution of $(\mathcal{R}h)(\theta, s)$ and $s_+^{-1/2}$. Consider the Fourier transform of ${}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)$ with respect to the variable s:

$$\mathcal{F}\{{}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)\}(\theta, \tau) = \int_{\mathbb{R}} {}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)\ e^{-is\tau}\ ds.$$

From Example 7.1.17 on Page 167 in [16],

$$\mathcal{F}\{s_+^{-1/2}\}(\tau) = \sqrt{\pi}\left(e^{-\frac{\pi i}{4}}\tau_+^{-1/2} + e^{\frac{\pi i}{4}}\tau_-^{-1/2}\right)$$

where

$$\tau_-^{-1/2} = |\tau|^{-1/2} \quad \text{if } \tau < 0, \quad \tau_-^{-1/2} = 0 \quad \text{if } \tau \geq 0,$$

we have

$$\mathcal{F}\{{}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)\}(\theta, \tau) = (i\tau)^{n/2}\mathcal{F}\{(\mathcal{R}h)(\theta, s)\}(\theta, \tau)\left(e^{-\frac{\pi i}{4}}\tau_+^{-1/2} + e^{\frac{\pi i}{4}}\tau_-^{-1/2}\right).$$

(A similar expression of $\mathcal{F}\{{}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)\}(\theta, \tau)$ can also be found on Page 111 in [28].) Now $\left|(i\tau)^{n/2}\left(e^{-\frac{\pi i}{4}}\tau_+^{-1/2} + e^{\frac{\pi i}{4}}\tau_-^{-1/2}\right)\right| = |\tau|^{\frac{(n-1)}{2}}$, and since $\mathcal{R}h \in \mathscr{S}(S^{n-1} \times \mathbb{R})$ and the Fourier transform is a bijection from $\mathscr{S}(\mathbb{R})$ onto $\mathscr{S}(\mathbb{R})$, $\mathcal{F}\{(\mathcal{R}h)(\theta, s)\}(\theta, \tau) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$. Hence, for fixed $\theta \in S^{n-1}$,

$$(i\tau)^{n/2}\mathcal{F}\{(\mathcal{R}h)(\theta, s)\}(\theta, \tau)\left(e^{-\frac{\pi i}{4}}\tau_+^{-1/2} + e^{\frac{\pi i}{4}}\tau_-^{-1/2}\right) \in L_2(\mathbb{R}).$$

Since the Fourier transform is a bijection from $L_2(\mathbb{R})$ onto $L_2(\mathbb{R})$, for fixed $\theta \in S^{n-1}$, ${}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)$ is also a function in $L_2(\mathbb{R})$. By Plancherel's theorem, we have

$$\begin{aligned} 2\pi \int_{\mathbb{R}} \left| {}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)\right|^2 ds &= \int_{\mathbb{R}} \left|\mathcal{F}\{{}_{-\infty}D_s^{\frac{n-1}{2}}(\mathcal{R}h)(\theta, s)\}(\theta, \tau)\right|^2 d\tau \\ &= \int_{\mathbb{R}} |\tau|^{n-1} \left|\mathcal{F}\{(\mathcal{R}h)(\theta, s)\}(\theta, \tau)\right|^2 d\tau. \end{aligned}$$

By Theorem 2.1 in [33],

$$\int_{\mathbb{R}^n} |h(x)|^2\, dx = \frac{1}{2(2\pi)^n}\int_{S^{n-1}}\int_{\mathbb{R}} |\tau|^{n-1} \left|\mathcal{F}\{(\mathcal{R}h)(\theta, s)\right|^2 d\tau\, d\theta.$$

Therefore,

$$\int_{\mathbb{R}^n} |h(x)|^2 \, dx = \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} \int_{\mathbb{R}} \left| {}_{-\infty}D_s^{\frac{n-1}{2}} (\mathcal{R}h)(\theta, s) \right|^2 \, ds \, d\theta,$$

□

The range of the Radon transform map $\mathcal{R}$ is given by the following theorem:

Theorem 2.3 (Theorem 4.1, Theorem 4.2 in [25]). *The Radon transform $\mathcal{R}$ is an injective linear map from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(S^{n-1} \times \mathbb{R})$. The range of $\mathcal{R}$ consists of precisely those functions $q(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$ such that $q(-\theta, -s) = q(\theta, s)$ for all $(s, \theta) \in S^{n-1} \times \mathbb{R}$ and*

$$\int_{\mathbb{R}} s^k q(\theta, s) \, ds = p_k(\theta)$$

is a homogeneous polynomial of degree k in θ, for each $k = 0, 1, 2 \cdots$.

Further, $(\mathcal{R}h)(\theta, s) = 0$ for all $(\theta, s) \in S^{n-1} \times (\sigma, \infty)$, for some $\sigma > 0$, iff $h(x)$ is supported in the ball $|x| \leq \sigma$.

The inverse of $\mathcal{R}$ is given by the following theorem.

Theorem 2.4 (Page 20 in [25]). *For any $h \in \mathscr{S}(\mathbb{R}^n)$ we have*

$$h(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \partial_s^{n-1} (\mathcal{R}h)(\theta, s)|_{s = x\cdot\theta} \, d\theta, \qquad n \text{ odd},$$

and

$$h(x) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{(\mathcal{R}h)(\theta, s)}{(\theta \cdot x - s)^n} \, ds \, d\theta, \qquad n \text{ even}$$

where the inner integral is a Cauchy principal value integral. The even n inversion formula may also be expressed as an ordinary integral

$$h(x) = \frac{(-1)^{n/2}}{2(2\pi)^n} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{(\mathcal{R}h)^{(n-1)}(\theta, x\cdot\theta + q) - (\mathcal{R}h)^{(n-1)}(\theta, x\cdot\theta - q)}{q} \, dq \, d\theta,$$

where $(\mathcal{R}h)^{(n-1)}$ means the $(n-1)$-th order partial derivative of $(\mathcal{R}h)(\theta, s)$ with respect to the variable s.

2.5 Spherical Harmonics Expansion

We use spherical harmonic expansions to transfer the isometries for $\mathcal{U}$ and $\mathcal{V}$ for $n = 2, 3$ to the general n case. The space $L^2(S^{n-1})$ of square integrable real valued functions on S^{n-1} is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{S^{n-1}} f(\theta)\, g(\theta)\, d\theta.$$

It has (see [32]) a **complete orthonormal basis** $\{\phi_k(\theta)\}_{k=0}^{\infty}$ where each $\phi_k(\theta)$ is the restriction to S^{n-1} of a homogeneous harmonic polynomial $\phi_k(x)$, $x \in \mathbb{R}^n$. Below d_k will denote the degree of $\phi_k(x)$ and we assume that the $\phi_k(\theta)$ have been arranged in a sequence so that $d_k \le d_{k+1}$ for all k. Of course it may happen that $d_k = d_{k+1}$ for some of the k. One can show (see [32]) that for each integer $d \ge 0$, the number of $\phi_k(x)$ with degree d is

$$\binom{n+d-1}{d} - \binom{n+d-3}{d-2}.$$

The Laplacian in $\mathbb{R}^n$ has the decomposition

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_s.$$

where $x = r\theta$ with $r \ge 0$ and $|\theta| = 1$, $\theta \in S^{n-1}$, and Δ_s is the Laplace-Beltrami operator on S^{n-1}. Spherical harmonics are perfect for problems involving the Laplacian because the $\phi_k(\theta)$ are eigenfunctions of Δ_s. Since $\phi_k(x)$ is a homogeneous harmonic polynomial, one can show (Theorem 1 in [32]) that if d_k is the degree of $\phi_k(x)$ then $\phi_k(\theta)$ is an eigenfunction of Δ_s with eigenvalue $-d_k(d_k + n - 2)$, that is

$$\Delta_s \phi_k(\theta) = -d_k(d_k + n - 2)\phi_k(\theta). \tag{2.9}$$

Given any $h \in C^\infty(\mathbb{R}^n)$, for any fixed $r \ge 0$, the function $h(r\theta)$, $\theta \in S^{n-1}$ is in $C^\infty(S^{n-1})$ so is in $L^2(S^{n-1})$. Hence $h(r\theta)$ has a spherical harmonic expansion

$$h(x) = h(r\theta) = \sum_{k=0}^{\infty} h_k(r) r^{d_k} \phi_k(\theta)$$

where $h_k(r) r^{d_k} = \int_{S^{n-1}} h(r\theta)\phi_k(\theta)\, d\theta$ is infinitely differentiable.

To understand the effect of Δ on each term in the spherical harmonic expansion of $h(x)$, we study $\Delta(h(r)r^d\phi(\theta))$, where $\phi(\theta)$ is a spherical harmonic, homogeneous of degree d, and $h(r)$ is arbitrary and smooth. We have

$$\begin{aligned}\Delta(h(r)r^d\phi(\theta)) &= (h(r)r^d)_{rr}\phi(\theta) + \frac{n-1}{r}(h(r)r^d)_r\phi(\theta) + \frac{h(r)r^d}{r^2}\Delta_s\phi_k(\theta) \\ &= \phi(\theta)\left((h(r)r^d)_{rr} + \frac{n-1}{r}(h(r)r^d)_r - \frac{d(d+n-2)}{r^2}h(r)r^d\right) \\ &= r^d\phi(\theta)\left(h_{rr}(r) + \frac{n+2d-1}{r}h_r(r)\right). \qquad (2.10)\end{aligned}$$

Chapter 3

GEOMETRIC INVERSION

Consider the geometric inversion map, consisting of inversion across the unit sphere, which maps

$$\begin{array}{ccc} \mathbb{R}^n \cup \{\infty\} & \longmapsto & \mathbb{R}^n \cup \{\infty\} \\ x & \longmapsto & X = x/|x|^2. \end{array}$$

Note that this map is its own inverse. In this chapter, we study the properties of this geometric inversion map.

The first result is that geometric inversion maps spheres to spheres or planes (which may be considered as spheres through ∞).

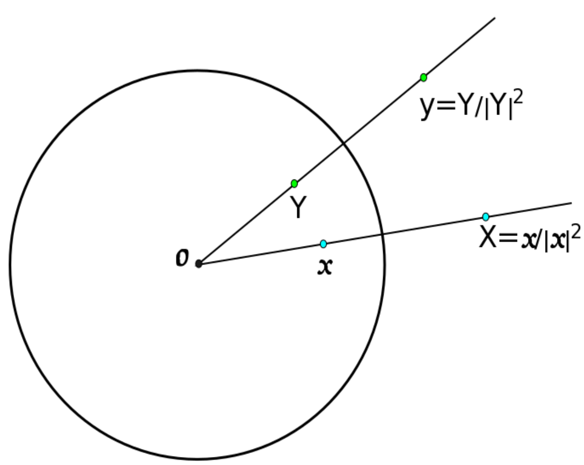

Figure 3.1: The geometric inversion mapping x to X.

Proposition 3.1. *The geometric inversion* $x \mapsto X = x/|x|^2$ *maps*

(a) the sphere $|x-c| = t,\ t \neq |c|$ *onto the sphere* $|X - C| = T$ *where*

$$C = \frac{c}{|c|^2 - t^2}, \quad T = \frac{t}{||c|^2 - t^2|},$$

(b) the sphere $|x - c| = |c|$ *onto the hyper-plane* $2X \cdot c = 1$.

Proof. Since $x = X/|X|^2$, the image of the sphere satisfies the relation

$$t^2 = |x-c|^2 = |x|^2 + |c|^2 - 2x \cdot c = \frac{1}{|X|^2} + |c|^2 - \frac{2X \cdot c}{|X|^2},$$

which simplifies to

$$(t^2 - |c|^2)|X|^2 + 2X \cdot c = 1.$$

which may be rewritten as (when $t \neq |c|$)

$$\left| X - \frac{c}{|c|^2 - t^2} \right|^2 = \frac{t^2}{(|c|^2 - t^2)^2}.$$

So the sphere $|x - c| = t,\ t \neq |c|$ is mapped to the sphere $|X - C| = T$.

When $t = |c|$, we have

$$2X \cdot c = 1.$$

So the sphere $|x - c| = |c|$ is mapped to the hyperplane $2X \cdot c = 1$. $\square$

The geometric inversion map will play an important role for us. The solution of the wave equation may be expressed in terms of spherical mean values and the trace on the surface of the light cone corresponds to the spherical mean values over spheres with centers c and radius $|c|$. So geometric inversion will map these integrals on a sphere to integrals on a hyper-plane. We can then appeal to results for the Radon transform to obtain results for our problem.

Now we compute the Jacobian of the geometric inversion. If we introduce spherical coordinates via $x = r_x\theta$ and $X = r_X\theta$ then $r_x = 1/r_X$ and $dr_x = (-1/r_X^2)\, dr_X$ implying

$$dx = r_x^{n-1}\, dr_x\, d\theta = \frac{1}{r_X^{n-1}} |-1/r_X^2|\, dr_X\, d\theta = 1/r_X^{2n}\, dX = 1/|X|^{2n}\, dX.$$

Hence, for an integrable infinitely differentiable function $h(x)$ on $\mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)\,dx &= \int_{S^{n-1}}\int_0^\infty r_x^{n-1} h(r_x\theta)\,dr_x\,d\theta = \int_{S^{n-1}}\int_\infty^0 1/r_X^{n-1} h(\theta/r_X)\,(1/r_X^2)\,dr_X\,d\theta \\ &= \int_{\mathbb{R}^n} h(X/|X|^2)/|X|^{2n}\,dX.\end{aligned}$$

Proposition 3.2. *If $h(x)$ is a smooth integrable function on $\mathbb{R}^n$, then we have*

$$\int_{\mathbb{R}^n} h(x)\,dx = \int_{\mathbb{R}^n} h(X/|X|^2)/|X|^{2n}\,dX.$$

The above identity suggests that under geometric inversion, functions should be mapped to weighted functions if we want an isometry. The weighted functions will often have the form $h(X/|X|^2)/|X|^k$ for some non-negative integer k. The following map and its properties will be useful in later chapters.

For an integer k consider the linear map

$$\mathscr{S}(\mathbb{R}^n) \ni h(x) \quad \longmapsto \quad H(X) = h(X/|X|^2)/|X|^k \in C^\infty(\mathbb{R}^n)$$

where we take $H(0) = 0$ - $H(X)$ is infinitely differentiable at 0 because of the behavior of $h(x)$ and its derivatives for large x. However this map is not surjective and we wish to restrict the domain to a subspace of $\mathscr{S}(\mathbb{R}^n)$ so that this map is bijection onto its domain. We show that the subspace $\mathscr{S}_0(\mathbb{R}^n)$, defined in Chapter 2, fulfills that role. Also, our main results pertain to the inversion of the restrictions of $\mathcal{U}, \mathcal{V}, \mathcal{W}$ to $\mathscr{S}_0(\mathbb{R}^n)$ where the following theorem will be useful.

Theorem 3.1. *If $h(x) \in \mathscr{S}_0(\mathbb{R}^n)$ and k an integer, then*

$$H(X) = h(X/|X|^2)/|X|^k, \qquad X \in \mathbb{R}^n$$

is in $\mathscr{S}_0(\mathbb{R}^n)$ and the map

$$\mathscr{S}_0(\mathbb{R}^n) \ni h(x) \longmapsto H(X) = h(X/|X|^2)/|X|^k \in \mathscr{S}_0(\mathbb{R}^n)$$

is a bijection.

Proof. For $h(x) \in \mathscr{S}_0(\mathbb{R}^n)$, it is not hard to see that for any integer k, $h(x)|x|^k \in \mathscr{S}_0(\mathbb{R}^n)$ as a function of x. Since $H(X) = h(X/|X|^2)/|X|^k = h(x)|x|^k$ with $x = X/|X|^2$, to prove the theorem, it is enough to prove just the case $k = 0$. From now on, we assume $k = 0$, $H(X) = h(X/|X|^2)$.

Firstly, we claim that $H(X) \in C(\mathbb{R}^n)$ as a function of X, and it approaches 0 faster than any polynomial when X goes to the origin or the infinity.

When X goes to the origin (x goes to the infinity), for any $\alpha \in \mathbb{Z}_+^n$, $H(X)/X^\alpha = h(x)|x|^{2|\alpha|}/x^\alpha$. Since $h(x) \in \mathscr{S}(\mathbb{R}^n)$, $|h(x)|x|^{2|\alpha|}/x^\alpha| \leq |h(x)|x|^{2|\alpha|}|$ goes to 0.

When X goes to the infinity (x goes to 0), for any $\alpha \in \mathbb{Z}_+^n$, $X^\alpha H(X) = h(x)x^\alpha/|x|^{2|\alpha|}$. Since, as $x \to 0$, $h(x)$ goes to 0 faster than any polynomial, we see that $|h(x)x^\alpha/|x|^{2|\alpha|}| \leq |h(x)/|x|^{2|\alpha|}|$ goes to 0 as x goes to zero.

Now we consider the derivatives of $H(X) = h(X/|X|^2)$. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let $X = (X_1, X_2, \ldots, X_n) = x/|x|^2$. Notice that for any $\alpha \in \mathbb{Z}_+^n$, we have $x^\alpha = X^\alpha/|X|^{2|\alpha|}$, and

$$\frac{\partial x_i}{\partial X_j} = \frac{\partial}{\partial X_j}\left(\frac{X_i}{|X|^2}\right) = \frac{\delta_{ij}}{|X|^2} - \frac{2X_iX_j}{|X|^4} = \delta_{ij}|x|^2 - 2x_ix_j.$$

Hence

$$\frac{\partial H(X)}{\partial X_j} = \sum_{i=1}^n \frac{\partial}{\partial x_i}(h(x))\frac{\partial x_i}{\partial X_j} = |x|^2\frac{\partial}{\partial x_j}h(x) - 2x_j\sum_{i=1}^n x_i\frac{\partial h(x)}{\partial x_i}.$$

If we define the differential operator L_j, $j = 1, 2, \ldots, n$, by

$$L_jh(x) = (|x|^2\partial_j - 2x_j\, x\cdot\nabla)h(x),$$

then we have showed

$$\frac{\partial H(X)}{\partial X_j} = L_jh(x)|_{x=X/|X|^2}.$$

Hence for every multi-index $\beta \in \mathbb{Z}_+^n$,

$$\partial_X^\beta H(X) = L^\beta h(x)|_{x=X/|X|^2}.$$

For every multi-index β, we can show that

$$L^\beta = \sum_{|\gamma|\leq|\beta|} p_{\beta,\gamma}(x)\partial_x^\gamma$$

for some homogeneous polynomial $p_{\beta,\gamma}(x)$ of degree $|\beta|+|\gamma|$. We can see that for every multi-index β, $\partial_X^\beta H(X)$ is a continuous function of X on $\mathbb{R}^n \setminus \{0\}$. Now we examine its behavior as $X \to 0$ and $X \to \infty$.

Consider the following when $X \to 0$ (that is when $x \to \infty$). For any $\alpha \in \mathbb{Z}_+^n$,

$$\partial_X^\beta H(X)/X^\alpha = L^\beta h(x)\ |x|^{2|\alpha|}/x^\alpha = \sum_{|\gamma|\leq|\beta|} p_{\beta,\gamma}(x)\partial_x^\gamma h(x)\ |x|^{2|\alpha|}/x^\alpha \to 0$$

because $||x|^{2|\alpha|}/x^\alpha| \leq |x|^{2|\alpha|}$ and $\partial_x^\gamma h(x)$ goes to 0 faster than any polynomial when x goes to the infinity. Next consider the following when $X \to \infty$ (that is $x \to 0$). For any $\alpha \in \mathbb{Z}_+^n$,

$$X^\alpha \partial_X^\beta H(X) = L^\beta h(x)\ x^\alpha/|x|^{2|\alpha|} = \sum_{|\gamma|\leq|\beta|} p_{\beta,\gamma}(x)\partial_x^\gamma h(x)\ x^\alpha/|x|^{2|\alpha|} \to 0$$

because $|x^\alpha/|x|^{2|\alpha|}| \leq 1/|x|^{2|\alpha|}$ and $\partial_x^\gamma h(x)$ goes to 0 faster than any polynomial when x goes to 0. Therefore $H(X) \in \mathscr{S}_0(\mathbb{R}^n)$. $\square$

Chapter 4

INVERSION OF $\mathcal{U}$, $\mathcal{V}$ AND $\mathcal{W}$ WHEN $n = 3$

If $\mathcal{U}, \mathcal{V}$ are the forward operators defined in (1.8), (1.9), respectively, then our goal is the inversion of $\mathcal{U}$ and $\mathcal{V}$. In this chapter, we accomplish our goal when $n = 3$ because it is the easiest case and it gives us some sense of how the general case will work. We study the isometry and the inversion of $\mathcal{U}, \mathcal{V}$ when $n = 3$ - the isometry and the L^2 adjoint lead to the inversion formula. In fact we give two inversion formulas. In a later chapter, we show how the isometry for the $n = 3$ case leads to an isometry for the odd n case when $n > 3$.

We first study the operator $\mathcal{V}$ and then we study the operator $\mathcal{U}$.

4.1 The inversion of $\mathcal{V}$

We show that $\mathcal{V}$ is an isometry when restricted to $\mathscr{S}_0(\mathbb{R}^n)$.

Theorem 4.1 (Isometry identity for $\mathcal{V}$ in $\mathbb{R}^3$). *For all $g \in \mathscr{S}_0(\mathbb{R}^3)$ we have*

$$\int_{\mathbb{R}^3} (|x|g(x))^2 \ dx = 8 \int_{\mathbb{R}^3} (\partial_r(|x|(\mathcal{V}g)(x)))^2 \ dx$$

where $r = |x|$.

Proof. By Theorem 2.1, the solution $v(x,t)$ of the IVP (1.6) (1.7) is given by

$$v(x,t) = t(\mathcal{M}g)(x,t), \quad t \geq 0.$$

For any $x \in \mathbb{R}^3,\ x \neq 0$, define $r = |x|, \theta = x/|x|, s = \frac{1}{2r}$. For $g(x) \in \mathscr{S}_0(\mathbb{R}^3)$, define

$$G(X) = g(X/|X|^2)/|X|^4,$$

where $X = x/|x|^2$. Since $g(x) \in \mathscr{S}_0(\mathbb{R}^3)$, by Theorem 3.1, $G(X) \in \mathscr{S}_0(\mathbb{R}^3)$. Using that $\delta(\cdot)$ is homogeneous of degree -1 and that

$$\int_{\phi(x)=0} \frac{f(x)}{|\nabla \phi(x)|}\, dS_x = \int_{\mathbb{R}^n} f(x)\, \delta(\phi(x))\, dx,$$

we have

$$\begin{aligned}
4\pi|x|^2(\mathcal{M}g)(x,|x|) &= \int_{|y-x|=|x|} g(y)\, dS_y = 2|x| \int_{\mathbb{R}^3} g(y)\delta(|y-x|^2 - |x|^2)\, dy \\
&= \frac{1}{s}\int_{\mathbb{R}^3} g(y)\delta(|y|^2 - \theta\cdot y/s)\, dy \qquad\qquad \text{let } Y = y/|y|^2 \\
&= \int_{\mathbb{R}^3} g(Y/|Y|^2)|Y|^2\delta(s-\theta\cdot Y)\, (1/|Y|^6) dY \\
&= (\mathcal{R}G)(\theta, s).
\end{aligned}$$

Now $(\mathcal{V}g)(x) = |x|(\mathcal{M}g)(x,|x|)$, so

$$\mathcal{R}G(\theta, s) = 4\pi|x|(\mathcal{V}g)(x),$$

equivalently,

$$\mathcal{R}G(\theta, \frac{1}{2r}) = 4\pi r(\mathcal{V}g)(r\theta), \quad \forall\theta \in S^2,\ r > 0.$$

When $r = 0$, $4\pi r(\mathcal{V}g)(r\theta) = 0$, $\mathcal{R}G(\theta, \frac{1}{2r})$ goes to 0 since $(\mathcal{R}G) \in \mathscr{S}(S^2 \times \mathbb{R})$.

When r goes to the infinity, $\mathcal{R}G(\theta, \frac{1}{2r})$ goes to a finite number $\mathcal{R}G(\theta, 0)$. Since $4\pi r(\mathcal{V}g)(r\theta)$ is a continuous function of r, $\lim_{r\to\infty} 4\pi r(\mathcal{V}g)(r\theta)$ exists and equals $\mathcal{R}G(\theta, 0)$. Therefore

$$\mathcal{R}G(\theta, \frac{1}{2r}) = 4r(\mathcal{V}g)(r\theta), \quad \forall\theta \in S^2,\ r \in [0, \infty]. \tag{4.1}$$

From this fact, we can see that when x goes to the infinity, $(\mathcal{V}g)(x)$ will go to 0, and $\lim_{x\to\infty} |x|(\mathcal{V}g)(|x|\theta)$ exists for a fixed $\theta \in S^2$, which does not guarantee that $\lim_{x\to\infty} |x|(\mathcal{V}g)(x)$ exists. Notice that

$$\begin{aligned}
\int_{S^2}\int_0^\infty |\partial_s(\mathcal{R}G)(\theta, s)|^2\, ds\, d\theta &= \int_{S^2}\int_0^\infty 2r^2 \left|\partial_r\left((\mathcal{R}G)(\theta, 1/(2r)) \right)\right|^2\, dr\, d\theta \\
&= 32\pi^2 \int_{S^2}\int_0^\infty r^2|\partial_r(r(\mathcal{V}g)(r\theta))|^2\, dr\, d\theta \\
&= 32\pi^2 \int_{\mathbb{R}^3} |\partial_r(|x|(\mathcal{V}g)(x))|^2\, dx.
\end{aligned}$$

In $\mathbb{R}^3$, by Theorem 2.2 the Radon transform has the isometry

$$\begin{aligned}\int_{\mathbb{R}^3} |G(X)|^2 \, dX &= \frac{1}{8\pi^2} \int_{S^2} \int_{-\infty}^{\infty} |\partial_s(\mathcal{R}G)(\theta,s)|^2 \, ds \, d\theta \\ &= \frac{1}{4\pi^2} \int_{S^2} \int_{0}^{\infty} |\partial_s(\mathcal{R}G)(\theta,s)|^2 \, ds \, d\theta.\end{aligned}$$

Since

$$\int_{\mathbb{R}^3} |G(X)|^2 \, dX = \int_{\mathbb{R}^3} \left(g(x)|x|^4\right)^2 (1/|x|^6) \, dx = \int_{\mathbb{R}^3} (|x|g(x))^2 \, dx,$$

therefore

$$\int_{\mathbb{R}^3} (|x|g(x))^2 \, dx = 8\pi^2 \int_{\mathbb{R}^3} |\partial_r(|x|v(x,|x|))|^2 \, dx.$$

□

Since $\mathcal{V}$ is an isometry, one can compute the inverse of $\mathcal{V}$ if one could determine the adjoint $\mathcal{V}'$ of $\mathcal{V}$ in the associated inner-product corresponding to a weighted L^2 space. Specifically, we have

$$\langle g, g\rangle = \langle \mathcal{V}g, \mathcal{V}g\rangle = \langle \mathcal{V}'\mathcal{V}g, g\rangle, \qquad \forall g \in \mathscr{S}_0(\mathbb{R}^n)$$

and hence

$$g = \mathcal{V}'\mathcal{V}g$$

where $\mathcal{V}'$ represents the adjoint of $\mathcal{V}$ in the appropriate inner product. Towards computing $\mathcal{V}'$, we compute the L^2 adjoint of $\mathcal{V}$.

Proposition 4.1 (**L^2 adjoint of $\mathcal{V}$ in $\mathbb{R}^3$**). *When $n = 3$, the L^2 adjoint of $\mathcal{V}$ is given by*

$$(\mathcal{V}^*\phi)(x) = \frac{1}{4\pi|x|}(\mathcal{R}\phi)(x/|x|, |x|/2), \quad \forall \phi \in \mathscr{S}(\mathbb{R}^3), \; x \in \mathbb{R}^3.$$

Proof. Let $g(x) \in \mathscr{S}(\mathbb{R}^3)$, $v(x,t)$ be the solution of the IVP (1.6) (1.7) when $n = 3$. Then

$$\begin{aligned}(\mathcal{V}g)(x) = v(x,|x|) = |x|(\mathcal{M}g)(x,|x|) &= |x|\frac{1}{4\pi|x|^2} \int_{|y-x|=|x|} g(y) \, dS_y \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^3} g(y)\delta(2x\cdot y - |y|^2) \, dy.\end{aligned}$$

So for any $\phi \in \mathscr{S}(\mathbb{R}^3)$, using the L^2 inner product, we have

$$
\begin{aligned}
\langle (\mathcal{V}g)(x), \phi(x) \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}^3} \phi(x) \int_{\mathbb{R}^3} g(y) \delta(2x \cdot y - |y|^2) \, dy \, dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^3} g(y) \int_{\mathbb{R}^3} \phi(x) \delta(2x \cdot y - |y|^2) \, dx \, dy.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\mathcal{V}^* \phi)(y) &= \frac{1}{2\pi} \int_{\mathbb{R}^3} \phi(x) \delta(2x \cdot y - |y|^2) \, dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^3} \phi(x) \delta \left(2|y| (x \cdot y/|y| - |y|/2) \right) \, dx \\
&= \frac{1}{4\pi |y|} (\mathcal{R}\phi)(y/|y|, |y|/2).
\end{aligned}
$$

□

Having obtained the L^2 adjoint of $\mathcal{V}$ we now construct the inverse of $\mathcal{V}$. We give two inversion formulas for $\mathcal{V}$.

Theorem 4.2 (First inversion formula for $\mathcal{V}$ in $\mathbb{R}^3$). *For any $g \in \mathscr{S}_0(\mathbb{R}^3)$ we have*

$$
g(x) = -\frac{8}{|x|^2} \mathcal{V}^* \left(\partial_r^2 (|x|^2 (\mathcal{V}g)(x)) \right), \qquad x \in \mathbb{R}^3
$$

where $r = |x|$ and $\mathcal{V}^*$ is the adjoint operator defined in Proposition 4.1.

Proof. Notice that for any smooth function $h(x)$ on $\mathbb{R}^n$,

$$
\partial_r h(x) = \nabla h \cdot \frac{x}{r} = \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i} \frac{x_i}{r},
$$

and for any $h_1(x), h_2(x) \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that ∇h_1, $\nabla h_2 \in L^2(\mathbb{R}^n)$ and

$$
\lim_{r \to \infty} \int_{S^{n-1}} r^{n-1} h_1(r\theta) h_2(r\theta) \, d\theta = 0
$$

we have

$$
\begin{aligned}
\langle \partial_r h_1(x), h_2(x)\rangle &= \int_{S^{n-1}} \int_0^\infty r^{n-1} \partial_r h_1(r\theta) h_2(r\theta)\ dr\ d\theta \\
&= \lim_{r\to\infty} \int_{S^{n-1}} r^{n-1} h_1(r\theta) h_2(r\theta)\ d\theta - \int_{S^{n-1}} \int_0^\infty h_1(r\theta)\partial_r(r^{n-1} h_2(r\theta))\ dr\ d\theta \\
&= -\int_{S^{n-1}} \int_0^\infty h_1(r\theta)\partial_r(r^{n-1} h_2(r\theta))\ dr\ d\theta \\
&= -\int_{S^{n-1}} \int_0^\infty r^{n-1} h_1(r\theta)(\partial_r + \frac{n-1}{r}) h_2(r\theta)\ dr\ d\theta \\
&= -\left\langle h_1(x), (\partial_r + \frac{n-1}{r}) h_2(x) \right\rangle .
\end{aligned}
$$

In $\mathbb{R}^3$, by Theorem 4.1 and the linearity of $\mathcal{V}$, we have

$$
\langle |x| g(x), |x|\phi(x)\rangle = 8\langle \partial_r(|x|(\mathcal{V}g)(x)), \partial_r(|x|(\mathcal{V}\phi)(x))\rangle, \quad \forall \phi \in \mathscr{S}_0(\mathbb{R}^3).
$$

Note that $|x|(\mathcal{V}g)(x), |x|(\mathcal{V}\phi)(x) \in C^\infty(\mathbb{R}^n)$, and from (4.1) in the proof of Theorem 4.1,

$$
\begin{aligned}
|x|(\mathcal{V}g)(x) &= \frac{1}{4\pi}(\mathcal{R}G)(x/|x|, 1/(2|x|)), \\
|x|(\mathcal{V}\phi)(x) &= \frac{1}{4\pi}(\mathcal{R}\Phi)(x/|x|, 1/(2|x|)),
\end{aligned}
$$

where $G(x) = g(x/|x|^2)/|x|^4$, $\Phi(x) = \phi(x/|x|)/|x|^4$. Hence

$$
\partial_r(|x|(\mathcal{V}g)(x)) = -\frac{1}{8\pi r^2}(\partial_s(\mathcal{R}G))(x/|x|, 1/(2|x|)).
$$

By Theorem 3.1, $G \in \mathscr{S}_0(\mathbb{R}^n)$, hence $\mathcal{R}G \in \mathscr{S}(S^2\times\mathbb{R})$. Therefore

$$
\lim_{r\to\infty} r^2\left(|x|(\mathcal{V}\phi)(r\theta)\ \partial_r(|x|(\mathcal{V}g)(r\theta))\right) = -\frac{1}{32\pi^2}(\mathcal{R}\Phi)(\theta,0)\partial_s(\mathcal{R}G)(\theta,0).
$$

Notice that

$$
(\mathcal{R}\Phi)(\theta,0) = (\mathcal{R}\Phi)(-\theta,0)
$$

and

$$
\partial_s(\mathcal{R}G)(\theta,0) = -\partial_s(\mathcal{R}G)(-\theta,0),
$$

so

$$
\lim_{r\to\infty}\int_{S^2} r^2\left(|x|(\mathcal{V}\phi)(r\theta)\ \partial_r(|x|(\mathcal{V}g)(r\theta))\right)\ d\theta = 0,
$$

and

$$\begin{aligned}\langle \partial_r(|x|(\mathcal{V}g)(x)), \partial_r(|x|(\mathcal{V}\phi)(x))\rangle &= -\left\langle (\partial_r + \frac{2}{r})\partial_r(|x|(\mathcal{V}g)(x)), |x|(\mathcal{V}\phi)(x)\right\rangle \\ &= -\left\langle |x|(\partial_r^2 + \frac{2}{r}\partial_r)(|x|(\mathcal{V}g)(x)), (\mathcal{V}\phi)(x)\right\rangle \\ &= -\left\langle \mathcal{V}^*\left(|x|(\partial_r^2 + \frac{2}{r}\partial_r)(|x|(\mathcal{V}g)(x))\right), \phi(x)\right\rangle \\ &= -\langle \mathcal{V}^*\left(\partial_r^2(|x|^2(\mathcal{V}g)(x))\right), \phi(x)\rangle.\end{aligned}$$

Since

$$\langle |x|g(x), |x|\phi(x)\rangle = \langle |x|^2 g(x), \phi(x)\rangle = -8\langle \mathcal{V}^*\left(\partial_r^2(|x|^2(\mathcal{V}g)(x))\right), \phi(x)\rangle,$$

we have

$$g(x) = -\frac{8}{|x|^2}\mathcal{V}^*\left(\partial_r^2(|x|^2(\mathcal{V}g)(x))\right).$$

□

Note that in $\mathbb{R}^3$, from (4.1) in the proof of Theorem 4.1, we have

$$(\mathcal{R}G)(\theta, s) = \frac{2\pi}{s}(\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall \theta \in S^2, s \geq 0,$$

where $G(X) = g(x)|x|^4,\ X = x/|x|^2$. Since the inverse formula for the Radon transform is known in $\mathbb{R}^3$, this identity gives a more direct way to find the inverse of $\mathcal{V}$ without using the adjoint operator.

Theorem 4.3 (Second inversion formula for $\mathcal{V}$ in $\mathbb{R}^3$). *For any $g(x) \in \mathscr{S}_0(\mathbb{R}^3)$ we have*

$$g(x) = -\frac{|X|^4}{4\pi}\Delta_X \int_{S^2} \frac{(\mathcal{V}g)(\frac{\theta}{X\cdot\theta})}{|X\cdot\theta|}\, d\theta, \qquad x \in \mathbb{R}^3$$

where $X = x/|x|^2$.

Proof. For any $\theta \in S^2, s > 0$, define $x = \frac{\theta}{2s}$. For $g(x)$, define $G(X) = g(x)|x|^4$. Then

$$(\mathcal{R}G)(\theta, s) = 4\pi|x|(\mathcal{V}g)(x) = \frac{2\pi}{s}(\mathcal{V}g)(\frac{\theta}{2s}).$$

Fix $\theta \in S^2$. For any $X \in \mathbb{R}^3$, if $X\cdot\theta > 0$, then $(\mathcal{R}G)(\theta, X\cdot\theta) = \frac{2\pi}{X\cdot\theta}(\mathcal{V}g)(\frac{\theta}{2X\cdot\theta})$.

If $X \cdot \theta < 0$, noting that $\mathcal{R}G$ is even, we have

$$(\mathcal{R}G)(\theta, X \cdot \theta) = (\mathcal{R}G)(-\theta, -X \cdot \theta) = \frac{2\pi}{-X \cdot \theta}(\mathcal{V}g)(\frac{\theta}{2X \cdot \theta}).$$

Therefore, for any $X \in \mathbb{R}^3$, we have

$$(\mathcal{R}G)(\theta, X \cdot \theta) = \frac{2\pi}{|X \cdot \theta|}(\mathcal{V}g)(\frac{\theta}{2X \cdot \theta}).$$

By the inversion formula for the Radon transform in $\mathbb{R}^3$,

$$\begin{aligned} G(X) &= -\frac{1}{8\pi^2}\Delta_X \int_{S^2} (\mathcal{R}G)(\theta, X \cdot \theta)\, d\theta \\ &= -\frac{1}{8\pi^2}\Delta_X \int_{S^2} \frac{2\pi}{|X \cdot \theta|}(\mathcal{V}g)(\frac{\theta}{2X \cdot \theta})\, d\theta \\ &= -\frac{1}{4\pi}\Delta_X \int_{S^2} \frac{(\mathcal{V}g)(\frac{\theta}{2X \cdot \theta})}{|X \cdot \theta|}\, d\theta. \end{aligned}$$

Therefore

$$g(x) = G(X)|X|^4 = -\frac{|X|^4}{4\pi}\Delta_X \int_{S^2} \frac{(\mathcal{V}g)(\frac{\theta}{2X \cdot \theta})}{|X \cdot \theta|}\, d\theta,$$

where $X = x/|x|^2$. $\square$

4.2 The inversion of $\mathcal{U}$

We now study the inversion of $\mathcal{U}$ and we start by proving that $\mathcal{U}$ is an isometry.

Theorem 4.4 (Isometry identity for $\mathcal{U}$ in $\mathbb{R}^3$). *We have*

$$\int_{\mathbb{R}^3} \left(\frac{f(x)}{|x|}\right)^2 dx = 2\int_{\mathbb{R}^3} \left(\frac{(\mathcal{U}f)(x)}{|x|}\right)^2 dx, \qquad \forall f \in \mathscr{S}_0(\mathbb{R}^3). \tag{4.2}$$

Proof. For $f(x) \in \mathscr{S}_0(\mathbb{R}^3)$, by Theorem 2.1, the solution $u(x,t)$ of the IVP (1.4) (1.5) is given by

$$u(x,t) = \partial_t \left(t(\mathcal{M}f)(x,|t|)\right), \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Define

$$F(X) = f(X/|X|^2)/|X|^2, \qquad X \in \mathbb{R}^3;$$

by Theorem 3.1, $F(X) \in \mathscr{S}_0(\mathbb{R}^3)$. Since

$$(\mathcal{M}f)(x,|t|) = \frac{1}{4\pi|t|^2}\int_{|y-x|=|t|} f(y)\, dS_y = \frac{1}{2\pi|t|}\int_{\mathbb{R}^3} f(y)\delta(|y-x|^2 - t^2)\, dy, \quad \forall t \neq 0,$$

when $t > 0$, we have

$$u(x,t) = \partial_t\left(t(\mathcal{M}f)(x,t)\right) = \frac{1}{2\pi}\partial_t\left(\int_{\mathbb{R}^3} f(y)\delta(|y-x|^2 - t^2)\,dy\right)$$

so

$$\begin{aligned}
u(x,t) &= \frac{2t}{2\pi}\partial_{t^2}\left(\int_{\mathbb{R}^3} f(y)\delta(t^2 - |y-x|^2)\,dy\right)\\
&= \frac{t}{\pi}\int_{\mathbb{R}^3} f(y)\delta'(t^2 - |y-x|^2)\,dy.
\end{aligned}$$

For any $x \in \mathbb{R}^3$, $x \neq 0$, let $\theta = x/|x|$, $s = \frac{1}{2|x|}$; then

$$\begin{aligned}
u(x,|x|) &= \frac{|x|}{\pi}\int_{\mathbb{R}^3} f(y)\delta'(|x|^2 - |y-x|^2)\,dy\\
&= \frac{|x|}{\pi}\int_{\mathbb{R}^3} f(y)\delta'(\theta\cdot y/s - |y|^2)\,dy\\
&= \frac{|x|}{\pi}\int_{\mathbb{R}^3} f(y)\delta'\left(\frac{|y|^2}{s}(\theta\cdot y/|y^2| - s)\right)\,dy\\
&= \frac{1}{4\pi|x|}\int_{\mathbb{R}^3} f(y)/|y|^4\delta'(\theta\cdot y/|y^2| - s)\,dy\\
&= \frac{1}{4\pi|x|}\int_{\mathbb{R}^3} f(Y/|Y|^2)|Y|^4\delta'(\theta\cdot Y - s)\,(1/|Y|^6)dY\\
&= \frac{1}{4\pi|x|}\int_{\mathbb{R}^3} F(Y)\delta'(\theta\cdot Y - s)\,dY\\
&= \frac{-1}{4\pi|x|}\partial_s\left(\int_{\mathbb{R}^3} F(Y)\delta(\theta\cdot Y - s)\,dY\right)\\
&= \frac{-1}{4\pi|x|}\partial_s(\mathcal{R}F)(\theta,s) = \frac{-s}{2\pi}\partial_s(\mathcal{R}F)(\theta,s),
\end{aligned}$$

that is,

$$\partial_s(\mathcal{R}F)(\theta,s) = -\frac{2\pi}{s}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall\theta\in S^2, s>0,$$

or equivalently,

$$-2r^2\partial_r((\mathcal{R}F)(\theta,1/(2r))) = (\mathcal{R}F)^{(1)}(\theta,1/(2r)) = -4\pi r(\mathcal{U}f)(r\theta), \quad \forall\theta\in S^2, r>0,$$

where $(\mathcal{R}F)^{(1)}$ means the first order derivative with respect to the second variable.

For fixed $\theta \in S^2$, when r goes to the infinity, $(\mathcal{R}F)^{(1)}(\theta,1/(2r))$ will go to a finite constant $(\mathcal{R}F)^{(1)}(\theta,0)$. Since $-4\pi r(\mathcal{U}f)(r\theta)$ is in $C^\infty(\mathbb{R})$ as a function of r,

$$\lim_{r\to\infty} -4\pi r(\mathcal{U}f)(r\theta) = (\mathcal{R}F)^{(1)}(\theta,0).$$

From this fact, we also know that $\lim_{x\to\infty}(\mathcal{U}f)(x) = 0$ and $\lim_{x\to\infty}|x|(\mathcal{U}f)(|x|\theta)$ exists, which does not guarantee that $\lim_{x\to\infty}|x|(\mathcal{U}f)(x)$ exists.

When r goes to 0, since both $r(\mathcal{U}f)(r\theta)$ and $(\mathcal{R}F)^{(1)}(\theta, 1/(2r))$ will go to 0, we have

$$(\mathcal{R}F)^{(1)}(\theta, 1/(2r)) = -4\pi r(\mathcal{U}f)(r\theta), \quad \forall\theta \in S^2, r \in [0, \infty]. \tag{4.3}$$

So we have

$$\begin{aligned}\int_{S^2}\int_0^\infty |\partial_s(\mathcal{R}F)(\theta, s)|^2\ ds\ d\theta &= \int_{S^2}\int_0^\infty \left(2r^2\partial_r(\mathcal{R}F)(\theta, 1/(2r))\right)^2\ (\frac{1}{2r^2})dr\ d\theta \\ &= 8\pi^2\int_{S^2}\int_0^\infty |(\mathcal{U}f)(x)|^2\ \ dr\ d\theta \\ &= 8\pi^2\int_{\mathbb{R}^3}\left(\frac{(\mathcal{U}f)(x)}{|x|}\right)^2\ dx.\end{aligned}$$

Now, in $\mathbb{R}^3$, for the Radon transform, we have the isometry identity (see Theorem 2.2)

$$\begin{aligned}\int_{\mathbb{R}^3}|F(X)|^2\ dX &= \frac{1}{8\pi^2}\int_{S^2}\int_{\mathbb{R}}|\partial_s(\mathcal{R}F)(\theta, s)|^2\ ds\ d\theta \\ &= \frac{1}{4\pi^2}\int_{S^2}\int_0^\infty |\partial_s(\mathcal{R}F)(\theta, s)|^2\ ds\ d\theta,\end{aligned}$$

so

$$\begin{aligned}2\int_{\mathcal{R}^3}\left(\frac{(\mathcal{U}f)(x)}{|x|}\right)^2\ dx = \int_{\mathbb{R}^3}|F(X)|^2\ dX &= \int_{\mathbb{R}^3}\left(f(x)|x|^2\right)^2\ 1/|x|^6\ dx \\ &= \int_{\mathbb{R}^3}\left(\frac{f(x)}{|x|}\right)^2\ dx.\end{aligned}$$

□

Proposition 4.2 (The L^2 adjoint of $\mathcal{U}$ in $\mathbb{R}^3$). *The L^2 adjoint of $\mathcal{U}$ is*

$$(\mathcal{U}^*\phi)(x) = \frac{-1}{2\pi|x|^2}\partial_r\left((\mathcal{R}\phi_*)(x/|x|, |x|/2)\right), \quad \forall\phi \in \mathscr{S}_0(\mathbb{R}^3),$$

where $\phi_*(y) = \phi(y)|y|$ and $r = |x|$.

Proof. For any $f(x) \in \mathscr{S}_0(\mathbb{R}^3)$, by (4.3) in the proof of Theorem 4.4, we have

$$(\mathcal{U}f)(x) = \frac{|x|}{\pi}\int_{\mathbb{R}^3} f(y)\delta'(2x\cdot y - |y|^2)\ dy,$$

and

$$\begin{aligned}\langle (\mathcal{U}f)(x), \phi(x)\rangle &= \int_{\mathbb{R}^3} \phi(x)\frac{|x|}{\pi}\int_{\mathbb{R}^3} f(y)\delta'(2x\cdot y - |y|^2)\,dy\,dx \\ &= \int_{\mathbb{R}^3} f(y)\int_{\mathbb{R}^3}\frac{|x|}{\pi}\phi(x)\delta'(2x\cdot y-|y|^2)\,dx\,dy \qquad \text{interchange } x, y \\ &= \int_{\mathbb{R}^3} f(x)\int_{\mathbb{R}^3}\frac{|y|}{\pi}\phi(y)\delta'(2x\cdot y-|x|^2)\,dy\,dx.\end{aligned}$$

So

$$\begin{aligned}(\mathcal{U}^*\phi)(x) &= \int_{\mathbb{R}^3}\frac{|y|}{\pi}\phi(y)\delta'(2x\cdot y-|x|^2)\,dy \\ &= \int_{\mathbb{R}^3}\frac{|y|}{\pi}\phi(y)\delta'(2r\theta\cdot y - r^2)\,dy \\ &= \frac{1}{4\pi r^2}\int_{\mathbb{R}^3}|y|\phi(y)\delta'(\theta\cdot y - r/2)\,dy \\ &= \frac{-2}{4\pi r^2}\partial_r\left(\int_{\mathbb{R}^3}|y|\phi(y)\delta(\theta\cdot y-r/2)\,dy\right) \\ &= \frac{-1}{2\pi r^2}\partial_r\big(\mathcal{R}\phi_*(\theta, r/2)\big),\end{aligned}$$

where $r = |x|, \theta = x/|x|$. □

Now we use the L^2 adjoint of $\mathcal{U}$ and the isometry to construct the inverse of $\mathcal{U}$.

Theorem 4.5 (First inversion formula for $\mathcal{U}$ in $\mathbb{R}^3$). *If $f(x) \in \mathscr{S}_0(\mathbb{R}^3)$ then*

$$f(x) = 2|x|^2\mathcal{U}^*\left(\frac{(\mathcal{U}f)(x)}{|x|^2}\right),$$

where $\mathcal{U}^*$ is the L^2 adjoint of $\mathcal{U}$ defined in Theorem 4.2.

Proof. By Theorem 4.4 and the linearity of $\mathcal{U}$, for any $\phi(x) \in \mathscr{S}_0(\mathbb{R}^3)$, we have

$$\left\langle \frac{f(x)}{|x|}, \frac{\phi(x)}{|x|}\right\rangle = 2\left\langle \frac{(\mathcal{U}f)(x)}{|x|}, \frac{(\mathcal{U}\phi)(x)}{|x|}\right\rangle.$$

Since

$$\left\langle \frac{(\mathcal{U}f)(x)}{|x|}, \frac{(\mathcal{U}\phi)(x)}{|x|}\right\rangle = \left\langle \frac{(\mathcal{U}f)(x)}{|x|^2}, (\mathcal{U}\phi)(x)\right\rangle = \left\langle \mathcal{U}^*\left(\frac{(\mathcal{U}f)(x)}{|x|^2}\right), \phi(x)\right\rangle,$$

and

$$\left\langle \frac{f(x)}{|x|}, \frac{\phi(x)}{|x|} \right\rangle = \left\langle \frac{f(x)}{|x|^2}, \phi(x) \right\rangle,$$

we have

$$f(x) = 2|x|^2 \mathcal{U}^* \left(\frac{(\mathcal{U}f)(x)}{|x|^2} \right).$$

$\square$

Note that in $\mathbb{R}^3$, from (4.3) in the proof of Theorem 4.4, we have

$$\partial_s(\mathcal{R}F)(\theta, s) = -\frac{2\pi}{s}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^2, s \geq 0,$$

where $F(x) = f(x/|x|^2)/|x|^2$. From this relation, we can recover F and f from $\mathcal{U}f$.

Theorem 4.6 (**Second inversion formula for $\mathcal{U}$ in $\mathbb{R}^3$**)**.** *If $f(x) \in \mathscr{S}_0(\mathbb{R}^3)$, then*

$$f(x) = \frac{|X|^2}{4\pi} \int_{S^2} \partial_s \left(\frac{1}{s}(\mathcal{U}f)(\frac{\theta}{2s}) \right) \Big|_{s = X\cdot\theta} d\theta,$$

where $X = x/|x|^2$.

Proof. Define

$$F(x) = f(x/|x|^2)/|x|^2.$$

By Theorem 3.1, $F \in \mathscr{S}_0(\mathbb{R}^3)$. We have

$$\partial_s(\mathcal{R}F)(\theta, s) = -\frac{2\pi}{s}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^2, s \geq 0.$$

When $s < 0$, plug $(-\theta, -s)$ into the above identity, we have

$$\partial_s(\mathcal{R}F)(\theta, s) = -\frac{2\pi}{s}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^2, s < 0.$$

Therefore,

$$\partial_s(\mathcal{R}F)(\theta, s) = -\frac{2\pi}{s}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^2, s \in \mathbb{R}.$$

So

$$\partial_s^2(\mathcal{R}F)(\theta, s) = -2\pi\partial_s \left(\frac{1}{s}(\mathcal{U}f)(\frac{\theta}{2s}) \right), \quad \forall \theta \in S^2, s \in \mathbb{R}.$$

In $\mathbb{R}^3$, by Theorem 2.4, the Radon transform $\mathcal{R}F$ has the inversion formula

$$F(X) = -\frac{1}{8\pi^2}\int_{S^2} \partial_s^2(\mathcal{R}F)(\theta, s)|_{s=X\cdot\theta}\, d\theta,$$

so

$$F(X) = \frac{1}{4\pi}\int_{S^2} \partial_s \left(\frac{1}{s}(\mathcal{U}f)(\frac{\theta}{2s})\right)\Big|_{s=X\cdot\theta}\, d\theta,$$

and

$$f(x) = \frac{|X|^2}{4\pi}\int_{S^2} \partial_s \left(\frac{1}{s}(\mathcal{U}f)(\frac{\theta}{2s})\right)\Big|_{s=X\cdot\theta}\, d\theta,$$

where $X = x/|x|^2$. □

4.3 The isometry and inversion of $\mathcal{W}$

Having proved the isometry for $\mathcal{U}$ and $\mathcal{V}$ and constructed their inverses we can write an isometry for $\mathcal{W}$ and construct its inverse.

Theorem 4.7 (Isometry for $\mathcal{W}$). *If $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^3)$ and $w(x,t)$ is the solution of the IVP (1.1) (1.2), then*

$$\begin{aligned}\int_{\mathbb{R}^3} (|x|g(x))^2 + \left(\frac{f(x)}{|x|}\right)^2\, dx \\ = \int_{\mathbb{R}^3} 2\left(\partial_r\big(|x|(w(x,|x|) - w(x,-|x|))\big)\right)^2 + \frac{1}{2}\left(\frac{w(x,|x|)+w(x,-|x|)}{|x|}\right)^2\, dx.\end{aligned}$$

Theorem 4.8 (First inversion formula for $\mathcal{W}$). *If $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^3)$ and $w(x,t)$ is the solution of the IVP (1.1) (1.2), then*

$$\begin{aligned} f(x) &= |x|^2\mathcal{U}^*\left(\frac{w(x,|x| + w(x,-|x|)}{|x|^2}\right), \qquad x\in\mathbb{R}^3 \\ g(x) &= -\frac{4}{|x|^2}\mathcal{V}^*\left(\partial_r^2\big(|x|^2(w(x,|x|) - w(x,-|x|))\big)\right), \qquad x\in\mathbb{R}^3,\end{aligned}$$

where $\mathcal{U}^*, \mathcal{V}^*$ are the operators defined in Proposition (4.2) and Proposition (4.1) respectively.

Theorem 4.9 (Second inversion formula for $\mathcal{W}$). *If $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^3)$ and $w(x,t)$ is the solution of the IVP (1.1) (1.2), then*

$$f(x) = \frac{|X|^2}{8\pi} \int_{S^2} \partial_s \left(\frac{w(\frac{\theta}{2s}, \frac{\theta}{2|s|}) + w(\frac{\theta}{2s}, -\frac{\theta}{2|s|})}{s} \right) \Big|_{s=X\cdot\theta} d\theta,$$

$$g(x) = -\frac{|X|^4}{8\pi} \Delta_X \int_{S^2} \frac{w(\frac{\theta}{2X\cdot\theta}, \frac{\theta}{2|X\cdot\theta|}) - w(\frac{\theta}{2X\cdot\theta}, -\frac{\theta}{2|X\cdot\theta|})}{|X\cdot\theta|} d\theta,$$

where $X = x/|x|^2$.

Proof. Define

$$u(x,t) = \frac{w(x,t) + w(x,-t)}{2},$$
$$v(x,t) = \frac{w(x,t) - w(x,-t)}{2}.$$

Then $u(x,t), v(x,t)$ are the unique solutions of the IVP (1.4) (1.5) and the IVP (1.6) (1.7) respectively, and the proof follows immediately follows from Theorem 4.2, Theorem 4.3, Theorem 4.4 and Theorem 4.5. $\square$

Chapter 5

INVERSION OF $\mathcal{U}$, $\mathcal{V}$ AND $\mathcal{W}$ WHEN $n \geq 3$ IS ODD

In Chapter 4, we studied the inversion of $\mathcal{U}, \mathcal{V}$ in $\mathbb{R}^3$. In this chapter, we find their inverses for $\mathbb{R}^n$, n odd with $n > 3$. As before we prove $\mathcal{U}, \mathcal{V}$ are isometries and then use it to construct their inverses. We give two proofs of the isometry, one direct using the Radon transform and the other which uses the isometry obtained for $\mathbb{R}^3$. We first study the operator $\mathcal{V}$ and then the operator $\mathcal{U}$.

5.1 The inversion of $\mathcal{V}$

We first show that $\mathcal{V}$ is an isometry.

Theorem 5.1 (Isometry identity for $\mathcal{V}$). *For any $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$, n odd, we have*

$$\int_{\mathbb{R}^n} (|x| g(x))^2 \, dx = 8 \int_{\mathbb{R}^n} \frac{|\partial_r (|x|^{\frac{n-1}{2}} (\mathcal{V}g)(x))|^2}{|x|^{n-3}} \, dx$$

where $r = |x|$.

We prove the isometry two different ways - the first using the Radon transform and the second using spherical harmonics and the isometry proved for $\mathbb{R}^3$.

Proof. (Using the Radon transform) Since n is odd we have $n = 2m + 1$ for some integer $m \geq 1$. If $v(x,t)$ is the solution of the IVP (1.6) (1.7) then, from Theorem 2.1, $v(x,t)$ is given by

$$v(x,t) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n}{2})} (\frac{\partial}{\partial t^2})^{(n-3)/2} \left(t^{n-2} (\mathcal{M}g)(x,|t|) \right), \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

When $t > 0$,

$$(\mathcal{M}g)(x,t) = \frac{2}{\omega_{n-1} t^{n-2}} \int_{\mathbb{R}^n} g(y) \delta(|y-x|^2 - t^2) \, dy.$$

So we have

$$\begin{aligned}
v(x,t) &= \frac{\sqrt{\pi}}{2\Gamma(\frac{n}{2})}(\frac{\partial}{\partial t^2})^{(n-3)/2}\left(t^{n-2}(\mathcal{M}g)(x,t)\right)\\
&= \frac{\sqrt{\pi}}{\omega_{n-1}\Gamma(\frac{n}{2})}(\frac{\partial}{\partial t^2})^{m-1}\left(\int_{\mathbb{R}^n} g(y)\delta(|y-x|^2-t^2)\,dy\right)\\
&= \frac{(-1)^{m-1}\sqrt{\pi}}{\omega_{n-1}\Gamma(\frac{n}{2})}\int_{\mathbb{R}^n} g(y)\delta^{m-1}(|y-x|^2-t^2)\,dy.
\end{aligned}$$

Since

$$\omega_{n-1} = |S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

we have

$$\begin{aligned}
(\mathcal{V}g)(x) = v(x,|x|) &= \frac{(-1)^{m-1}}{2\pi^{n/2-1}}\int_{\mathbb{R}^n} g(y)\delta^{m-1}(|y-x|^2-|x|^2)\,dy\\
&= \frac{(-1)^{m-1}}{2\pi^{n/2-1}}\int_{\mathbb{R}^n} g(y)\delta^{m-1}(|y|^2-2x\cdot y)\,dy.
\end{aligned}$$

For any $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$, define

$$G(X) = g(X/|X|^2)/|X|^{n+1};$$

then by Theorem 3.1, $G(X) \in \mathscr{S}_0(\mathbb{R}^n)$. For any $x \neq 0 \in \mathbb{R}^n$, let $\theta = x/|x|,\ s = \frac{1}{2r}$. Noting that $\delta^{m-1}(\cdot)$ is a homogeneous function of degree $-m$,

$$\begin{aligned}
&(-2)^{m+1}\pi^{n/2-1}|x|^m(\mathcal{V}g)(x)\\
&\quad= \int_{\mathbb{R}^n} g(y)|2x|^m\delta^{m-1}(|y|^2-2x\cdot y)\,dy\\
&\quad= \int_{\mathbb{R}^n} g(y)\delta^{m-1}(\frac{|y|^2}{2|x|}-\frac{x}{|x|}\cdot y)\,dy\\
&\quad= \int_{\mathbb{R}^n} g(y)\delta^{m-1}(s|y|^2-\theta\cdot y)\,dy\\
&\quad= \int_{\mathbb{R}^n} \frac{g(y)}{|y|^{2m}}\delta^{m-1}(s-\theta\cdot\frac{y}{|y|^2})\,dy \qquad \text{let } Y = y/|y|^2\\
&\quad= \int_{\mathbb{R}^n} g(Y/|Y|^2)|Y|^{2m}\delta^{m-1}(s-\theta\cdot Y)\,(1/|Y|^{2n})dY\\
&\quad= \int_{\mathbb{R}^n} G(Y)\delta^{m-1}(s-\theta\cdot Y)\,dY\\
&\quad= \partial_s^{m-1}\int_{\mathbb{R}^n} G(Y)\delta(s-\theta\cdot Y)\,dY\\
&\quad= \partial_s^{m-1}(\mathcal{R}G)(\theta,s),
\end{aligned}$$

that is,

$$\partial_s^{m-1}(\mathcal{R}G)(\theta,s)=\frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s}),\quad \forall\theta\in S^{n-1},s>0.$$

For fixed $\theta\in S^{n-1}$, when s goes to 0, $\partial_s^{m-1}(\mathcal{R}G)(\theta,s)$ will go to a finite constant $(\mathcal{R}G)^{(m-1)}(\theta,0)$. Since $\frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s})$ is continuous when $s>0$, as $s\to\infty$, the limit of $\frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s})$ exists and equals $(\mathcal{R}G)^{(m-1)}(\theta,0)$.

When s goes to the infinity, both of $\partial_s^{m-1}(\mathcal{R}G)(\theta,s)$ and $\frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s})$ will go to 0. Therefore

$$\partial_s^{m-1}(\mathcal{R}G)(\theta,s)=\frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s}),\quad \forall\theta\in S^{n-1},s\in[0,\infty]. \tag{5.1}$$

From this fact, we can see when x goes to the infinity, $(\mathcal{V}g)(x)$ will go to 0, and $\lim_{r\to\infty}r^k(\mathcal{V}g)(r\theta)=0$ for any $k<m$ and $\lim_{r\to\infty}r^m(\mathcal{V}g)(r\theta)$ exists which is a finite constant.

Noting that $r=\frac{1}{2s}$, so

$$\partial_s^m(\mathcal{R}G)(\theta,s)=(-2)^{m+2}\pi^m r^2\partial_r(r^m(\mathcal{V}g)(r\theta)).$$

Hence

$$\begin{aligned}\frac{1}{(2\pi)^{n-1}}&\int_{S^{n-1}}\int_0^\infty|\partial_s^m(\mathcal{R}G)(\theta,s)|^2\,ds\,d\theta\\ &=\frac{2^{2m+5}\pi^{2m}}{2(2\pi)^{n-1}}\int_{S^{n-1}}\int_0^\infty r^4\left|\partial_r(r^m(\mathcal{V}g)(r\theta))\right|^2\,ds\,d\theta\\ &=8\int_{S^{n-1}}\int_0^\infty r^2\left|\partial_r(r^m(\mathcal{V}g)(r\theta))\right|^2\,dr\,d\theta\\ &=8\int_{\mathbb{R}^n}\frac{|\partial_r(|x|^{\frac{n-1}{2}}(\mathcal{V}g)(x))|^2}{|x|^{n-3}}\,dx.\end{aligned}$$

When $n=2m+1$, the Radon transform $\mathcal{R}$ obeys the isometry identity

$$\frac{1}{2(2\pi)^{n-1}}\int_{S^{n-1}}\int_{-\infty}^\infty|\partial_s^m(\mathcal{R}G)(\theta,s)|^2\,ds\,d\theta=\int_{\mathbb{R}^n}|G(X)|^2\,dX,$$

therefore

$$\begin{aligned}
8\int_{\mathbb{R}^n} \frac{|\partial_r(|x|^{\frac{n-1}{2}}(\mathcal{V}g)(x))|^2}{|x|^{n-3}}\,dx &= \frac{1}{(2\pi)^{n-1}}\int_{S^{n-1}}\int_0^\infty |\partial_s^m(\mathcal{R}G)(\theta,s)|^2\,ds\,d\theta \\
&= \frac{1}{2(2\pi)^{n-1}}\int_{S^{n-1}}\int_{-\infty}^\infty |\partial_s^m(\mathcal{R}G)(\theta,s)|^2\,ds\,d\theta \\
&= \int_{\mathbb{R}^n} |G(X)|^2\,dX \\
&= \int_{\mathbb{R}^n} \left(g(x)|x|^{n+1}\right)^2\,(1/|x|^{2n})dx \\
&= \int_{\mathbb{R}^n} \left(|x|g(x)\right)^2\,dx.
\end{aligned}$$

□

Now we give another proof of Theorem 5.1 using spherical harmonic expansions and the isometry obtained in the $n = 3$ case. Similar ideas can be used to obtain an isometry for all even $n \geq 4$ from the isometry for the $n = 2$ case.

Proof. (Using spherical harmonic expansions) Below, for any $x \in \mathbb{R}^n$, $x \neq 0$, let $\theta = x/|x|$ and $r = |x|$.

Suppose $\{\phi_k(\theta)\}_{k=0}^\infty$ is the complete orthonormal set of spherical harmonics on the unit sphere S^{n-1} in $\mathbb{R}^n$, where $\phi_k(\theta)$ is the restriction of a homogeneous harmonic polynomial of degree d_k onto S^{n-1}. Since $g(x) \in C^\infty(\mathbb{R}^n)$, $v(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$. They have the spherical harmonics expansions

$$\begin{aligned}
g(x) &= \sum_{k=0}^\infty g_k(r) r^{d_k}\phi_k(\theta), \\
v(x,t) &= \sum_{k=0}^\infty a_k(r,t) r^{d_k}\phi_k(\theta),
\end{aligned}$$

where $g_k(r)r^{d_k} = \int_{S^{n-1}} g(r\theta)\phi_k(\theta)\;d\theta$, $a_k(r,t)r^{d_k} = \int_{S^{n-1}} v(r\theta,t)\phi_k(\theta)\;d\theta$, and $g_k(r)$, $a_k(r,t)$ are smooth.

Applying the Laplacian to $v(x,t)$, and using (2.10), we have

$$\Delta v(x,t) = \sum_{k=0}^\infty \left(\partial_r^2 a_k(r,t) + \frac{2d_k+n-1}{r}\partial_r a_k(r,t)\right) r^{d_k}\phi_k(\theta). \tag{5.2}$$

Since $v(x,t)$ satisfies the wave equation (1.6), by the orthonormality of the spherical harmonics $\{\phi_k(\theta)\}_{k=0}^{\infty}$, we have that, for each $k \geq 0$, $a_k(r,t)$ and $g_k(r)$ satisfy

$$\partial_t^2 a_k - \partial_r^2 a_k - \frac{2d_k + n - 1}{r}\partial_r a_k = 0, \quad (r,t) \in [0,\infty) \times \mathbb{R}, \tag{5.3}$$

$$a_k(r,0) = 0, \quad \partial_t a_k(r,0) = g_k(r). \tag{5.4}$$

Taking even extensions of $g_k(r)$ and $a_k(r,t)$ when $r < 0$, we extend the above equations to the whole space:

$$\partial_t^2 a_k - \partial_r^2 a_k - \frac{2d_k + n - 1}{r}\partial_r a_k = 0, \quad (r,t) \in \mathbb{R} \times \mathbb{R}, \tag{5.5}$$

$$a_k(r,0) = 0, \quad \partial_t a_k(r,0) = g_k(r). \tag{5.6}$$

We try to find an isometry between $g_k(r)$ and $a_k(r,|r|)$ for any $d_k = 0, 1, \ldots$. The idea arises from the isometry identity of $\mathcal{V}$ in $\mathbb{R}^3$. When $n = 3$, we have an isometry identity between $v(x,|x|)$ and $g(x)$, which in turn gives an isometry between $a_k(r,|r|)$ and $g_k(r)$ in the $n = 3$ case. Notice that for any odd $n \geq 3$, $n = 2m+1$, we can write $2d_k + n - 1 = 2(d_k + m - 1) + 3 - 1$ for any $d_k = 0, 1, \ldots$, which will give us the idea that the isometry in $\mathbb{R}^3$ will help us to find a similar isometry for any odd n.

Consider the IVP

$$\partial_t^2 a(r,t) - \partial_r^2 a(r,t) - \frac{2l+2}{r}\partial_r a(r,t) = 0, \quad (r,t) \in \mathbb{R} \times \mathbb{R}, \tag{5.7}$$

$$a(r,0) = 0, \quad \partial_t a(r,0) = h(r), \tag{5.8}$$

where $h(r) \in \mathscr{S}_0(\mathbb{R})$, $l \geq 0$ is an integer. Using any spherical harmonic function $\phi_l(\theta)$ of degree l in S^2, we can construct a function in $\mathscr{S}_0(\mathbb{R}^3)$:

$$g_l(x) = g_l(r\theta) = h(r) r^l \phi_l(\theta),$$

and a function in $\mathscr{S}_0(\mathbb{R}^3 \times \mathbb{R})$:

$$v_l(x,t) = v_l(r\theta, t) = a(r,t) r^l \phi_l(\theta).$$

Then $g_l(x), v_l(x,t)$ satisfies the IVP in $\mathbb{R}^3 \times \mathbb{R}$:

$$\partial_t^2 v_l(x,t) - \Delta v_l(x,t) = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

$$v_l(x,0) = 0, \quad \partial_t v_l(x,0) = g_l(x).$$

By the isometry identity of the forward operator $\mathcal{V}$ in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} (|x|g_l(x))^2 \ dx = 8 \int_{\mathbb{R}^3} |\partial_r(|x|v_l(x,|x|))|^2 \ dx.$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^3} (|x|g_l(x))^2 \ dx &= \int_{\mathbb{R}^3} \left(|x|h(r)r^l\phi_l(\theta)\right)^2 \ dx \\ &= \int_{S^2} (\phi_l(\theta))^2 \ d\theta \int_0^\infty r^{2l+4}h^2(r) \ dr,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} |\partial_r(|x|v_l(x,|x|))|^2 \ dx &= \int_{\mathbb{R}^3} \left|\partial_r\left(|x|a(r,|r|)r^l\phi_l(\theta)\right)\right|^2 \ dx \\ &= \int_{S^2} (\phi_l(\theta))^2 \ d\theta \int_0^\infty r^2 \left|\partial_r\left(r^{l+1}a(r,|r|)\right)\right|^2 \ dr.\end{aligned}$$

By the orthonormality of the spherical harmonics on S^2, we have

$$\int_0^\infty r^{2l+4}h^2(r) \ dr = 8\int_0^\infty r^2 \left|\partial_r\left(r^{l+1}a(r,|r|)\right)\right|^2 \ dr. \tag{5.9}$$

That is, when $h(r) \in \mathscr{S}_0(\mathbb{R})$, for any integer $l \geq 0$, the solution $a(r,t)$ of the IVP (5.7) (5.8) has the isometry identity (5.9).

Now consider the IVP (1.6) (1.7) in $\mathbb{R}^n$ where $n = 2m+1$. If $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ has the spherical harmonics expansion with only one term $g(x) = g_k(r)r^{d_k}\phi_k(\theta)$, where $d_k \geq 0$ is an integer, $g_k(r) \in \mathscr{S}_0(\mathbb{R})$, then the solution $v(x,t)$ of the IVP (1.6) (1.7) has the form $v(x,t) = a_k(r,t)r^{d_k}\phi_k(\theta)$ for some function $a_k(r,t)$. And $a_k(r,t), g_k(r)$ satisfy the IVP

$$\partial_t^2 a_k(r,t) - \partial_r^2 a_k(r,t) - \frac{2(d_k+m-1)+2}{r}\partial_r a_k(r,t) = 0, \quad (r,t) \in \mathbb{R}\times\mathbb{R} \tag{5.10}$$

$$a_k(r,0) = 0, \quad \partial_t a_k(r,0) = g_k(r), \tag{5.11}$$

which has the form of the IVP (5.7) (5.8). So by (5.9), we have

$$\int_0^\infty r^{2d_k+2m+2}g_k^2(r) \ dr = 8\int_0^\infty r^2 \left|\partial_r\left(r^{d_k+m}a_k(r,|r|)\right)\right|^2 \ dr. \tag{5.12}$$

Now consider any $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ with the spherical harmonics expansion

$$g(x) = \sum_{k=0}^{\infty} g_k(r) r^{d_k} \phi_k(\theta).$$

Let $v(x,t)$ be the solution of the IVP (1.6) (1.7) with the spherical harmonics expansion

$$v(x,t) = \sum_{k=0}^{\infty} a_k(r,t) r^{d_k} \phi_k(\theta).$$

By the orthonormality of the spherical harmonics, $g_k(r), a_k(r,t)$ satisfy the IVP (5.10) (5.11) for any $k \geq 0$. Therefore, $g_k(r), a_k(r,t)$ satisfy the equation (5.12), for any $k \geq 0$.

Now

$$\begin{aligned}
\left|\partial_r\left(|x|^m v(x,|x|)\right)\right|^2 &= \left|\sum_{k=0}^{\infty} \partial_r\left(r^{d_k+m} a_k(r,|r|)\phi_k(\theta)\right)\right|^2 \\
&= \sum_{k,l=0}^{\infty} \partial_r(r^{d_k+m} a_k(r,|r|))\, \partial_r(r^{d_l+m} a_l(r,|r|))\, \phi_k(\theta)\phi_l(\theta)
\end{aligned}$$

Hence using the orthonormality of $\{\phi_k(\theta)\}_{k=0}^{\infty}$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\left|\partial_r\left(|x|^m v(x,|x|)\right)\right|^2}{|x|^{2m-2}}\, dx &= \sum_{k=0}^{\infty} \int_0^{\infty} r^{n-1-(2m-2)} \left|\partial_r(r^{d_k+m} a_k(r,|r|))\right|^2\, dr \\
&= \sum_{k=0}^{\infty} \int_0^{\infty} r^2 \left|\partial_r(r^{d_k+m} a_k(r,|r|))\right|^2\, dr.
\end{aligned}$$

While, again using the orthonormality of $\{\phi_k(\theta)\}_{k=0}^{\infty}$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(|x| g(x)\right)^2\, dx &= \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} r^{d_k+1} g_k(r)\phi_k(\theta)\right)^2\, dx \\
&= \sum_{k,l=0}^{\infty} \int_0^{\infty} r^{d_k+d_l+2m+2} g_k(r) g_l(r)\, dr \int_{S^{n-1}} \phi_k(\theta)\phi_l(\theta)\, d\theta \\
&= \sum_{k=0}^{\infty} \int_0^{\infty} r^{2d_k+2m+2} g_k^2(r)\, dr.
\end{aligned}$$

For each $k \geq 0$, we have the identity (5.12), therefore we have proved the theorem. $\square$

Remark 5.1. *For future use we note a relation obtained in the first proof of Theorem 5.1. For any odd $n \geq 3$, $g(x) \in C^{\infty}(\mathbb{R}^n)$ such that $g(x/|x|^2)/|x|^{n+1} \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$\partial_s^{\frac{n-3}{2}} (\mathcal{R}G)(\theta,s) = \frac{(-1)^{\frac{n+1}{2}} 2\pi^{\frac{n-1}{2}}}{s^{\frac{n-1}{2}}} (\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall \theta \in S^2, s \geq 0.$$

We will use this relation again when characterizing the range of $\mathcal{V}$.

We now construct the L^2 adjoint of $\mathcal{V}$ with the goal of using it and the isometry for $\mathcal{V}$ to construct an inverse for $\mathcal{V}$.

Proposition 5.1 (The L^2 adjoint of $\mathcal{V}$). *The L^2 adjoint of $\mathcal{V}$ is given by*

$$\mathcal{V}^*\phi(x) = \frac{(-1)^{\frac{n+1}{2}}}{4\pi^{\frac{n-1}{2}}|x|^{\frac{n-1}{2}}}\partial_r^{\frac{n-3}{2}}(\mathcal{R}\phi)(x/|x|, |x|/2), \quad \forall \phi \in \mathscr{S}_0(\mathbb{R}^n).$$

Proof. Denote $n = 2m+1$. Let $g \in \mathscr{S}_0(\mathbb{R}^n)$. Then from (5.1) in the proof of Theorem 5.1, we have

$$(\mathcal{V}g)(x) = \frac{(-1)^{m-1}}{2\pi^m}\int_{\mathbb{R}^n} g(y)\delta^{m-1}(|y|^2 - 2x\cdot y)\, dy.$$

For any $\phi \in \mathscr{S}_0(\mathbb{R}^n)$,

$$\begin{aligned}\langle(\mathcal{V}g)(x), \phi(x)\rangle &= \int_{\mathbb{R}^n}(\mathcal{V}g)(x)\ \phi(x)\ dx \\ &= \frac{(-1)^{m-1}}{2\pi^m}\int_{\mathbb{R}^n}\phi(x)\int_{\mathbb{R}^n} g(y)\ \delta^{m-1}(|y|^2 - 2x\cdot y)\ dy\ dx \\ &= \frac{(-1)^{m-1}}{2\pi^m}\int_{\mathbb{R}^n} g(y)\int_{\mathbb{R}^n}\phi(x)\ \delta^{m-1}(|y|^2 - 2x\cdot y)\ dx\ dy,\end{aligned}$$

therefore

$$(\mathcal{V}^*\phi)(y) = \frac{(-1)^{m-1}}{2\pi^m}\int_{\mathbb{R}^n}\phi(x)\delta^{m-1}(|y|^2 - 2x\cdot y)\ dx.$$

Interchange x, y,

$$(\mathcal{V}^*\phi)(x) = \frac{(-1)^{m-1}}{2\pi^m}\int_{\mathbb{R}^n}\phi(y)\delta^{m-1}(|x|^2 - 2x\cdot y)\ dy.$$

Let us see the meaning of this integral.

For any $x \in \mathbb{R}^n$, let $r = |x|, \theta = x/|x|$. Then

$$\begin{aligned}\int_{\mathbb{R}^n}\phi(y)\delta^{m-1}(|x|^2 - 2x\cdot y)\ dy &= \int_{\mathbb{R}^n}\phi(y)\delta^{m-1}(r^2 - 2r\theta\cdot y)\ dy \\ &= \frac{1}{r^m}\int_{\mathbb{R}^n}\phi(y)\delta^{m-1}(r - 2\theta\cdot y)\ dy \\ &= \frac{1}{r^m}\partial_r^{m-1}\int_{\mathbb{R}^n}\phi(y)\delta(r - 2\theta\cdot y)\ dy \\ &= \frac{1}{2r^m}\partial_r^{m-1}\int_{\mathbb{R}^n}\phi(y)\delta(\theta\cdot y - r/2)\ dy \\ &= \frac{1}{2r^m}\partial_r^{m-1}\left((\mathcal{R}\phi)(\theta, r/2)\right).\end{aligned}$$

So

$$
\begin{aligned}
(\mathcal{V}^*\phi)(x) &= \frac{(-1)^{m-1}}{2\pi^m}\frac{1}{2r^m}\partial_r^{m-1}\left((\mathcal{R}\phi)(\theta, r/2)\right) \\
&= \frac{(-1)^{m-1}}{4\pi^m r^m}\partial_r^{m-1}\left((\mathcal{R}\phi)(\theta, r/2)\right).
\end{aligned}
$$

□

We now use the isometry for $\mathcal{V}$ and the L^2 adjoint of $\mathcal{V}$ to construct an inverse for $\mathcal{V}$.

Theorem 5.2 (First inversion formula for $\mathcal{V}$). *If $n = 2m+1$ with $m \geq 1$ then for any $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ we have*

$$
g(x) = -\frac{8}{|x|^2}\mathcal{V}^*\left(|x|^{1-m}\partial_r^2(|x|^{m+1}(\mathcal{V}g)(x))\right),
$$

where $\mathcal{V}^*$ is the L^2 adjoint of $\mathcal{V}$ given by Proposition 5.1.

Proof. For $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ where $n = 2m+1$, by Theorem 5.1, we have

$$
\int_{\mathbb{R}^n} (|x|g(x))^2 \; dx = 8\int_{\mathbb{R}^n}\left|\frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{m-1}}\right|^2 \; dx.
$$

For fixed $\theta \in S^{n-1}$, by Remark 5.1, we have

$$
r^m(\mathcal{V}g)(r\theta) = \frac{1}{(-2)^{m+1}\pi^m}(\partial_s^{m-1}(\mathcal{R}G))(\theta, 1/(2r)), \quad \forall r \geq 0,
$$

where $G(x) = g(x/|x|^2)/|x|^{n+1}$, $G \in \mathscr{S}_0(\mathbb{R}^n)$, $\mathcal{R}G \in \mathscr{S}(S^{n-1}\times\mathbb{R})$. Hence

$$
\begin{aligned}
\partial_r(r^m(\mathcal{V}g)(r\theta)) &= \frac{1}{(-2)^{m+2}\pi^m r^2}(\partial_s^m(\mathcal{R}G))(\theta, 1/(2r)). \\
\frac{\partial_r(r^m(\mathcal{V}g)(r\theta))}{r^{2m-2}} &= \frac{1}{(-2)^{m+2}\pi^m r^{2m}}(\partial_s^m(\mathcal{R}G))(\theta, 1/(2r)).
\end{aligned}
$$

Therefore, for any $\phi \in \mathscr{S}_0(\mathbb{R}^n)$,

$$
\begin{aligned}
&\frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{2m-2}}|x|^m(\mathcal{V}\phi)(x) \\
&= \frac{1}{(-2)^{2m+3}\pi^{2m}r^{2m}}(\partial_s^m(\mathcal{R}G))(\theta, 1/(2r))(\partial_s^{m-1}(\mathcal{R}\Phi))(\theta, 1/(2r)),
\end{aligned}
$$

where $\Phi(x) = \phi(x/|x|^2)/|x|^{n+1}$. When $r \to \infty$,

$$\lim_{r\to\infty} r^{n-1} \frac{\partial_r(|x|^m(\mathcal{V}g)(r\theta))}{|x|^{2m-2}} |x|^m(\mathcal{V}\phi)(r\theta) = \frac{1}{(-2)^{2m+3}\pi^{2m}} (\partial_s^m(\mathcal{R}G))(\theta,0)(\partial_s^{m-1}(\mathcal{R}\Phi))(\theta,0).$$

Since $(\partial_s^m(\mathcal{R}G))(\theta,0)$ and $(\partial_s^{m-1}(\mathcal{R}\Phi))(\theta,0)$ has opposite even-odd parity with respect to θ, that is,

$$(\partial_s^m(\mathcal{R}G))(\theta,0)(\partial_s^{m-1}(\mathcal{R}\Phi))(\theta,0) = -(\partial_s^m(\mathcal{R}G))(-\theta,0)(\partial_s^{m-1}(\mathcal{R}\Phi))(-\theta,0),$$

so

$$\int_{S^{n-1}} (\partial_s^m(\mathcal{R}G))(\theta,0)(\partial_s^{m-1}(\mathcal{R}\Phi))(\theta,0)\, d\theta = 0,$$

and

$$\lim_{r\to\infty} \int_{S^{n-1}} r^{n-1} \frac{\partial_r(|x|^m(\mathcal{V}g)(r\theta))}{|x|^{2m-2}} |x|^m(\mathcal{V}\phi)(r\theta)\, d\theta = 0.$$

Then by the linearity of $\mathcal{V}$, we have

$$\begin{aligned}
\langle |x|g(x), |x|\phi(x)\rangle &= 8\left\langle \frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{m-1}}, \frac{\partial_r(|x|^m(\mathcal{V}\phi)(x))}{|x|^{m-1}} \right\rangle \\
&= 8\left\langle \frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{2m-2}}, \partial_r(|x|^m(\mathcal{V}\phi)(x)) \right\rangle \\
&= -8\left\langle (\partial_r + \frac{n-1}{r})(\frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{2m-2}}), |x|^m(\mathcal{V}\phi)(x) \right\rangle \\
&= -8\left\langle (r^m\partial_r + (n-1)r^{m-1})(\frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{2m-2}}), (\mathcal{V}\phi)(x) \right\rangle \\
&= -8\left\langle \mathcal{V}^*\left((r^m\partial_r + (n-1)r^{m-1})(\frac{\partial_r(|x|^m(\mathcal{V}g)(x))}{|x|^{2m-2}}) \right), \phi(x) \right\rangle \\
&== -8\left\langle \mathcal{V}^*\left(|x|^{1-m}\partial_r^2(|x|^{m+1}(\mathcal{V}g)(x))\right), \phi(x) \right\rangle.
\end{aligned}$$

While

$$\langle |x|g(x), |x|\phi(x)\rangle = \langle |x|^2 g(x), \phi(x)\rangle,$$

therefore

$$g(x) = -\frac{8}{|x|^2}\mathcal{V}^*\left(|x|^{1-m}\partial_r^2(|x|^{m+1}(\mathcal{V}g)(x))\right).$$

□

Using Remark 5.1, we can give another direct inversion formula for $\mathcal{V}$.

Theorem 5.3 (**Second inversion formula for $\mathcal{V}$**)**.** *Let $n = 2m+1$ with $m \geq 1$, then for any $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$, we have*

$$g(x) = -\frac{|X|^{2m+2}}{2^{2m}\pi^m} \int_{S^{n-1}} \partial_s^{m+1} \left(\frac{1}{s^{m-1}|s|} (\mathcal{V}g)(\frac{\theta}{2s}) \right) \Big|_{s=X\cdot\theta} d\theta,$$

where $X = x/|x|^2$.

Proof. Define

$$G(X) = g(X/|X|^2)/|X|^{n+1}.$$

Then by Remark 5.1 we have

$$\partial_s^{m-1}(\mathcal{R}G)(\theta, s) = \frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall\theta \in S^2, s \geq 0.$$

When $s < 0$, plug $(-\theta, -s)$ into the above identity, we have

$$\partial_{-s}^{m-1}(\mathcal{R}G)(-\theta, -s) = \frac{(-1)^{m+1}2\pi^m}{(-s)^m}(\mathcal{V}g)(\frac{-\theta}{-2s}), \quad \forall\theta \in S^2, s \leq 0.$$

Since

$$\partial_{-s}^{m-1}(\mathcal{R}G)(-\theta, -s) = (-1)^{m-1}\partial_s^{m-1}(\mathcal{R}G)(-\theta, -s) = (-1)^{m-1}\partial_s^{m-1}(\mathcal{R}G)(\theta, s),$$

and

$$\frac{(-1)^{m+1}2\pi^m}{(-s)^m}(\mathcal{V}g)(\frac{-\theta}{-2s}) = \frac{(-1)\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s}),$$

so

$$\partial_s^{m-1}(\mathcal{R}G)(\theta, s) = \frac{(-1)^{m+1}2\pi^m}{s^{m-1}|s|}(\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall\theta \in S^{n-1}, s \in \mathbb{R},$$

and

$$\partial_s^{n-1}(\mathcal{R}G)(\theta, s) = \partial_s^{m+1}\left(\frac{(-1)^{m+1}2\pi^m}{s^{m-1}|s|}(\mathcal{V}g)(\frac{\theta}{2s})\right), \quad \forall\theta \in S^{n-1}, s \in \mathbb{R}.$$

By Theorem 2.4, the Radon transform $\mathcal{R}G$ has the inversion formula

$$G(X) = \frac{(-1)^m}{2(2\pi)^{2m}} \int_{S^{n-1}} \partial_s^{n-1}(\mathcal{R}G)(\theta, s)|_{s=X\cdot\theta}\, d\theta,$$

hence

$$G(X) = -\frac{1}{2^{2m}\pi^m} \int_{S^{n-1}} \partial_s^{m+1} \left(\frac{1}{s^{m-1}|s|} (\mathcal{V}g)(\frac{\theta}{2s}) \right) \Big|_{s=X\cdot\theta} d\theta.$$

Therefore

$$g(x) = -\frac{|X|^{n+1}}{2^{2m}\pi^m} \int_{S^{n-1}} \partial_s^{m+1} \left(\frac{1}{s^{m-1}|s|} (\mathcal{V}g)(\frac{\theta}{2s}) \right) \Big|_{s=X\cdot\theta} d\theta,$$

where $X = x/|x|^2$. □

5.2 The inversion of $\mathcal{U}$

Now we study the inversion of $\mathcal{U}$. As before we first obtain an isometry identity, construct the L^2 adjoint and then use these two to construct the inverse of $\mathcal{U}$.

Theorem 5.4 (Isometry identity for $\mathcal{U}$). *If n is odd, we have*

$$\int_{\mathbb{R}^n} \left(\frac{f(x)}{|x|}\right)^2 dx = 2\int_{\mathbb{R}^n} \left(\frac{(\mathcal{U}f)(x)}{|x|}\right)^2 dx, \qquad \forall f \in \mathscr{S}_0(\mathbb{R}^n).$$

Proof. Since n is odd, we write $n = 2m+1$ for some integer $m \geq 1$. If $u(x,t)$ is the solution of the IVP (1.4) (1.5) then by Theorem 2.1, $u(x,t)$ is given by

$$u(x,t) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n}{2})}(\frac{\partial}{\partial t})(\frac{\partial}{\partial t^2})^{(n-3)/2}\left(t^{n-2}(\mathcal{M}f)(x,|t|)\right), \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

When $t > 0$,

$$(\mathcal{M}f)(x,t) = \frac{2}{\omega_{n-1}t^{n-2}} \int_{\mathbb{R}^n} f(y)\delta(|y-x|^2 - t^2)\, dy.$$

Noting that $\frac{\partial}{\partial t} = 2t(\frac{\partial}{\partial t^2})$,

$$\begin{aligned}
u(x,t) &= \frac{2\sqrt{\pi}t}{2\Gamma(\frac{n}{2})}(\frac{\partial}{\partial t^2})^m \left(t^{n-2}(\mathcal{M}f)(x,t)\right)\\
&= \frac{2\sqrt{\pi}t}{\omega_{n-1}\Gamma(\frac{n}{2})}(\frac{\partial}{\partial t^2})^m \left(\int_{\mathbb{R}^n} f(y)\delta(|y-x|^2 - t^2)\, dy\right)\\
&= \frac{(-1)^m 2\sqrt{\pi}t}{\omega_{n-1}\Gamma(\frac{n}{2})} \int_{\mathbb{R}^n} f(y)\delta^m(|y-x|^2 - t^2)\, dy\\
&= \frac{(-1)^m t}{\pi^m} \int_{\mathbb{R}^n} f(y)\delta^m(|y-x|^2 - t^2)\, dy.
\end{aligned}$$

So

$$(\mathcal{U}f)(x) = \frac{(-1)^m|x|}{\pi^m}\int_{\mathbb{R}^n} f(y)\delta^m(|y|^2 - 2x\cdot y)\, dy, \qquad \forall x \in \mathbb{R}^n,\ x \neq 0. \tag{5.13}$$

For any $x \in \mathbb{R}^n$, $x \neq 0$, let $\theta = x/|x|, s = \frac{1}{2|x|}$ and define

$$F(X) = f(X/|X|^2)/|X|^{n-1};$$

then by Theorem (3.1), $F(X) \in \mathscr{S}_0(\mathbb{R}^n)$. We have

$$\begin{aligned}(-1)^m 2^{m+1}\pi^m |x|^m (\mathcal{U}f)(x) &= \int_{\mathbb{R}^n} f(y)|2x|^{m+1}\delta^m(|y|^2 - 2x\cdot y)\, dy \\ &= \int_{\mathbb{R}^n} f(y)\delta^m(\frac{|y|^2}{2|x|} - \frac{x}{|x|}\cdot y)\, dy \\ &= \int_{\mathbb{R}^n} f(y)\delta^m(s|y|^2 - \theta\cdot y)\, dy \\ &= \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{2m+2}}\delta^m(s - \theta\cdot\frac{y}{|y|^2})\, dy \qquad\qquad \text{let } Y = y/|y|^2 \\ &= \int_{\mathbb{R}^n} f(Y/|Y|^2)|Y|^{2m+2}\delta^m(s-\theta\cdot Y)\ (1/|Y|^{2n})dY \\ &= \int_{\mathbb{R}^n} F(Y)\delta^m(s-\theta\cdot Y)\, dY \\ &= \partial_s^m(\mathcal{R}F)(\theta,s),\end{aligned}$$

that is,

$$\partial_s^m(\mathcal{R}F)(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ s>0.$$

For fixed $\theta \in S^{n-1}$, when s goes to 0, $\partial_s^m(\mathcal{R}F)(\theta,s)$ will go to a finite constant $(\mathcal{R}F)^{(m)}(\theta,0)$. Since $\frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s})$ is a continuous as a function of s, hence the limit of $\frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s})$ exists which is equal to $(\mathcal{R}F)^{(m)}(\theta,0)$.

When s goes to the infinity, since both $\partial_s^m(\mathcal{R}F)(\theta,s)$ and $\frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s})$ will go to 0, therefore

$$\partial_s^m(\mathcal{R}F)(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ s\geq 0. \tag{5.14}$$

From this fact, we can see that when x goes to the infinity, $(\mathcal{U}f)(x)$ will go to 0, so as $|x|^k(\mathcal{U}f)(x)$, for any $k<m$, and the limit of $r^m(\mathcal{U}f)(r\theta)$ exists when r goes to the infinity.

Since

$$\begin{aligned}\int_{S^{n-1}}\int_{-\infty}^0 |\partial_s^m(\mathcal{R}F)(\theta,s)|^2\, ds\, d\theta &= \int_{S^{n-1}}\int_{-\infty}^0 |\partial_s^m(\mathcal{R}F)(-\theta,-s)|^2\, ds\, d\theta \quad \text{let } t=-s \\ &= \int_{S^{n-1}}\int_0^\infty |\partial_t^m(\mathcal{R}F)(-\theta,t)|^2\, dt\, d\theta \\ &= \int_{S^{n-1}}\int_0^\infty |\partial_t^m(\mathcal{R}F)(\theta,t)|^2\, dt\, d\theta,\end{aligned}$$

so

$$\frac{1}{2(2\pi)^{n-1}}\int_{S^{n-1}}\int_{-\infty}^{\infty}|\partial_s^m(\mathcal{R}F)(\theta,s)|^2\,ds\,d\theta=\frac{1}{(2\pi)^{n-1}}\int_{S^{n-1}}\int_{0}^{\infty}|\partial_s^m(\mathcal{R}F)(\theta,s)|^2\,ds\,d\theta.$$

When $n=2m+1$, the Radon transform has isometry identity (see Theorem 2.2)

$$\int_{\mathbb{R}^n}|F(X)|^2\,dX=\frac{1}{2(2\pi)^{n-1}}\int_{S^{n-1}}\int_{-\infty}^{\infty}|\partial_s^m(\mathcal{R}F)(\theta,s)|^2\,ds\,d\theta,$$

and since

$$\int_{\mathbb{R}^n}|F(X)|^2\,dX=\int_{\mathbb{R}^n}(f(x)|x|^{n-1})^2\,(1/|x|^{2n})dx=\int_{\mathbb{R}^n}\left(\frac{f(x)}{|x|}\right)^2\,dx,$$

therefore

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\frac{f(x)}{|x|}\right)^2\,dx&=\frac{1}{(2\pi)^{n-1}}\int_{S^{n-1}}\int_{0}^{\infty}|\partial_s^m(\mathcal{R}F)(\theta,s)|^2\,ds\,d\theta\qquad\text{let } r=\tfrac{1}{2s}\\&=\frac{1}{(2\pi)^{n-1}}\int_{S^{n-1}}\int_{\infty}^{0}|(-2\pi)^m2|x|^mu(x,|x|)|^2\;(\frac{-1}{2r^2})dr\,d\theta\\&=2\int_{S^{n-1}}\int_{\infty}^{0}r^{n-3}|u(x,|x|)|^2\,dr\,d\theta\\&=2\int_{\mathbb{R}^n}\left(\frac{u(x,|x|)}{|x|}\right)^2\,dx.\end{aligned}$$

□

Remark 5.2. *We showed in the proof that if $n=2m+1\geq 3$, $f(x)\in C^\infty(\mathbb{R}^n)$ such that $f(x/|x|^2)/|x|^{n-1}\in\mathscr{S}(\mathbb{R}^n)$, then*

$$\partial_s^m(\mathcal{R}F)(\theta,s)=\frac{(-1)^m2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}),\quad\forall\theta\in S^{n-1},\;s\geq 0.$$

We will use this relation again when characterizing the range of $\mathcal{U}$.

Now we find the L^2 adjoint of $\mathcal{U}$.

Proposition 5.2 (The L^2 adjoint of $\mathcal{U}$). *If n is odd then the L^2 adjoint of $\mathcal{U}$ is given by*

$$(\mathcal{U}^*\phi)(x)=\frac{(-1)^{\frac{n-1}{2}}}{2\pi^{\frac{n-1}{2}}|x|^{\frac{n+1}{2}}}\partial_r^{\frac{n-1}{2}}(\mathcal{R}\phi_*)(x/|x|,|x|/2),\quad\forall\phi\in\mathscr{S}_0(\mathbb{R}^n),\quad x\in\mathbb{R}^n,$$

where $\phi_*(y)=\phi(y)|y|$ and $r=|x|$.

Proof. Let $n = 2m+1$ and $f(x) \in \mathscr{S}_0(\mathbb{R}^n)$. From (5.13) in the proof of Theorem 5.4, we have

$$(\mathcal{U}f)(x) = \frac{(-1)^m 2|x|}{\pi^m} \int_{\mathbb{R}^n} f(y)\delta^m(|y|^2 - 2x \cdot y)\, dy.$$

For any $\phi(x) \in \mathscr{S}_0(\mathbb{R}^n)$,

$$\begin{aligned}
\langle (\mathcal{U}f)(x), \phi(x)\rangle &= \int_{\mathbb{R}^n} (\mathcal{U}f)(x)\, \phi(x)\, dx \\
&= \frac{(-1)^m}{\pi^m} \int_{\mathbb{R}^n} |x|\phi(x) \int_{\mathbb{R}^n} f(y)\, \delta^m(|y|^2 - 2x \cdot y)\, dy\, dx \\
&= \frac{(-1)^m}{\pi^m} \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} |x|\phi(x)\, \delta^m(|y|^2 - 2x \cdot y)\, dx\, dy,
\end{aligned}$$

therefore

$$(\mathcal{U}^*\phi)(x) = \frac{(-1)^m}{\pi^m} \int_{\mathbb{R}^n} \phi(y)|y|\delta^m(|x|^2 - 2x \cdot y)\, dy = \frac{(-1)^m}{\pi^m} \int_{\mathbb{R}^n} \phi_*(y)\delta^m(|x|^2 - 2x \cdot y)\, dy.$$

Let us see the meaning of this integral.

For any $x \in \mathbb{R}^n$, let $r = |x|,\ \theta = x/|x|$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi_*(y)\delta^m(|x|^2 - 2x \cdot y)\, dy &= \int_{\mathbb{R}^n} \phi_*(y)\delta^m(r^2 - 2r\theta \cdot y)\, dy \\
&= \frac{1}{r^{m+1}} \int_{\mathbb{R}^n} \phi_*(y)\delta^m(r - 2\theta \cdot y)\, dy \\
&= \frac{1}{r^{m+1}} \partial_r^m \int_{\mathbb{R}^n} \phi_*(y)\delta(r - 2\theta \cdot y)\, dy \\
&= \frac{1}{2r^{m+1}} \partial_r^m \int_{\mathbb{R}^n} \phi_*(y)\delta(\theta \cdot y - r/2)\, dy \\
&= \frac{1}{2r^{m+1}} \partial_r^m (\mathcal{R}\phi_*)(\theta, r/2).
\end{aligned}$$

So we have

$$(\mathcal{U}^*\phi)(x) = \frac{(-1)^m}{2\pi^m r^{m+1}} \partial_r^m (\mathcal{R}\phi_*)(\theta, r/2),$$

where $\phi_*(y) = \phi(y)|y|$. □

Now we obtain the inverse of $\mathcal{U}$.

Theorem 5.5 (First inversion formula for $\mathcal{U}$). *If n is odd then*

$$f(x) = 2|x|^2 \mathcal{U}^* \left(\frac{(\mathcal{U}f)(x)}{|x|^2} \right), \qquad \forall f \in \mathscr{S}_0(\mathbb{R}^n),\ x \in \mathbb{R}^n$$

where $\mathcal{U}^*$ is the L^2 adjoint of $\mathcal{U}$ given by Proposition 5.1.

Proof. By Theorem 5.4, we have

$$\int_{\mathbb{R}^n} \left(\frac{f(x)}{|x|}\right)^2 \, dx = 2 \int_{\mathbb{R}^n} \left(\frac{(\mathcal{U}f)(x)}{|x|}\right)^2 \, dx.$$

By the linearity of $\mathcal{U}$, for any $\phi(x) \in \mathscr{S}_0(\mathbb{R}^n)$, we have

$$\begin{aligned} \left\langle \frac{f(x)}{|x|}, \frac{\phi(x)}{|x|} \right\rangle &= 2 \left\langle \frac{(\mathcal{U}f)(x)}{|x|}, \frac{(\mathcal{U}\phi)(x)}{|x|} \right\rangle \\ &= 2 \left\langle \frac{(\mathcal{U}f)(x)}{|x|^2}, (\mathcal{U}\phi)(x) \right\rangle \\ &= 2 \left\langle \mathcal{U}^* \left(\frac{(\mathcal{U}f)(x)}{|x|^2}\right), \phi(x) \right\rangle \end{aligned}$$

Since

$$\left\langle \frac{f(x)}{|x|}, \frac{\phi(x)}{|x|} \right\rangle = \left\langle \frac{f(x)}{|x|^2}, \phi(x) \right\rangle$$

therefore

$$f(x) = 2|x|^2 \mathcal{U}^* \left(\frac{(\mathcal{U}f)(x)}{|x|^2}\right).$$

□

From Remark 5.2, there is a direct relation bewteen $\mathcal{R}F$ and $\mathcal{U}f$, where $F(x) = f(x/|x|^2)/|x|^{n-1}$. Using this fact, we can obtain another inversion formula for $\mathcal{U}$.

Theorem 5.6 (**Second inversion formula for $\mathcal{U}$**)**.** *Let $n = 2m+1$ with $m \geq 1$. For any $f(x) \in \mathscr{S}_0(\mathbb{R}^n)$, we have*

$$f(x) = \frac{|X|^{2m}}{(4\pi)^m} \int_{S^{n-1}} \partial_s^m \left(\frac{1}{s^m} (\mathcal{U}f)(\frac{\theta}{2s})\right)\Big|_{s=X\cdot\theta} \, d\theta,$$

where $X = x/|x|^2$.

Proof. Define

$$F(x) = f(x/|x|^2)/|x|^{n-1}.$$

By Theorem 3.1, $F \in \mathscr{S}_0(\mathbb{R}^n)$. By Remark 5.2, we have

$$\partial_s^m (\mathcal{R}F)(\theta, s) = \frac{(-1)^m 2\pi^m}{s^m} (\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^{n-1}, \; s \geq 0.$$

When $s < 0$, plug $(-\theta, -s)$ into the above identity, we get

$$\partial_s^m (\mathcal{R}F)(\theta, s) = \frac{(-1)^m 2\pi^m}{s^m} (\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^{n-1},\ s < 0.$$

Therefore

$$\partial_s^m (\mathcal{R}F)(\theta, s) = \frac{(-1)^m 2\pi^m}{s^m} (\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall \theta \in S^{n-1},\ s \in \mathbb{R},$$

and

$$\partial_s^{2m} (\mathcal{R}F)(\theta, s) = (-1)^m 2\pi^m \partial_s^m \left(\frac{1}{s^m} (\mathcal{U}f)(\frac{\theta}{2s}) \right), \quad \forall \theta \in S^{n-1},\ s \in \mathbb{R}.$$

By Theorem 2.4, in $\mathbb{R}^n$ of odd dimension, the Radon transform $\mathcal{R}F$ has the inversion formula

$$F(X) = \frac{(-1)^m}{2^{2m+1}\pi^{2m}} \int_{S^{n-1}} \partial_s^{2m} (\mathcal{R}F)(\theta, s)|_{s = X\cdot\theta}\, d\theta.$$

Therefore,

$$F(X) = \frac{1}{(4\pi)^m} \int_{S^{n-1}} \partial_s^m \left(\frac{1}{s^m} (\mathcal{U}f)(\frac{\theta}{2s}) \right) \Big|_{s = X\cdot\theta} d\theta,$$

and

$$f(x) = \frac{|X|^{n-1}}{(4\pi)^m} \int_{S^{n-1}} \partial_s^m \left(\frac{1}{s^m} (\mathcal{U}f)(\frac{\theta}{2s}) \right) \Big|_{s = X\cdot\theta} d\theta,$$

where $X = x/|x|^2$. □

5.3 The isometry and inverse for $\mathcal{W}$

Having the isometries and inverses for $\mathcal{U}$ and $\mathcal{V}$ we can easily construct the isometry and inverse of $\mathcal{W}$.

Theorem 5.7 (Isometry for $\mathcal{W}$). *If $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ and $w(x,t)$ is the solution of the IVP (1.1) (1.2), where $n = 2m+1$, then*

$$\int_{\mathbb{R}^n} (|x| g(x))^2 + \left(\frac{f(x)}{|x|} \right)^2 dx$$
$$= \int_{\mathbb{R}^n} 2 \left(\frac{\partial_r \big(|x|^m (w(x,|x|) - w(x,-|x|)) \big)}{|x|^{m-1}} \right)^2 + \frac{1}{2} \left(\frac{w(x,|x|) + w(x,-|x|)}{|x|} \right)^2 dx.$$

Theorem 5.8 (**First inverse formula of $\mathcal{W}$**). *Let $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$, $w(x,t)$ be the solution of the IVP (1.1) (1.2), where $n = 2m+1 \geq 3$ is odd. Then we have*

$$f(x) = |x|^2 \mathcal{U}^* \left(\frac{w(x, |x| + w(x, -|x|)}{|x|^2} \right),$$
$$g(x) = -\frac{4}{|x|^2} \mathcal{V}^* \Big(|x|^{1-m} \partial_r^2 \big(|x|^{m+1} (w(x,|x|) - w(x,-|x|)) \big) \Big),$$

where $\mathcal{U}^*, \mathcal{V}^*$ are the operators defined in Proposition 5.2 and Proposition 5.1 respectively.

Theorem 5.9 (**Second inverse formula of $\mathcal{W}$**). *Let $f(x), g(x) \in \mathscr{S}_0(\mathbb{R}^n)$, $w(x,t)$ be the solution of the IVP (1.1) (1.2), where $n = 2m+1 \geq 3$ is odd. Then we have*

$$f(x) = \frac{|X|^{2m}}{2^{2m+1}\pi^m} \int_{S^{n-1}} \partial_s^m \left(\frac{w(\frac{\theta}{2s}, \frac{\theta}{2|s|}) + w(\frac{\theta}{2s}, -\frac{\theta}{2|s|})}{s^m} \right) \Bigg|_{s=X\cdot\theta} d\theta,$$
$$g(x) = -\frac{|X|^{2m+2}}{2^{2m+1}\pi^m} \int_{S^{n-1}} \partial_s^{m+1} \left(\frac{w(\frac{\theta}{2s}, \frac{\theta}{2|s|}) - w(\frac{\theta}{2s}, -\frac{\theta}{2|s|})}{s^{m-1}|s|} \right) \Bigg|_{s=X\cdot\theta} d\theta,$$

where $X = x/|x|^2$.

Proof. Define

$$u(x,t) = \frac{w(x,t) + w(x,-t)}{2},$$
$$v(x,t) = \frac{w(x,t) - w(x,-t)}{2}.$$

Then $u(x,t), v(x,t)$ are the unique solutions of the IVP (1.4) (1.5) and the IVP (1.6) (1.7) respectively, and the proof immediately follows from Theorem 5.2, Theorem 5.3, Theorem 5.4, Theorem 5.5 and Theorem 5.6. □

Chapter 6

THE RANGES OF $\mathcal{U}$ AND $\mathcal{V}$ WHEN n IS ODD.

In Chapters 4 and 5, we studied the inversion of $\mathcal{U}, \mathcal{V}$ when n is odd. We found isometries and inversion formulas for $\mathcal{U}$ and $\mathcal{V}$. In this chapter, we characterize the ranges of $\mathcal{U}, \mathcal{V}$ when n is odd. We obtain necessary conditions on the range of $\mathcal{U}, \mathcal{V}$ if the domain of $\mathcal{U}$ and $\mathcal{V}$ is restricted to $\mathscr{S}_0(\mathbb{R}^n)$. For $\mathcal{U}$ and $\mathcal{V}$ with domains restricted to $C^\infty(\mathbb{R}^n)$, n odd, we give sufficient conditions for functions to be in the ranges of $\mathcal{U}$ or $\mathcal{V}$. Unfortunately, we do not have complete characterizations of the ranges of $\mathcal{U}$ and $\mathcal{V}$.

6.1 The Range of $\mathcal{V}$

In this section, we characterize the range of $\mathcal{V}$. The idea is to use the fact that, if $n = 2m+1$, then from (5.1) we have

$$\frac{(-1)^{m+1} 2\pi^m}{s^m} (\mathcal{V}g)(\frac{\theta}{2s}) = \partial_s^{m-1} (\mathcal{R}G)(\theta, s), \quad \forall \theta \in S^{n-1}, \ s \geq 0,$$

where $G(X) = g(X/|X|^2)/|X|^{n+1}$. Since the range of the Radon transform has been characterized, we can use it to characterize the range of $\mathcal{V}$. Unfortunately, it is difficult to analyze the relationship between the behavior of g at ∞ and G at the origin, which is the reason for our incomplete characterization of the range of $\mathcal{V}$.

Theorem 6.1 (**Necessary conditions on the range of $\mathcal{V}$**). *If $n = 2m+1$ and $g(x) \in \mathscr{S}_0(\mathbb{R}^n)$ then $(\mathcal{V}g)(x) \in C^\infty(\mathbb{R}^n)$ and, for $\theta \in S^{n-1}$, we have*

- $\lim_{r\to\infty} r^k (\mathcal{V}g)(r\theta) = 0, \quad \forall k = 0, 1, \ldots m-1,$
- $\lim_{r\to\infty} r^m (\mathcal{V}g)(r\theta)$ *exists,*
- $\int_{\mathbb{R}} r^{k-2} |r| (\mathcal{V}g)(r\theta)\, dr = 0, \quad \forall k = 1, 2, \ldots, m-1,$

- $\int_{\mathbb{R}} \frac{1}{r^k|r|}(\mathcal{V}g)(r\theta)\,dr = p_k(\theta)$ *is the restriction to the unit sphere of a homogeneous polynomial of degree* k*, for any* $k = 0, 1, \ldots$*.*

Proof. If we define

$$G(X) = g(X/|X|^2)/|X|^{n+1},$$

then by Theorem 3.1, $G(X) \in \mathscr{S}_0(\mathbb{R}^n)$ and, by Theorem 2.3, $\mathcal{R}G$ is in $\mathscr{S}(\mathbb{R}^n)$. From (5.1) in the proof of Theorem 5.1, we have

$$\partial_s^{m-1}(\mathcal{R}G)(\theta, s) = \frac{(-1)^{m+1}2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall\theta \in S^{n-1},\ s \in [0, \infty].$$

When $s < 0$, noting that $(\mathcal{R}G)(\theta, s) = (\mathcal{R}G)(-\theta, -s)$,

$$\begin{aligned}\partial_s^{m-1}(\mathcal{R}G)(\theta, s) &= \partial_s^{m-1}\left((\mathcal{R}G)(-\theta, -s)\right) = (-1)^{m-1}\partial_s^{m-1}(\mathcal{R}G)(-\theta, -s)\\ &= \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{V}g)(\frac{\theta}{2s}),\end{aligned}$$

therefore

$$\partial_s^{m-1}(\mathcal{R}G)(\theta, s) = \frac{(-1)^{m+1}2\pi^m}{s^{m-1}|s|}(\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall\theta \in S^{n-1},\ \forall s \in [-\infty, \infty], \tag{6.1}$$

equivalently,

$$\begin{aligned}(\mathcal{V}g)(r\theta) &= \frac{1}{(-2)^{m+1}\pi^m r^m}\partial_s^{m-1}(\mathcal{R}G)(\theta, s)|_{s=1/(2r)}, \quad \forall\theta \in S^{n-1},\ \forall r \in [0, \infty],\\ (\mathcal{V}g)(x) &= \frac{1}{(-2)^{m+1}\pi^m |x|^m}\partial_s^{m-1}(\mathcal{R}G)(x/|x|, s)|_{s=1/(2|x|)}, \quad \forall x \in \mathbb{R}^n.\end{aligned}$$

Since $\mathcal{R}G \in \mathscr{S}(S^{n-1} \times \mathbb{R})$, similar to the proof of Theorem 3.1, it is not hard to show that $(\mathcal{V}g)(x) \in C^\infty(\mathbb{R}^n)$. Furthermore, for fixed $\theta \in S^{n-1}$,

$$r^k(\mathcal{V}g)(r\theta) = \frac{r^{k-m}}{(-2)^{m+1}\pi^m}\partial_s^{m-1}(\mathcal{R}G)(\theta, s)|_{s=1/(2r)}, \quad \forall\theta \in S^{n-1},\ \forall r \in [0, \infty].$$

Hence

$$\begin{aligned}&\lim_{r\to\infty} r^k(\mathcal{V}g)(r\theta) = 0, \quad \forall k = 0, 1, \ldots, m-1,\\ &\lim_{r\to\infty} r^m(\mathcal{V}g)(r\theta) = \frac{(\mathcal{R}G)^{(m-1)}(\theta, 0)}{(-2)^{m+1}\pi^m}.\end{aligned}$$

Now consider $\int_{\mathbb{R}} s^k \partial_s^{m-1}(\mathcal{R}G)(\theta, s)\, ds$. Since $(\mathcal{R}G)(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$,

$$\lim_{s \to \pm\infty} s^j \partial_s^k (\mathcal{R}G)(\theta, s) = 0, \quad \forall j, k \in \mathbb{Z}_+.$$

Hence, when $k = 0, 1, \ldots, m-2$, using integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}} s^k\, \partial_s^{m-1}(\mathcal{R}G)(\theta, s)\, ds &= s^k\, \partial_s^{m-2}(\mathcal{R}G)(\theta, s)|_{-\infty}^{\infty} - k \int_{\mathbb{R}} s^{k-1}\, \partial_s^{m-2}(\mathcal{R}G)(\theta, s)\, ds \\
&= -k \int_{\mathbb{R}} s^{k-1}\, \partial_s^{m-2}(\mathcal{R}G)(\theta, s)\, ds \\
&= k(k-1) \int_{\mathbb{R}} s^{k-2}\, \partial_s^{m-3}(\mathcal{R}G)(\theta, s)\, ds \\
&\cdots \\
&= (-1)^k k! \partial_s^{m-2-k}(\mathcal{R}G)(\theta, s)|_{-\infty}^{\infty} = 0,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} s^k\, \partial_s^{m-1}(\mathcal{R}G)(\theta, s)\, ds &= \int_{\mathbb{R}} s^k\, \frac{(-1)^{m+1} 2\pi^m}{s^{m-1}|s|} (\mathcal{V}g)(\frac{\theta}{2s}) \\
&= (-1)^{m+1} 2\pi^m \int_{\mathbb{R}} \frac{1}{s^{m-1-k}|s|} (\mathcal{V}g)(\frac{\theta}{2s})\, ds,
\end{aligned}$$

that is, for any $k = 1, 2, \ldots, m-1$, for any $\theta \in S^{n-1}$,

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \frac{1}{s^k |s|} (\mathcal{V}g)(\frac{\theta}{2s})\, ds \qquad \text{let } r = \tfrac{1}{2s} \\
&= \int_{\mathbb{R}} 2^k r^{k-2} |r| (\mathcal{V}g)(r\theta)\, dr.
\end{aligned}$$

When $k \geq m-1$,

$$\begin{aligned}
\int_{\mathbb{R}} s^k\, \partial_s^{m-1}(\mathcal{R}G)(\theta, s)\, ds &= (-1)^{m-1} \int_{\mathbb{R}} (\partial_s^{m-1} s^k)\, (\mathcal{R}G)(\theta, s)\, ds \\
&= (-1)^{m-1} \frac{k!}{(k-m+1)!} \int_{\mathbb{R}} s^{k-m+1} (\mathcal{R}G)(\theta, s)\, ds,
\end{aligned}$$

which is a homogeneous polynomial of degree $k - m + 1$ in θ. Noting that

$$\int_{\mathbb{R}} s^k\, \partial_s^{m-1}(\mathcal{R}G)(\theta, s)\, ds = (-1)^{m+1} 2\pi^m \int_{\mathbb{R}} \frac{s^{k-m+1}}{|s|} (\mathcal{V}g)(\frac{\theta}{2s})\, ds,$$

therefore, for any $k = 0, 1, \ldots$

$$\begin{aligned} p_k(\theta) &= \int_{\mathbb{R}} \frac{s^k}{|s|} (\mathcal{V}g)(\frac{\theta}{2s})\, ds && \text{let } r = \tfrac{1}{2s} \\ &= \int_{\mathbb{R}} \frac{1}{(2r)^k |r|} (\mathcal{V}g)(r\theta)\, dr \end{aligned}$$

is a homogeneous polynomial of degree k in θ. $\square$

Remark 6.1. *There is a difficulty to show these necessary conditions for $(\mathcal{V}g)(x)$ are also sufficient. By (6.1) in the proof of Theorem 6.1, we have*

$$\partial_s^{m-1}(\mathcal{R}G)(\theta, s) = \frac{(-1)^{m+1} 2\pi^m}{s^{m-1}|s|} (\mathcal{V}g)(\frac{\theta}{2s}), \quad \forall \theta \in S^{n-1},\ \forall s \in [-\infty, \infty],$$

Suppose there is a function $\bar{v}(x) \in C^\infty(\mathbb{R}^n)$ satisfying the necessary conditions in Theorem 6.1, we can construct a Radon transform $q(\theta, s)$ such that

$$\partial_s^{m-1} q(\theta, s) = \frac{(-1)^{m+1} 2\pi^m}{s^{m-1}|s|} \bar{v}(\frac{\theta}{2s}), \quad \forall \theta \in S^{n-1},\ \forall s \in [-\infty, \infty].$$

But the difficulty, if we require $\bar{v}(x)$ satisfies only the conditions in Theorem 6.1, it is not enought to show that the function $q(\theta, s)$ we obtained is in $\mathscr{S}(S^{n-1} \times \mathbb{R})$, therefore, there may be not a function $G \in \mathscr{S}(\mathbb{R}^n)$ such that $q = \mathcal{R}G$ and we may not construct g using the relation

$$g(x) = G(x/|x|^2)/|x|^{n+1}$$

such that

$$\partial_s^{m-1}(\mathcal{R}G)(\theta, s) = \frac{(-1)^{m+1} 2\pi^m}{s^{m-1}|s|} \bar{v}(\frac{\theta}{2s}), \quad \forall \theta \in S^{n-1},\ \forall s \in [-\infty, \infty],$$

To overcome this difficulty, we need to impose stronger conditions on $\mathcal{V}g$.

Now we restrict the domain of $\mathcal{V}$ to $C^\infty(\mathbb{R}^n)$ and seek good sufficient conditions for a function to be in the range of $\mathcal{V}$.

Theorem 6.2 (Sufficient conditions for the range of $\mathcal{V}$). *Suppose n is odd and $n = 2m + 1$. For any $\bar{v}(x) \in C^\infty(\mathbb{R}^n)$ with*

(a) $\bar{v}(r\theta) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$ as a function of $(\theta, r) \in S^{n-1} \times \mathbb{R}$;

(b) $\int_{\mathbb{R}} r^{k-2}|r|\bar{v}(r\theta)\, dr = 0, \quad \forall\theta \in S^{n-1},\ k = 1, 2, \ldots, m-1;$

(c) $\int_{\mathbb{R}} \frac{1}{r^k|k|}\bar{v}(r\theta)\, dr = p_k(\theta)$ *is a homogeneous polynomial of degree* k *for any* $k = 0, 1, 2\cdots;$

there is a $g(x) \in C^\infty(\mathbb{R}^n)$*, such that* $(\mathcal{V}g)(x) = \bar{v}(x)$*.*

Proof. For use below, we show that

$$\int_{-\infty}^{\infty} \frac{(\tau - s)^{m-2-k_2}}{\tau^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\, d\tau = 0, \quad \forall k_2 = 0, 1, 2, \ldots, m-2.$$

For any $k_2 = 0, 1, 2, \ldots, m-2$,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{(\tau - s)^{m-2-k_2}}{\tau^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\, d\tau &= \sum_{l=0}^{m-2-k_2} \frac{(m-2-k_2)!}{l!\,(m-2-k_2-l)!}(-s)^{m-2-k_2-l}\int_{\mathbb{R}} \frac{\tau^l}{\tau^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\, d\tau \\ &= \sum_{l=0}^{m-2-k_2} \frac{(m-2-k_2)!}{l!\,(m-2-k_2-l)!}(-s)^{m-2-k_2-l}\int_{\mathbb{R}} \frac{1}{\tau^{m-1-l}|\tau|}\bar{v}(\frac{\theta}{2\tau})\, d\tau.\end{aligned}$$

Using hypothesis (b), which is equivalent to

$$\int_{\mathbb{R}} \frac{1}{s^k|s|}\bar{v}(\frac{\theta}{2s}) = 0, \quad \forall\theta \in S^{n-1}, \quad k = 1, 2, \ldots, m-1,$$

each term of $\int_{\mathbb{R}} \frac{1}{\tau^{m-1-l}|\tau|}\bar{v}(\frac{\theta}{2\tau})\, d\tau$ is equal to 0, hence

$$\int_{-\infty}^{\infty} \frac{(\tau - s)^{m-2-k_2}}{\tau^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\, d\tau = 0, \quad \forall k_2 = 0, 1, 2, \ldots, m-2.$$

For any $\theta \in S^{n-1}, s \in \mathbb{R}$, define

$$\begin{aligned}q(\theta, s) &= \frac{1}{(m-2)!}\int_{-\infty}^{s} (s-t)^{m-2}\frac{(-1)^{m+1}2\pi^m}{t^{m-1}|t|}\bar{v}(\frac{\theta}{2t})\, dt \\ &= \frac{(-1)^{m+1}2\pi^m}{(m-2)!}\int_{-\infty}^{s} \frac{(s-t)^{m-2}}{t^{m-1}|t|}\bar{v}(\frac{\theta}{2t})\, dt.\end{aligned}$$

We show that $q(\theta, s) = (\mathcal{R}G)(\theta, s)$ for some $G(X) \in \mathscr{S}(\mathbb{R}^n)$.

First we show $q(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R}^n)$. Using hypothesis (a) and taking $s = \frac{1}{2r}$, by Theorem 3.1 $\bar{v}(\frac{\theta}{2s}) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$ as a function of $(\theta, s) \in S^{n-1} \times \mathbb{R}$, so it is not hard to see that $q(\theta, s) \in C^\infty(S^{n-1} \times \mathbb{R})$. Now we show that, for any $k = 0, 1, \ldots,$ $s^k q(\theta, s)$ goes to 0 as s goes to the infinity.

For any $k = 0, 1, \ldots,$

$$s^k q(\theta, s) = \frac{(-1)^{m+1} 2\pi^m}{(m-2)!} \left(s^k \int_{-\infty}^{s} \frac{(s-t)^{m-2}}{t^{m-1}|t|} \bar{v}(\frac{\theta}{2t})\, dt \right)$$
$$= \frac{(-1)^{m} 2\pi^m}{(m-2)!} \left(s^k \int_{s}^{\infty} \frac{(s-t)^{m-2}}{t^{m-1}|t|} \bar{v}(\frac{\theta}{2t})\, dt \right).$$

When $t \to \infty$, since $\bar{v}(\frac{\theta}{2t}) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$, we have $|t^{k+m}\bar{v}(\frac{\theta}{2t})| \leq 1$ when t is large enough. Therefore

$$|s^k q(\theta, s)| \leq \frac{2\pi^m}{(m-2)!} \left(\int_{s}^{\infty} \left| s^k \frac{(s-t)^{m-2}}{t^{m-1}|t|} \bar{v}(\frac{\theta}{2t}) \right| \, dt \right) \leq \frac{2\pi^m}{(m-2)!} \int_{s}^{\infty} \frac{1}{t^2}\, dt$$

goes to 0 when s goes to the infinity.

Now we show that for all $k_1, k_2 = 0, 1, \ldots$, as s goes to the infinity, $s^{k_1}\partial_s^{k_2} q(\theta, s)$ will go to 0. Notice that when $k_2 = 0, 1, \ldots, m-2$,

$$\partial_s^{k_2} q(\theta, s) = \frac{(-1)^{m+1} 2\pi^m}{(m-2-k_2)!} \int_{-\infty}^{s} \frac{(s-t)^{m-2-k_2}}{t^{m-1}|t|} \bar{v}(\frac{\theta}{2t})\, dt,$$

and when $k_2 \geq m-1$,

$$\partial_s^{k_2} q(\theta, s) = \frac{(-1)^{m+1} 2\pi^m}{(m-2-k_2)!} \partial_s^{k_2-m+1} \left(\frac{1}{s^{m-1}|s|} \bar{v}(\frac{\theta}{2s}) \right).$$

Since we have shown that $\int_{-\infty}^{\infty} \frac{(s-t)^{m-2-k_2}}{t^{m-1}|t|} \bar{v}(\frac{\theta}{2t})\, dt = 0$ for any $k_2 = 0, 1, \ldots, m-2$ and any $s \in \mathbb{R}$, Using the fact that $\bar{v}(\frac{\theta}{2s}) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$, we can show that for all $k_1, k_2 = 0, 1, \ldots$, as s goes to the infinity, $s^{k_1}\partial_s^{k_2} q(\theta, s)$ will go to 0. Therefore, $q(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$.

Secondly, we show $q(\theta, s)$ is even, that is, $q(\theta, s) = q(-\theta, -s)$. Consider

$$q(-\theta, -s) = \frac{(-1)^{m+1} 2\pi^m}{(m-2)!} \int_{-\infty}^{-s} \frac{(-s-t)^{m-2}}{t^{m-1}|t|} \bar{v}(\frac{-\theta}{2t})\, dt.$$

Let $\tau = -t$, we have

$$\int_{-\infty}^{-s} \frac{(-s-t)^{m-2}}{t^{m-1}|t|} \bar{v}(\frac{-\theta}{2t})\, dt = \int_{s}^{\infty} \frac{(\tau - s)^{m-2}}{(-\tau)^{m-1}|\tau|} \bar{v}(\frac{\theta}{2\tau})\, d\tau.$$

Therefore

$$\begin{aligned}
q(-\theta,-s) &= \frac{(-1)^{m+1}2\pi^m}{(m-2)!}\int_{-\infty}^{-s}\frac{(-s-t)^{m-2}}{t^{m-1}|t|}\bar{v}(\frac{-\theta}{2t})\,dt\\
&= \frac{(-1)^{m+1}2\pi^m}{(m-2)!}\int_{s}^{\infty}\frac{(\tau-s)^{m-2}}{(-\tau)^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\,d\tau\\
&= -\frac{2\pi^m}{(m-2)!}\int_{-\infty}^{s}\frac{(\tau-s)^{m-2}}{\tau^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\,d\tau\\
&= \frac{(-1)^{m+1}2\pi^m}{(m-2)!}\int_{-\infty}^{s}\frac{(s-\tau)^{m-2}}{\tau^{m-1}|\tau|}\bar{v}(\frac{\theta}{2\tau})\,d\tau\\
&= q(\theta,s).
\end{aligned}$$

Finally, we show $\int_{\mathbb{R}} s^k q(\theta,s)\,ds = p_k(\theta)$ is a homogeneous polynomial of degree k in θ. Noting that

$$\partial_s^{m-1}q(\theta,s) = \frac{(-1)^{m+1}2\pi^m}{s^{m-1}|s|}\bar{v}(\frac{\theta}{2s}),$$

for any $k=0,1,\ldots$, we have

$$\begin{aligned}
\int_{\mathbb{R}} s^k q(\theta,s)\,ds &= (-1)^{m-1}\frac{k!}{(k+m-1)!}\int_{\mathbb{R}} s^{k+m-1}\partial_s^{m-1}q(\theta,s)\,ds\\
&= \frac{k!2\pi^m}{(k+m-1)!}\int_{\mathbb{R}}\frac{s^k}{|s|}\bar{v}(\frac{\theta}{2s})\,ds.
\end{aligned}$$

Taking $s=\frac{1}{2r}$, hypothesis (c) is equivalent to the statement that

$$\int_{\mathbb{R}}\frac{s^k}{|s|}\bar{v}(\frac{\theta}{2s}) = p_k(\theta)$$

is a homogeneous polynomial of degree k for any $k=0,1,2\cdots$. Hence

$$\int_{\mathbb{R}} s^k q(\theta,s)\,ds = p_k(\theta)$$

is a homogeneous polynomial of degree k in θ. Therefore, from Theorem 2.3, $q(\theta,s)$ is in the range of the Radon transform, and $q(\theta,s)=\mathcal{R}G(\theta,s)$ for some function $G\in\mathscr{S}(\mathbb{R}^n)$. Define

$$g(x) = G(x/|x|^2)/|x|^{n+1};$$

then, as in the proof of Theorem 3.1, we can show $g(x)\in C^\infty(\mathbb{R}^n)$. Applying Remark 5.1 we have

$$\partial_s^{m-1}(\mathcal{R}G)(\theta,s) = \frac{(-1)^{m+1}2\pi^m}{s^{m-1}|s|}(\mathcal{V}g)(\frac{\theta}{2s}),\quad \forall\theta\in S^{n-1},\ \forall s\in\mathbb{R}.$$

Noting that

$$\partial_s^{m-1}(\mathcal{R}G)(\theta,s) = \partial_s^{m-1}q(\theta,s) = \frac{(-1)^{m+1}2\pi^m}{s^{m-1}|s|}\bar{v}(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ \forall s\in\mathbb{R},$$

we have

$$\bar{v}(x) = (\mathcal{V}g)(x), \quad \forall x\in\mathbb{R}^n.$$

□

6.2 The Range of $\mathcal{U}$

We now attempt to characterize the range of $\mathcal{U}$. Again, we rely on the connection between $\mathcal{U}$ and the Radon transform - if $n=2m+1$, then from (5.14) we have

$$\partial_s^m(\mathcal{R}F)(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ s\geq 0.$$

First we find necessary conditions on the range of $\mathcal{U}$ if $\mathcal{U}$ is restricted to $\mathscr{S}_0(\mathbb{R}^n)$.

Theorem 6.3 **(Necessary conditions on the range of $\mathcal{U}$).** *Suppose $n=2m+1$. If $f(x)\in\mathscr{S}_0(\mathbb{R}^n)$ then $(\mathcal{U}f)(x)\in C^\infty(\mathbb{R}^n)$ and, for $\theta\in S^{n-1}$, we have*

- $\lim_{r\to\infty} r^k(\mathcal{U}f)(r\theta) = 0, \quad \forall k=0,1,\ldots,m-1,$
- $\lim_{r\to\infty} r^m(\mathcal{U}f)(r\theta)$ *exists,*
- $\int_{\mathbb{R}} r^{k-2}(\mathcal{U}f)(r\theta)\,dr = 0, \quad \forall k=1,2,\ldots,m,$
- $\int_{\mathbb{R}} \frac{1}{r^{k-2}}(\mathcal{U}f)(r\theta)\,dr = p_k(\theta)$ *is a homogeneous polynomial of degree k, $k=0,1,\ldots$.*

Proof. For any $x\in\mathbb{R}^n$, let $r=|x|, \theta=x/|x|$. Define

$$F(X) = f(X/|X|^2)/|X|^{n-1},$$

where $X=x/|x|^2$. Let $\mathcal{R}F$ be its Radon transform. Then from (5.14) in the proof of Theorem 5.4, we have

$$\partial_s^m(\mathcal{R}F)(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ s\in[0,\infty].$$

When $s<0$, define $t=-s>0$. Noting that

$$\begin{aligned}\partial_s^m(\mathcal{R}F)(\theta,s) &= \partial_s^m(\mathcal{R}F)(-\theta,-s) = (-1)^m\partial_t^m(\mathcal{R}F)(-\theta,t)\\ &= \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}),\end{aligned}$$

therefore

$$\partial_s^m(\mathcal{R}F)(\theta,s)=\frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}),\quad \forall\theta\in S^{n-1},\ s\in[-\infty,\infty],$$

equivalently,

$$(\mathcal{U}f)(x)=\frac{(-1)^m}{2^{m+1}\pi^m|x|^m}\partial_s^m(\mathcal{R}F)(x/|x|,s)|_{s=1/(2|x|)},\quad \forall x\in\mathbb{R}^n.$$

Since $\mathcal{R}F\in\mathscr{S}(S^{n-1}\times\mathbb{R})$, it is not hard to show that $(\mathcal{U}f)(x)\in C^\infty(\mathbb{R}^n)$. Furthermore, for fixed $\theta\in S^{n-1}$,

$$r^k(\mathcal{U}f)(r\theta)=\frac{(-1)^m r^{k-m}}{2^{m+1}\pi^m}\partial_s^m(\mathcal{R}F)(\theta,s)|_{s=1/(2|x|)},\quad \forall r\in[0,\infty].$$

Hence

$$\begin{aligned}\lim_{r\to\infty}r^k(\mathcal{U}f)(r\theta)&=0,\quad k=0,1,\ldots,m-1\\ \lim_{r\to\infty}r^m(\mathcal{U}f)(r\theta)&=\frac{(-1)^m}{2^{m+1}\pi^m}(\mathcal{R}F)^{(m)}(\theta,0).\end{aligned}$$

Consider $\int_{\mathbb{R}}s^k\partial_s^m(\mathcal{R}F)(\theta,s)\,ds$. Since $\mathcal{R}F(\theta,s)\in\mathscr{S}(S^{n-1}\times\mathbb{R})$,

$$\lim_{s\to\pm\infty}s^\alpha\partial_s^\beta(\mathcal{R}F)(\theta,s)=0,\quad \forall\alpha,\beta\in\mathbb{Z}_+^n.$$

So when $k=0,1,\ldots,m-1$,

$$\begin{aligned}\int_{\mathbb{R}}s^k\ \partial_s^m(\mathcal{R}F)(\theta,s)\ ds&=s^k\ \partial_s^{m-1}(\mathcal{R}F)(\theta,s)|_{-\infty}^{\infty}-k\int_{\mathbb{R}}s^{k-1}\ \partial_s^{m-1}(\mathcal{R}F)(\theta,s)\ ds\\ &=-k\int_{\mathbb{R}}s^{k-1}\ \partial_s^{m-1}(\mathcal{R}F)(\theta,s)\ ds\\ &=k(k-1)\int_{\mathbb{R}}s^{k-2}\ \partial_s^{m-2}(\mathcal{R}F)(\theta,s)\ ds\\ &=(-1)^k k!\partial_s^{m-k-1}(\mathcal{R}F)(\theta,s)|_{-\infty}^{\infty}=0,\end{aligned}$$

and

$$\int_{\mathbb{R}}s^k\ \partial_s^m(\mathcal{R}F)(\theta,s)\ ds=(-1)^m2\pi^m\int_{\mathbb{R}}\frac{1}{s^{m-k}}(\mathcal{U}f)(\frac{\theta}{2s})\ ds,$$

that is, for any $k=1,2,\ldots,m$

$$\int_{\mathbb{R}}\frac{1}{s^k}(\mathcal{U}f)(\frac{\theta}{2s})\ ds=0,\quad \forall\theta\in S^{n-1}.$$

Equivalently,

$$\int_{\mathbb{R}} r^{k-2}(\mathcal{U}f)(r\theta)\, dr = 0, \quad \forall k = 1, 2, \ldots, m.$$

When $k \geq m$,

$$\begin{aligned}\int_{\mathbb{R}} s^k\ \partial_s^m(\mathcal{R}F)(\theta, s)\, ds &= (-1)^m \int_{\mathbb{R}} (\partial_s^m s^k)(\mathcal{R}F)(\theta, s)\, ds \\ &= (-1)^m \frac{k!}{(k-m)!} \int_{\mathbb{R}} s^{k-m}(\mathcal{R}F)(\theta, s)\, ds,\end{aligned}$$

which is a homogeneous polynomial of degree $k - m$ in θ. Noting that

$$\begin{aligned}\int_{\mathbb{R}} s^k\ \partial_s^m \mathcal{R}F(\theta, s)\, ds &= (-1)^m 2\pi^m \int_{\mathbb{R}} s^{k-m}(\mathcal{U}f)(\frac{\theta}{2s})\, ds \qquad \text{let } r = \tfrac{1}{2s} \\ &= \frac{(-1)^m \pi^m}{2^{k-m}} \int_{\mathbb{R}} \frac{1}{r^{k-m+2}}(\mathcal{U}f)(r\theta)\, dr\end{aligned}$$

so, for any $k = 0, 1, \ldots$

$$\int_{\mathbb{R}} \frac{1}{r^{k-2}}(\mathcal{U}f)(r\theta)\, dr = p_k(\theta)$$

which is a homogeneous polynomial of degree k in θ. $\square$

Remark 6.2. *As in Remark 6.1, we encounter similar issues when we attempt to show the necessary conditions in Theorem 6.1 are sufficient. To overcome these issues, we need to impose stronger conditions on* $\mathcal{U}f$.

Now, we find good sufficient conditions for a function to be in the range of $\mathcal{U}$ if $\mathcal{U}$ is restricted to $C^\infty(\mathbb{R}^n)$.

Theorem 6.4 (Sufficient conditions for range of $\mathcal{U}$). *Suppose* $n = 2m + 1$. *If* $\bar{u}(x) \in C^\infty(\mathbb{R}^n)$ *such that*

- *(a)* $\bar{u}(r\theta) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$ *as function of* $(\theta, r) \in S^{n-1} \times \mathbb{R}$.,
- *(b)* $\int_{\mathbb{R}} r^{k-2}\bar{u}(r\theta)\, dr = 0, \quad \forall \theta \in S^{n-1}$ *for any* $k = 1, 2, \ldots, m$,
- *(c)* $\int_{\mathbb{R}} \frac{1}{r^{k+2}}\bar{u}(r\theta)\, dr = p_k(\theta)$ *is a homogeneous polynomial of degree* k, $k = 0, 1, 2, \cdots$,

then there is a $f(x) \in C^\infty(\mathbb{R}^n)$ *such that* $(\mathcal{U}f)(x) = \bar{u}(x)$.

Proof. For any $\theta \in S^{n-1}, s \in \mathbb{R}$, define

$$\begin{aligned} q(\theta, s) &= \frac{1}{(m-1)!} \int_{-\infty}^{s} (s-t)^{m-1} \frac{(-1)^m 2\pi^m}{t^m} \bar{u}(\frac{\theta}{2t})\, dt \\ &= \frac{(-1)^m 2\pi^m}{(m-1)!} \int_{-\infty}^{s} \frac{(s-t)^{m-1}}{t^m} \bar{u}(\frac{\theta}{2t})\, dt. \end{aligned}$$

We want to show $q(\theta, s)$ is in the range of the Radon transform.

Firstly, we show $q(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$. Since $\bar{u}(r\theta) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$ as a function of $(\theta, r) \in S^{n-1} \times \mathbb{R}$, by Theorem 3.1, $\bar{u}(\frac{\theta}{2s}) \in \mathscr{S}_0(S^{n-1} \times \mathbb{R})$ as a function of $(\theta, s) \in S^{n-1} \times \mathbb{R}$, so it is not hard to show $q(\theta, s) \in C^\infty(\mathbb{R}^n)$.

For any $k = 1, 2 \ldots m$, using the hypothesis (b), which is equivalent to

$$\int_{\mathbb{R}} \frac{1}{s^k} \bar{u}(\frac{\theta}{2s})\, ds = 0, \quad \forall \theta \in S^{n-1},$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{(\tau - s)^{m-1}}{\tau^m} \bar{u}(\frac{\theta}{2\tau})\, d\tau &= \sum_{l=0}^{m-1} \frac{(m-1)!}{l!\,(m-1-l)!} (-s)^{m-1-l} \int_{\mathbb{R}} \frac{\tau^l}{\tau^m} \bar{u}(\frac{\theta}{2\tau})\, d\tau \\ &= \sum_{l=0}^{m-1} \frac{(m-1)!}{l!\,(m-1-l)!} (-s)^{m-1-l} \int_{\mathbb{R}} \frac{1}{\tau^{m-l}} \bar{u}(\frac{\theta}{2\tau})\, d\tau \\ &= 0, \quad \forall s \in \mathbb{R}. \end{aligned}$$

Hence $q(\theta, s)$ goes to 0 as s goes to the infinity. Similar to the proof of Theorem 6.2, we can show that, for any $k_1, k_2 = 0, 1, 2, \ldots$, when s goes to the infinity, $s^{k_1} \partial_s^{k_2} q(\theta, s)$ will to 0. Therefore, $q(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$.

Secondly, we show $q(\theta, s)$ is even. Consider

$$q(-\theta, -s) = \frac{(-1)^m 2\pi^m}{(m-1)!} \int_{-\infty}^{-s} \frac{(-s-t)^{m-1}}{t^m} \bar{u}(\frac{-\theta}{2t})\, dt.$$

Let $\tau = -t$, we have

$$\int_{-\infty}^{-s} \frac{(-s-t)^{m-1}}{t^m} \bar{u}(\frac{-\theta}{2t})\, dt = \int_{s}^{\infty} \frac{(\tau - s)^{m-1}}{(-\tau)^m} \bar{u}(\frac{\theta}{2\tau})\, d\tau.$$

So

$$
\begin{aligned}
q(-\theta,-s) &= \frac{(-1)^m 2\pi^m}{(m-1)!}\int_s^\infty \frac{(\tau-s)^{m-1}}{(-\tau)^m}\bar{u}(\frac{\theta}{2\tau})\,d\tau \\
&= -\frac{2\pi^m}{(m-1)!}\int_{-\infty}^s \frac{(\tau-s)^{m-1}}{\tau^{m-1}}\bar{u}(\frac{\theta}{2\tau})\,d\tau \\
&= \frac{(-1)^m 2\pi^m}{(m-1)!}\int_{-\infty}^s \frac{(s-\tau)^{m-1}}{\tau^m}\bar{u}(\frac{\theta}{2\tau})\,d\tau \\
&= q(\theta,s).
\end{aligned}
$$

Finally, we show that for any $k=0,1,\ldots$, $\int_{\mathbb{R}} s^k q(\theta,s)\,ds = p_k(\theta)$ which is a homogeneous polynomial of degree k in θ. Noting that

$$
\partial_s^m q(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}\bar{u}(\frac{\theta}{2s}),
$$

for any $k=0,1,\ldots$, we have

$$
\begin{aligned}
\int_{\mathbb{R}} s^k q(\theta,s)\,ds &= (-1)^m\frac{k!}{(k+m)!}\int_{\mathbb{R}} s^{k+m}\partial_s^m q(\theta,s)\,ds \\
&= \frac{k!2\pi^m}{(k+m)!}\int_{\mathbb{R}} s^k \bar{u}(\frac{\theta}{2s})\,ds \qquad \text{let } r=\tfrac{1}{2s} \\
&= \frac{k!\pi^m}{2^k(k+m)!}\int_{\mathbb{R}} \frac{1}{r^{k+2}}\bar{u}(r\theta)\,dr \\
&= p_k(\theta)
\end{aligned}
$$

is a homogeneous polynomial of degree k in θ, where we use the hypothesis (c). Therfore, $q(\theta,s)\in\mathscr{S}(S^{n-1}\times\mathbb{R})$ is in the range of the Radon transform, that is, there is a function $F\in\mathscr{S}(\mathbb{R}^n)$, such that $q(\theta,s)=(\mathcal{R}F)(\theta,s)$. Define

$$
f(x) = F(x/|x|^2)/|x|^{n-1}.
$$

Then $f(x)\in C^\infty(\mathbb{R}^n)$. Let $(\mathcal{U}f)(x)$ be the forward operator defined in (1.8). By Remark 5.2,

$$
\partial_s^m(\mathcal{R}F)(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}(\mathcal{U}f)(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ \forall s\in\mathbb{R}.
$$

Noting that

$$
\partial_s^m(\mathcal{R}F)(\theta,s) = \partial_s^m q(\theta,s) = \frac{(-1)^m 2\pi^m}{s^m}\bar{u}(\frac{\theta}{2s}), \quad \forall\theta\in S^{n-1},\ \forall s\in\mathbb{R},
$$

we have

$$\bar{u}(x) = (\mathcal{U}f)(x), \quad \forall x \in \mathbb{R}.$$

$\square$

Chapter 7

INVERSION OF $\mathcal{V}$ WHEN $n = 2$

In Chapter 5, we found isometry identities and inversion formulas for $\mathcal{U}$ and $\mathcal{V}$ when n was odd. In Chapter 6, we studied the ranges of $\mathcal{U}$ and $\mathcal{V}$ when n was odd. Now our goal is to study these questions for $\mathcal{U}$ and $\mathcal{V}$ when n is even. As in Chapter 5, we hope to get an isometry when $n = 2$ and use that to obtain an isometry when n is even, $n > 2$. The trace on $t = |x|$, of solution of the IVP for the wave equation, for the odd n case is a surface integral on a sphere through the origin, but, for the even n case it is a an integral on a ball whose boundary passes through the origin. This makes it harder, when n is even, to establish a connection between the trace of the solution and the Radon transform. So the results for the even n case may be more complicated and not as appealing as the odd n case. Perhaps further manipulations may result in appealing results for the even n case but we have not obtained them yet. When $n = 2$, the inversion formula for $\mathcal{V}$ is not too complicated and we derive that and an isometry identity in this chapter.

We have the following inversion formula for $\mathcal{V}$ when $n = 2$.

Theorem 7.1 (Inversion formula for $\mathcal{V}$ when $n = 2$). *For any $g(x) \in \mathscr{S}_0(\mathbb{R}^2)$ we have*

$$g(x) = \frac{-1}{\sqrt{2}\pi^2} \int_{S^1} \int_{\mathbb{R}} \frac{1}{\theta \cdot x|x| - s|x|^3} \frac{\partial}{\partial s} \left(\sqrt{|s|} \frac{\partial}{\partial s} \int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2s} - \tau\right|}} \, d\tau \right) \, ds \, d\theta.$$

Proof. For $g(x) \in \mathscr{S}_0(\mathbb{R}^2)$, let $v(x,t)$ be the solution of the IVP (1.6) (1.7). By Theorem 2.1, $v(x,t)$ is given by

$$v(x,t) = \int_0^t \frac{\tau(\mathcal{M}g)(x, |\tau|)}{\sqrt{t^2 - \tau^2}} \, d\tau, \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Define

$$G(X) = g(X/|X|^2)/|X|^3;$$

then by Theorem 3.1, $G(X) \in \mathscr{S}_0(\mathbb{R}^2)$. For any $x \in \mathbb{R}^2$, let $r = |x|, \theta = x/|x|$. When $x \neq 0$, we have

$$\begin{aligned}
(\mathcal{V}g)(x) = (\mathcal{V}g)(r\theta) &= \int_0^r \frac{\tau(\mathcal{M}g)(x,\tau)}{\sqrt{r^2-\tau^2}}\, d\tau \\
&= \frac{1}{\pi}\int_0^r \frac{\tau}{\sqrt{r^2-\tau^2}} \int_{\mathbb{R}^2} g(y)\delta(\tau^2 - |y-x|^2)\, dy\, d\tau \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2} g(y) \int_0^r \frac{\tau}{\sqrt{r^2-\tau^2}}\delta(\tau^2 - |y-x|^2)\, d\tau\, dy \qquad \text{let } p = \tau^2 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} g(y) \int_0^{r^2} \frac{1}{\sqrt{r^2-p}}\delta(p - |y-x|^2)\, dp\, dy.
\end{aligned}$$

Noting that

$$\int_0^{r^2} \frac{1}{\sqrt{r^2-p}}\delta(p-|y-x|^2)\, dp = \frac{H(r^2-|y-x|^2)}{\sqrt{r^2-|y-x|^2}} = \frac{H(2r\theta\cdot y - |y|^2)}{\sqrt{2r\theta\cdot y-|y|^2}},$$

where $H(\cdot)$ is the Heaviside function. Hence, for $r > 0,\ \theta \in S^1$, we have

$$\begin{aligned}
(\mathcal{V}g)(r\theta) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{g(y)}{\sqrt{2r}|y|\sqrt{\theta\cdot\frac{y}{|y|^2}-\frac{1}{2r}}} H(\theta\cdot\frac{y}{|y|^2}-\frac{1}{2r})\, dy \qquad \text{let } Y = y/|y|^2 \\
&= \frac{1}{2\pi\sqrt{2r}}\int_{\mathbb{R}^2} \frac{G(Y)}{\sqrt{\theta\cdot Y - \frac{1}{2r}}} H(\theta\cdot Y - \frac{1}{2r})\, dY \\
&= \frac{1}{2\pi\sqrt{2r}}\int_{\mathbb{R}^2} G(Y)\int_{\frac{1}{2r}}^\infty \frac{\delta(a-\theta\cdot Y)}{\sqrt{a-\frac{1}{2r}}}\, da\, dY \\
&= \frac{1}{2\pi\sqrt{2r}}\int_{\frac{1}{2r}}^\infty \frac{(\mathcal{R}G)(\theta,a)}{\sqrt{a-\frac{1}{2r}}}\, da,
\end{aligned}$$

which is the 1/2 order (right side) Riemann-Liouville integral of $(\mathcal{R}G)(\theta, 1/(2r))$.

From this we can see that for fixed $\theta \in S^1$, $(\mathcal{V}g)(r\theta)$ is a continuous function of r. Since $v(x,t) \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$, $(\mathcal{V}g)(r\theta) = v(r\theta, r)$ is continuous in r, θ. Furthermore, when $r \to \infty$, since $\frac{1}{2\pi\sqrt{2r}}\int_{\frac{1}{2r}}^\infty \frac{(\mathcal{R}G)(\theta,a)}{\sqrt{a-\frac{1}{2r}}}\, da$ will go to 0, hence $(\mathcal{V}g)(r\theta)$ will go to 0, and $\sqrt{r}(\mathcal{V}g)(r\theta)$ will go to a finite constant $\frac{1}{2\sqrt{2}\pi}\int_0^\infty (\mathcal{R}G)(\theta,\tau)/\sqrt{\tau}\, d\tau$.

When $r \to 0^+$, $1/(2r)$ will go to ∞, and

$$\left| \frac{1}{2\pi\sqrt{2r}} \int_{\frac{1}{2r}}^{\infty} \frac{(\mathcal{R}G)(\theta, a)}{\sqrt{a - \frac{1}{2r}}} \, da \right| = \frac{1}{2\pi} \int_{\frac{1}{2r}}^{\infty} \frac{(\mathcal{R}G)(\theta, a)}{\sqrt{2ra - 1}} \, da \leq \frac{1}{2\pi} \int_{\frac{1}{2r}}^{\infty} (\mathcal{R}G)(\theta, a) \, da$$

which will go to 0, and $(\mathcal{V}g)(r\theta)$ will go to $v(0,0) = g(0) = 0$, therefore

$$(\mathcal{V}g)(r\theta) = \frac{1}{2\pi\sqrt{2r}} \int_{\frac{1}{2r}}^{\infty} \frac{(\mathcal{R}G)(\theta, a)}{\sqrt{a - \frac{1}{2r}}} \, da, \quad \forall \theta \in S^1, \; r \in [0, \infty].$$

Integrating both sides with kernel $1/\sqrt{r - \tau}$, we get

$$\begin{aligned}
\int_0^r \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{r - \tau}} \, d\tau &= \int_0^r \frac{1}{2\pi\sqrt{2\tau(r - \tau)}} \int_{\frac{1}{2\tau}}^{\infty} \frac{(\mathcal{R}G)(\theta, a)}{\sqrt{a - \frac{1}{2\tau}}} \, da \, d\tau \qquad \text{let } b = \tfrac{1}{2a} \\
&= \frac{1}{4\pi} \int_0^r \frac{1}{\sqrt{r - \tau}} \int_0^{\tau} \frac{1}{b^{3/2}\sqrt{\tau - b}} (\mathcal{R}G)(\theta, \frac{1}{2b}) \, db \, dr \\
&= \frac{1}{4\pi} \int_0^r (\mathcal{R}G)(\theta, \frac{1}{2b}) \frac{1}{b^{3/2}} \int_b^r \frac{1}{\sqrt{r - \tau}\sqrt{\tau - b}} \, d\tau \, db \\
&= \frac{1}{4} \int_0^r \frac{(\mathcal{R}G)(\theta, \frac{1}{2b})}{b^{3/2}} \, db.
\end{aligned}$$

Take derivative with respect to r for both sides, we get

$$(\mathcal{R}G)(\theta, \frac{1}{2r}) = 4r^{3/2} \frac{\partial}{\partial r} \left(\int_0^r \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{r - \tau}} \, d\tau \right),$$

where both sides are continuous functions of r for fixed $\theta \in S^1$.

When r goes to the infinity, $(\mathcal{R}G)(\theta, \frac{1}{2r})$ will go to a finite constant $(\mathcal{R}G)(\theta, 0)$, hence the limit of the right hand side also exists and will also go to$(\mathcal{R}G)(\theta, 0)$.

When r goes to 0, both sides will go to 0. Therefore

$$(\mathcal{R}G)(\theta, \frac{1}{2r}) = 4r^{3/2} \frac{\partial}{\partial r} (\int_0^r \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{r - \tau}} \, d\tau), \quad \forall \theta \in S^1, \; r \in [0, \infty].$$

Let $s = \frac{1}{2r}$, then we have

$$(\mathcal{R}G)(\theta, s) = -2\sqrt{2s} \frac{\partial}{\partial s} (\int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\frac{1}{2s} - \tau}} \, d\tau).$$

When $s < 0$, noting that $(\mathcal{R}G)(\theta, s) = (\mathcal{R}G)(-\theta, -s)$, we have

$$\begin{aligned}
(\mathcal{R}G)(\theta, s) &= (\mathcal{R}G)(-\theta, -s) \\
&= -2\sqrt{-2s}\frac{\partial}{\partial(-s)}\Big(\int_0^{-\frac{1}{2s}} \frac{(\mathcal{V}g)(-\tau\theta)}{\sqrt{-\frac{1}{2s} - \tau}}\, d\tau\Big) \\
&= -2\sqrt{-2s}\frac{\partial}{\partial s}\Big(\int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{|\frac{1}{2s} - \tau|}}\, d\tau\Big).
\end{aligned}$$

Therefore

$$(\mathcal{R}G)(\theta, s) = -2\sqrt{2|s|}\frac{\partial}{\partial s}\Big(\int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{|\frac{1}{2s} - \tau|}}\, d\tau\Big), \quad \forall\theta \in S^1,\ s \in \mathbb{R}. \tag{7.1}$$

In $\mathbb{R}^2$, the Radon transform $(\mathcal{R}G)(\theta, s)$ has the inversion formula

$$G(X) = \frac{1}{4\pi^2}\int_{S^1}\int_{\mathbb{R}} \frac{\partial_s(\mathcal{R}G)(\theta, s)}{\theta \cdot X - s}\, ds,$$

so we have

$$G(X) = -\frac{1}{\sqrt{2}\pi^2}\int_{S^1}\int_{\mathbb{R}} \frac{1}{\theta \cdot X - s}\frac{\partial}{\partial s}\left(\sqrt{|s|}\frac{\partial}{\partial s}\int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{|\frac{1}{2s} - \tau|}}\, d\tau\right)\, ds\, d\theta,$$

where the integral is a Cauchy principal value integral, and it can be expressed as

$$\frac{1}{2\sqrt{2}\pi^2}\int_{S^1}\int_{\mathbb{R}} \frac{1}{q}\left(\frac{\partial}{\partial s}\left(\sqrt{|s|}\frac{\partial}{\partial s}\int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{|\frac{1}{2s} - \tau|}}\, d\tau\right)\Bigg|_{s=X\cdot\theta-q}^{s=X\cdot\theta+q}\right)\, dq\, d\theta.$$

Therefore

$$\begin{aligned}
g(x) &= G(x/|x|^2)/|x|^3 \\
&= \frac{-1}{\sqrt{2}\pi^2}\int_{S^1}\int_{\mathbb{R}} \frac{1}{\theta \cdot x|x| - s|x|^3}\frac{\partial}{\partial s}\left(\sqrt{|s|}\frac{\partial}{\partial s}\int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{|\frac{1}{2s} - \tau|}}\, d\tau\right)\, ds\, d\theta.
\end{aligned}$$

□

As for odd n, we have an isometry identity for $\mathcal{V}$ in $\mathbb{R}^2$, which we hope to extend to the general n case.

Theorem 7.2 (Isometry identity for $\mathcal{V}$ in $\mathbb{R}^2$). *If $g(x) \in \mathscr{S}_0(\mathbb{R}^2)$ then*

$$\frac{\pi^2}{4}\int_{S^1}\int_{\mathbb{R}} \left(|s|^{3/2} g(s\theta)\right)^2 \, ds \, d\theta = \int_{S^1}\int_{\mathbb{R}} \left(\frac{\partial}{\partial s} \int_{-\infty}^{s} \frac{\sqrt{|a|}}{\sqrt{s-a}} \frac{\partial}{\partial a} \int_0^{\frac{1}{2a}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2a} - \tau\right|}} \, d\tau \, da \right)^2 ds \, d\theta.$$

Proof. For any $X \in \mathbb{R}^2$, define $G(X) = g(X/|X|^2)/|X|^3$. By (7.1) in the proof of Theorem 7.1, we have

$$(\mathcal{R}G)(\theta, s) = -2\sqrt{2|s|} \frac{\partial}{\partial s} \int_0^{\frac{1}{2s}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2s} - \tau\right|}} \, d\tau, \quad \forall \theta \in S^1, \; s \in \mathbb{R}.$$

Denote $D_s^{1/2}(\mathcal{R}G)(\theta, s)$ to be the 1/2-th order Riemann-Liouville fractional derivative of $(\mathcal{R}G)(\theta, s)$ with respect to s:

$$\begin{aligned} D_s^{1/2}(\mathcal{R}G)(\theta, s) &= \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial s} \int_{-\infty}^{s} \frac{(\mathcal{R}G)(\theta, a)}{\sqrt{s-a}} \, da \\ &= -\frac{2\sqrt{2}}{\sqrt{\pi}} \frac{\partial}{\partial s} \left(\int_{-\infty}^{s} \frac{\sqrt{|a|}}{\sqrt{s-a}} \frac{\partial}{\partial a} \int_0^{\frac{1}{2a}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2a} - \tau\right|}} \, d\tau \, da \right). \end{aligned}$$

By Theorem 2.2, the 1/2-th order Riemann-Liouville fractional derivative obeys the identity

$$\int_{S^1}\int_{\mathbb{R}} |D_s^{1/2}(\mathcal{R}G)(\theta, s)|^2 \, ds \, d\theta = 4\pi \int_{\mathbb{R}} |G(X)|^2 \, dX,$$

noting that

$$\int_{S^1}\int_{\mathbb{R}} |D_s^{1/2}(\mathcal{R}G)(\theta, s)|^2 \, ds \, d\theta = \frac{8}{\pi}\int_{S^1}\int_{\mathbb{R}} \left(\frac{\partial}{\partial s} \int_{-\infty}^{s} \frac{\sqrt{|a|}}{\sqrt{s-a}} \frac{\partial}{\partial a} \int_0^{\frac{1}{2a}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2a} - \tau\right|}} \, d\tau \, da \right)^2 ds \, d\theta$$

and

$$\begin{aligned} \int_{\mathbb{R}} |G(X)|^2 \, dX &= \int_{\mathbb{R}} \left(|x| g(x)\right)^2 \, dx = \int_{S^1}\int_0^{\infty} s^3 |g(s\theta)|^2 \, ds \, d\theta \\ &= \frac{1}{2}\int_{S^1}\int_{\mathbb{R}} |s|^3 |g(s\theta)|^2 \, ds \, d\theta, \end{aligned}$$

we have

$$\frac{\pi^2}{4}\int_{S^1}\int_{\mathbb{R}} \left(|s|^{3/2} g(s\theta)\right)^2 \, ds \, d\theta = \int_{S^1}\int_{\mathbb{R}} \left(\frac{\partial}{\partial s} \int_{-\infty}^{s} \frac{\sqrt{|a|}}{\sqrt{s-a}} \frac{\partial}{\partial a} \int_0^{\frac{1}{2a}} \frac{(\mathcal{V}g)(\tau\theta)}{\sqrt{\left|\frac{1}{2a} - \tau\right|}} \, d\tau \, da \right)^2 ds \, d\theta.$$

$\square$

One can attempt to extend the above inversion formula and the isometry identity for $n=2$ case to the general even n case. Unfortunately, the isometry for $\mathcal{V}$ for the $n=2$ case is too complicated to easily extend to the general even n case and the inversion formulas for the general even n case are complicated involving Riemann-Liouville fractional derivatives. The inversion formula for $\mathcal{U}$ is also more difficult. We give some calculations for the readers' consideration.

Remark 7.1. *For any $f(x)\in\mathscr{S}_0(\mathbb{R}^2)$, let $u(x,t)$ be the solution of the IVP (1.4) (1.5). By Theorem 2.1, when $t>0$, $u(x,t)$ is given by*

$$\begin{aligned}
u(x,t) &= \partial_t\left(\int_0^t \frac{s(\mathcal{M}f)(x,s)}{\sqrt{t^2-s^2}}\,ds\right)\\
&= \partial_t\left(\int_0^t \frac{1}{\pi\sqrt{t^2-s^2}}\int_{\mathbb{R}^2} f(y)\delta(s^2-|y-x|^2)\,dy\,ds\right)\\
&= \partial_t\left(\int_{\mathbb{R}^2} f(y)\int_0^t \frac{1}{\pi\sqrt{t^2-s^2}}\delta(s^2-|y-x|^2)\,dy\,ds\right)\\
&= \frac{1}{2\pi}\partial_t\left(\int_{\mathbb{R}^2}\frac{f(y)}{\sqrt{t^2-|y-x|^2}}H(t^2-|y-x|^2)\,dy\right)\\
&= \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{tf(y)}{\sqrt{t^2-|y-x|^2}}\delta(t^2-|y-x|^2)\,dy - \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{tf(y)}{(\sqrt{t^2-|y-x|^2})^3}H(t^2-|y-x|^2)\,dy,
\end{aligned}$$

where $H(\cdot)$ is the Heaviside function.

For any $x\in\mathbb{R}^2$, let $r=|x|,\ \theta=x/|x|$ and define

$$F(x)=f(x/|x|^2)/|x|.$$

By Theorem 3.1, $F\in\mathscr{S}_0(\mathbb{R}^2)$. When $t=|x|$, we have

$$u(x,|x|)=u(r\theta,r)=\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{rf(y)\delta(2r\theta\cdot y-|y|^2)}{\sqrt{2r\theta\cdot y-|y|^2}}\,dy-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{rf(y)H(2r\theta\cdot y-|y|^2)}{(\sqrt{2r\theta\cdot y-|y|^2})^3}\,dy$$

let $Y=y/|y|^2$

$$\begin{aligned}
&= \frac{r^{3/2}}{\sqrt{2}\pi}\partial_r\left(\int_{\mathbb{R}^2}\frac{F(Y)H(\theta\cdot Y-\frac{1}{2r})}{\sqrt{\theta\cdot Y-\frac{1}{2r}}}\,dY\right)-\frac{1}{4\pi\sqrt{2r}}\int_{\frac{1}{2r}}^{\infty}\frac{(\mathcal{R}F)(\theta,\tau)}{(\sqrt{\tau-\frac{1}{2r}})^3}\,d\tau\\
&= \frac{r^{3/2}}{\sqrt{2}\pi}\partial_r\left(\int_{\frac{1}{2r}}^{\infty}\frac{(\mathcal{R}F)(\theta,\tau)}{\sqrt{\tau-\frac{1}{2r}}}\,d\tau\right)-\frac{1}{4\pi\sqrt{2r}}\int_{\frac{1}{2r}}^{\infty}\frac{(\mathcal{R}F)(\theta,\tau)}{(\sqrt{\tau-\frac{1}{2r}})^3}\,d\tau,
\end{aligned}$$

where the first term is related to the right side $1/2$*-th order Riemann-Liouville fractional derivative of* $(\mathcal{R}F)(\theta, 1/(2r))$ *with upper limit* ∞*, and the second term is related to the Riemann-Liouville fractional integral of* $(\mathcal{R}F)(\theta, 1/(2r))$ *of a negative order, which needs regularization to make it well defined.*

Chapter 8

SPHERICAL MEANS OVER SPHERES THROUGH THE ORIGIN

In this chapter, we study the problem of recovering a function on $\mathbb{R}^n$ given the spherical averages of this function over all spheres through the origin (spheres centered at an arbitrary point x and of radius $|x|$). Using geometrical inversion across the unit sphere, this problem reduces to inverting the Radon transform of a function which results in an isometry and an inversion formula for the problem under consideration.

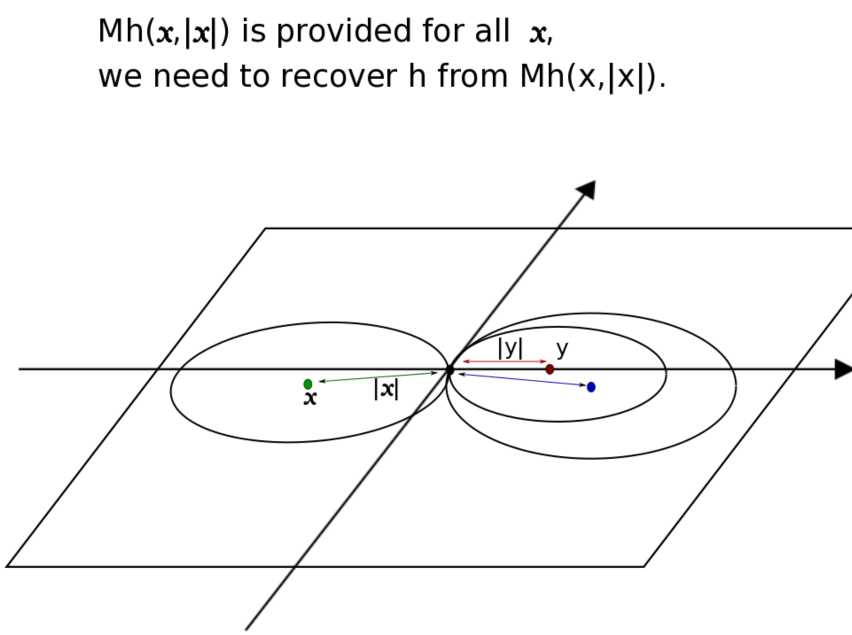

Figure 8.1: Spheres centered at x of radius $|x|$.

We first establish a connection between spherical averages over spheres through the origin and the Radon transform. Recall that the operator $(\mathcal{M}h)(a,r)$ represents the spherical average of h over the sphere of radius r centered at a and $(\mathcal{R}h)(\theta, s)$ represents the Radon transform of h over the plane $x \cdot \theta = s$.

Lemma 8.1. *Given $h \in \mathscr{S}_0(\mathbb{R}^n)$, if we define $H(X) = h(X/|X|^2)/|X|^{2n-2}$, $X \in \mathbb{R}^n$, then*

$$(\mathcal{M}h)(x, |x|) = \frac{1}{\omega_{n-1} r^{n-1}} (\mathcal{R}H)(\theta, \frac{1}{2r}), \quad \forall x \in \mathbb{R}^n$$

where $r = |x|$ and $\theta = x/|x|$.

Proof. For any $x \in \mathbb{R}^n$, $x \neq 0$, let $r = |x|, \theta = x/|x|$ we have

$$\begin{aligned}
(\mathcal{M}h)(x, |x|) &= \frac{1}{\omega_{n-1} r^{n-1}} \int_{|y-x|=r} h(y)\, dS_y \\
&= \frac{2}{\omega_{n-1} r^{n-2}} \int_{\mathbb{R}^n} h(y) \delta(|y|^2 - 2r\theta \cdot y)\, dy \qquad\qquad \text{let } Y = y/|y|^2 \\
&= \frac{2}{\omega_{n-1} r^{n-2}} \int_{\mathbb{R}^n} \frac{h(Y/|Y|^2)}{|Y|^{2n}} \delta(\frac{2r}{|Y|^2}(\frac{1}{2r} - \theta \cdot Y)\, dY \\
&= \frac{1}{\omega_{n-1} r^{n-1}} \int_{\mathbb{R}^n} H(Y) \delta(\frac{1}{2r} - \theta \cdot Y)\, dY \\
&= \frac{1}{\omega_{n-1} r^{n-1}} (\mathcal{R}H)(\theta, \frac{1}{2r}).
\end{aligned}$$

Since $(\mathcal{R}H)(\theta, s) \in \mathscr{S}(S^{n-1} \times \mathbb{R})$, when r goes to 0, $\frac{1}{\omega_{n-1} r^{n-1}} (\mathcal{R}H)(\theta, \frac{1}{2r})$ will go to 0 for any $\theta \in S^{n-1}$. Also, since $(\mathcal{M}h)(x, |x|)$ is a continuous function of x, when x goes to 0, $(\mathcal{M}h)(x, |x|)$ will go to $h(0)$ which is equal to 0. Hence

$$(\mathcal{M}h)(x, |x|) = \frac{1}{\omega_{n-1} r^{n-1}} (\mathcal{R}H)(\theta, \frac{1}{2r})$$

holds for any $x \in \mathbb{R}^n$. $\square$

Remark 8.1. *For future use we make the following observations. Since $h \in \mathscr{S}_0(\mathbb{R}^n)$, as x approaches 0, $(\mathcal{M}h)(x, |x|)$ and all of its derivatives approach 0 faster than any polynomial. Also, from Lemma 8.1, we can deduce that*

$$\lim_{x \to \infty} (\mathcal{R}H)(\theta, \frac{1}{2r}) = (\mathcal{R}H)(\theta, 0)$$

and

$$\lim_{x \to \infty} |x|^k (\mathcal{M}h)(x, |x|) = \lim_{x \to \infty} \frac{|x|^{k+1-n}}{\omega_{n-1}} (\mathcal{R}H)(\theta, \frac{1}{2r}) = 0, \qquad \text{if } k = 0, 1, \cdots, n-2.$$

We now show the isometry between a function h and $(\mathcal{M}h)(x,|x|)$ - the spherical averages of h over spheres through the origin.

Theorem 8.1 (Isometry for spherical means). *Let $h(x) \in \mathscr{S}_0(\mathbb{R}^n)$. For odd n we have*

$$\int_{\mathbb{R}^n} \left(h(x)|x|^{n-2}\right)^2 \, dx = \frac{\pi}{2^{3n-4}\Gamma^2(n/2)} \int_{S^{n-1}} \int_{\mathbb{R}} \left(\partial_s^{\frac{n-1}{2}} \left(\frac{1}{|s|^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{|2s|}) \right) \right)^2 \, ds \, d\theta.$$

and for even n we have

$$\int_{\mathbb{R}^n} \left(h(x)|x|^{n-2}\right)^2 \, dx =$$
$$\frac{(-1)^{\frac{n}{2}}(n-1)!}{2^{3n-4}\Gamma^2(n/2)} \int_{S^{n-1}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\mathcal{M}h)(\frac{\theta}{2(s+t)}, \frac{1}{2|s+t|})(\mathcal{M}h)(\frac{\theta}{2t}, \frac{1}{|2t|}) \, dt \right) s^{-n} \, ds \, d\theta.$$

Proof. For $h \in \mathscr{S}_0(\mathbb{R}^n)$, define

$$H(X) = h(X/|X|^2)/|X|^{2n-2}.$$

From Lemma 8.1, for any $\theta \in S^{n-1}$, $s \in \mathbb{R}, s \geq 0$, we have

$$(\mathcal{R}H)(\theta, s) = \frac{\omega_{n-1}}{(2s)^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2s}).$$

When $s < 0$, noting that

$$\begin{aligned}(\mathcal{R}H)(\theta, s) &= (\mathcal{R}H)(-\theta, -s) \\ &= \frac{\omega_{n-1}}{(-2s)^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{-2s}),\end{aligned}$$

we obtain

$$(\mathcal{R}H)(\theta, s) = \frac{\omega_{n-1}}{|2s|^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{|2s|}), \quad \forall \theta \in S^{n-1}, \ s \in \mathbb{R}.$$

By Theorem 2.2, when n is odd, $\mathcal{R}H$ has the isometry

$$\int_{\mathbb{R}^n} |H(X)|^2 \, dX = \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} \int_{\mathbb{R}} \left| \partial_s^{\frac{n-1}{2}} (\mathcal{R}H)(\theta, s) \right|^2 \, ds \, d\theta.$$

Since

$$\int_{\mathbb{R}^n} |H(X)|^2 \, dX = \int_{\mathbb{R}^n} \left(h(x)|x|^{2n-2}\right)^2 \frac{1}{|x|^{2n}} dx = \int_{\mathbb{R}^n} \left(h(x)|x|^{n-2}\right)^2 \, dx,$$

and

$$\int_{S^{n-1}}\int_{\mathbb{R}}\left|\partial_s^{\frac{n-1}{2}}(\mathcal{R}H)(\theta,s)\right|^2\ ds\ d\theta=\frac{\omega_{n-1}^2}{2^{2n-2}}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\partial_s^{\frac{n-1}{2}}\left(\frac{1}{|s|^{n-1}}(\mathcal{M}h)(\frac{\theta}{2s},\frac{1}{|2s|})\right)\right)^2\ ds\ d\theta,$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\left(h(x)|x|^{n-2}\right)^2\ dx&=\frac{\omega_{n-1}^2}{2^{3n-2}\pi^{n-1}}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\partial_s^{\frac{n-1}{2}}\left(\frac{1}{|s|^{n-1}}(\mathcal{M}h)(\frac{\theta}{2s},\frac{1}{|2s|})\right)\right)^2\ ds\ d\theta\\&=\frac{\pi}{2^{3n-4}\Gamma^2(n/2)}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\partial_s^{\frac{n-1}{2}}\left(\frac{1}{|s|^{n-1}}(\mathcal{M}h)(\frac{\theta}{2s},\frac{1}{|2s|})\right)\right)^2\ ds\ d\theta.\end{aligned}$$

By Theorem 2.2, when n is even, $\mathcal{R}H$ has the isometry

$$\int_{\mathbb{R}^n}(H(X))^2\ dX=\frac{(-1)^{\frac{n}{2}}(n-1)!}{(2\pi)^n}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mathcal{R}H)(\theta,s+t)(\mathcal{R}H)(\theta,t)\ dt\right)s^{-n}\ ds\ d\theta.$$

Since

$$\begin{aligned}&\int_{S^{n-1}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mathcal{R}H)(\theta,s+t)(\mathcal{R}H)(\theta,t)\ dt\right)s^{-n}\ ds\ d\theta\\&=\frac{\omega_{n-1}^2}{2^{2n-2}}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mathcal{M}h)(\frac{\theta}{2(s+t)},\frac{1}{2|s+t|})(\mathcal{M}h)(\frac{\theta}{2t},\frac{1}{|2t|})\ dt\right)s^{-n}\ ds\ d\theta,\\&=\frac{\pi^n}{2^{2n-4}\Gamma^2(n/2)}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mathcal{M}h)(\frac{\theta}{2(s+t)},\frac{1}{2|s+t|})(\mathcal{M}h)(\frac{\theta}{2t},\frac{1}{|2t|})\ dt\right)s^{-n}\ ds\ d\theta,\end{aligned}$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(h(x)|x|^{n-2}\right)^2\ dx=\\&\frac{(-1)^{\frac{n}{2}}(n-1)!}{2^{3n-4}\Gamma^2(n/2)}\int_{S^{n-1}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mathcal{M}h)(\frac{\theta}{2(s+t)},\frac{1}{2|s+t|})(\mathcal{M}h)(\frac{\theta}{2t},\frac{1}{|2t|})\ dt\right)s^{-n}\ ds\ d\theta.\end{aligned}$$

□

Next we show how to recover a function from its spherical mean values over all spheres through the origin.

Theorem 8.2 (Inversion formula). *Let $h\in\mathscr{S}_0(\mathbb{R}^n)$. For odd n we have*

$$h(x)=\frac{(-1)^{\frac{n-1}{2}}}{2^{2n-2}\pi^{n/2-1}\Gamma(n/2)|x|^{2n-2}}\int_{S^{n-1}}\frac{\partial^{n-1}}{\partial s^{n-1}}\left(\frac{(\mathcal{M}h)(\frac{\theta}{2s},\frac{1}{2|s|})}{|s|^{n-1}}\right)\Big|_{s=X\cdot\theta}\ d\theta,$$

and for even n we have

$$h(x) = \frac{(-1)^{\frac{n}{2}}}{2^{2n-2}\pi^{n/2}\Gamma(n/2)|x|^{2n-2}} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{1}{q} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}} \right) \Big|_{s=X\cdot\theta+q} \, dq \, d\theta.$$

where $X = x/|x|^2$.

Proof. For $h \in \mathscr{S}_0(\mathbb{R}^n)$, define $H(X) = h(X/|X|^2)/|X|^{2n-2}$; then $H(X) \in \mathscr{S}_0(\mathbb{R}^n)$. By Lemma 8.1, we have

$$(\mathcal{R}H)(\theta, \frac{1}{2r}) = \omega_{n-1} r^{n-1} (\mathcal{M}h)(r\theta, r), \quad \forall \theta \in S^{n-1}, r \in [0, \infty].$$

Let $s = \frac{1}{2r}$, then

$$(\mathcal{R}H)(\theta, s) = \frac{\omega_{n-1}}{(2s)^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2s}), \quad \forall s \geq 0.$$

When $s < 0$, noting that $(\mathcal{R}H)(\theta, s) = (\mathcal{R}H)(-\theta, -s)$, hence

$$(\mathcal{R}H)(\theta, s) = (\mathcal{R}H)(\theta, s) = \frac{\omega_{n-1}}{(-2s)^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{-2s}), \quad \forall s \leq 0. \tag{8.1}$$

Hence

$$(\mathcal{R}H)(\theta, s) = \frac{\omega_{n-1}}{(2|s|)^{n-1}} (\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|}), \quad \forall s \geq \mathbb{R}.$$

The Radon inversion formula (see Theorem 2.4) for odd n is

$$H(X) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial s^{n-1}} (\mathcal{R}H)(\theta, s)|_{s=X\cdot\theta} \, d\theta.$$

Therefore

$$\begin{aligned} h(x) &= \frac{(-1)^{\frac{n-1}{2}} \omega_{n-1}}{2^{2n-1}\pi^{n-1}|x|^{2n-2}} \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}} \right) \Big|_{s=X\cdot\theta} \, d\theta \\ &= \frac{(-1)^{\frac{n-1}{2}}}{2^{2n-2}\pi^{n/2-1}\Gamma(n/2)|x|^{2n-2}} \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}} \right) \Big|_{s=X\cdot\theta} \, d\theta, \end{aligned}$$

when n is odd.

When n is even, the Radon inversion formula is (see Theorem 2.4)

$$H(X) = \frac{(-1)^{\frac{n}{2}}}{(2\pi)^n} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{1}{q} \frac{\partial^{n-1}}{\partial s^{n-1}} (\mathcal{R}H)(\theta, s)|_{s=X\cdot\theta+q} \, d\theta,$$

which is a Cauchy principal value integral. So when n is even, we have the inversion formula

$$
\begin{aligned}
h(x) &= \frac{(-1)^{\frac{n}{2}}\omega_{n-1}}{2^{2n-1}\pi^n|x|^{2n-2}} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{1}{q} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}} \right) \Big|_{s=X\cdot\theta+q} \, dq \, d\theta \\
&= \frac{(-1)^{\frac{n}{2}}}{2^{2n-2}\pi^{n/2}\Gamma(n/2)|x|^{2n-2}} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{1}{q} \frac{\partial^{n-1}}{\partial s^{n-1}} \left(\frac{(\mathcal{M}h)(\frac{\theta}{2s}, \frac{1}{2|s|})}{|s|^{n-1}} \right) \Big|_{s=X\cdot\theta+q} \, dq \, d\theta.
\end{aligned}
$$

□

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