

“I’M LIKE THE SHERPA GUIDE”: ON LEARNING TO TEACH PROOF IN SCHOOL MATHEMATICS

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This article describes the experiences of a beginning mathematics teacher, Matt, across his first three years of teaching proof in a high school geometry course. Matt’s past experiences with mathematics influenced his beliefs about what he could and could not do to help his students learn how to prove. During his first year of teaching proof, Matt claimed that you cannot teach someone to write a proof. Over time, however, Matt eventually developed some strategies for teaching proof to his students. Within this work is an interest in learning more about how a teacher learns to teach proof to students who are just learning how to construct a formal proof. This case highlights the importance of pedagogical content knowledge.

Learning to think and reason both formally and informally is an important goal in the mathematics classroom. On the formal end of reasoning, students must learn to understand and write a proof (NCTM, 2009). Over the past few decades, proof has been given increased attention in many countries around the world (see, e.g., Knipping, 2004). This is primarily because “proof is the basis of mathematical understanding and is essential for developing, establishing, and communicating mathematical knowledge” (Stylianides, 2007, p. 191). In the *Reasoning and Sense-Making* document (NCTM, 2009), formal reasoning (i.e., proof) was situated as the final of three stages in the reasoning progression required for increasing levels of understanding in the high school mathematics classroom. The authors pointed out that the effort to help students progress from less formal to more formal reasoning requires that “teachers play an essential role in encouraging students to explore more sophisticated levels of reasoning and sense making” (p. 11). One might wonder, however, how and how well are teachers being prepared to play this essential role? A more relevant question to this study might be: Is prior experience with mathematical proof as a student sufficient preparation for teaching it?

In this paper, I use data from a longitudinal case study designed to learn more about how a beginning teacher learns to teach proof in Euclidean geometry to address this question. At the onset of the study, Matt (a pseudonym) was teaching proof in geometry for the first time. Here I address the following research questions: (1) How did Matt introduce proof to his students? (2) What limitations did Matt believe that he had with regard to teaching proof? (3) What strategies did Matt develop to overcome these limitations?

Before I explore these questions, I review some literature on learning to teach mathematics and on proof as problem solving. After, discussing the methodology of the study, I present and discuss some findings.

LEARNING TO TEACH MATHEMATICAL PROOF

Shulman (1986) described three types of knowledge that are necessary for effective teaching: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. According to Shulman (1986), to present specific content to particular students, teachers need a special blend of content and pedagogy that he referred to as “pedagogical content knowledge.” This includes the ways of representing and reformulating the subject that make it comprehensible to students (Shulman, 1986). Influenced, in part, by Shulman’s conceptualization of pedagogical content knowledge, researchers in the 1980s and 90s sought to identify what teachers know (or should know) to teach mathematics (Hill, Sleep, Lewis, & Ball, 2007). This area is an important body of work that has provided frameworks to investigate the various kinds of knowledge that teachers must acquire to maximize student learning. In the interest of brevity, however, in this paper, discussions of mathematics knowledge for teaching will include only references to Shulman’s subject matter content knowledge (CT) and pedagogical content knowledge (PCK). Of particular interest in this paper is the knowledge needed to teach mathematical proof.

A number of studies have already reported that proof is a difficult topic, both for students to learn (e.g., Senk, 1985) and for teachers to teach (e.g., Knuth, 2002). Some research has suggested that perhaps the reason that teachers have not moved their students beyond the traditional two-column approach to proof is related to teachers’ beliefs about the purpose of proof and their students’ abilities to complete a proof (Knuth, 2002). Additionally, teachers may not have had opportunities to consider alternative ways of teaching proof that fall outside of the “apprenticeship of observation” (Lortie, 1975) experienced in their own mathematics backgrounds. Finally, when we think of proof as problem solving, it is easy to understand why it is a challenging area in mathematics education.

PROOF AS PROBLEM SOLVING

A review of the current proof literature illustrates that some researchers are beginning to take the stance that proving is a form of problem solving. By its very definition, a task is only a “problem” when there is no immediate, clear solution or a known path or strategy that sheds light on the appropriate mathematical action required to complete the task (Weber, 2005). Weber (2005) argued that “focusing on the problem-solving aspects of proving allows insight into some important themes that other perspectives on proving do not address” (p. 352). One example of such a theme is the exploration of reasons that students reach impasses in proof where they do not know how to proceed (Schoenfeld, 1985).

In order to solve a proving task that is truly a “problem” as described above, successful students eventually have a breakthrough where they progress from not seeing a path or strategy to developing one that will assist them in writing a correct proof. These kinds of breakthroughs have been described in the literature. For example, Barnes (2002) wrote about a student named Naidra who described his lack

of insight on one particular day as not having “anything magical” happen. When pressed further, Naidra said that “flashes of understanding can happen” and “lots of different things can spark that off” (p. 83). This sudden flash of understanding that Naidra described as magical is often referred to as an ‘Aha!’ or ‘Eureka!’ experience (Barnes, 2002). Mathematicians writing about the creative process have also described these kinds of moments. For example, Polya (1965) wrote about “a sudden clarification that brings light, order, connection and purpose to details which before appeared obscure, confused, scattered, and elusive” (p. 54). In the context of this study, these descriptions beg the question: What can teachers do to support their students in having these kinds “magical,” “aha” discoveries when they are first learning to prove?

“Discovering” a Proof

The idea that there are different phases or activities in proving has been tacitly acknowledged by various sources. For example, in textbooks, the problem solving aspect of proving has been called developing a “Plan for Proof” (Larson, Boswell, & Stiff, 2001), “analyzing a proof” (CME project), “scratch work” (Velleman, 2006), and so forth. The idea that *doing* a proof and *writing* a proof are two different activities was explicitly noted by Farrell (1987) who portrayed both of these activities as important. The *doing* requires good problem solving skills, and the *writing* requires rigor and precision. Farrell claimed, however, that prospective teachers needed to learn that the *writing* takes a back seat to the generation of ideas. Because I call on Herbst and Brach’s (2006) work related to “doing proofs,” which they describe as the range of practices carried out by students and their teacher, I do not reference the problem solving part of proving as “doing” a proof as Farrell did. Rather, I refer to the problem solving, finding a proof phase of proving as *developing* a proof. As Farrell noted, this activity is the more difficult phase of proving. The development precedes writing up the proof, an activity that is important in terms of mathematical communication, however, it is more about expressing yourself clearly, rather than a problem solving endeavour. This construct is useful for describing a practice of this study’s participating teacher.

METHODOLOGY

This longitudinal interpretive case study (Stake, 1995) focuses on the classroom experiences of a high school geometry teacher, Matt, over a three-year period (2005-2007) in which Matt taught Euclidean proof to students (ages 15-16) in the regular track of a geometry course. At the beginning of the study, he had just taken a new position at a public high school in a suburb of a large U.S. city. Matt was chosen to participate in this study because of his new teacher status, his willingness to share his experiences, and his interest in studying his own practice. He provides an interesting case because he had a strong mathematics background as well as a Masters degree in teaching. According to the teacher preparation literature, Matt’s background represents the “best-case scenario” (Gay, 1994) in terms of beginning high school

teacher preparation. Therefore, the case of Matt presents a “well-prepared” teacher who is learning to teach proof in school mathematics.

Data Collection and Analysis

Data was collected across three years and analysed using qualitative methods. The primary data sources were classroom observations, in-situ field notes, and interviews with Matt. All interviews were semi-structured and audio recorded. For three years, I visited Matt’s classroom during lessons when he introduced proof to his students. Each lesson was audio and video recorded. In the interest of illustrating change over time, I only report on the classroom observations from Years 1 and 3 here.

FINDINGS AND DISCUSSION

In the interest of space, in this section, I present data intertwined with some brief discussion. I first describe Matt’s early experiences with mathematical proof in high school and at the university in order to shed some light on Matt’s preparation for teaching proof. I then describe the ways that Matt introduced proof to his students during Year 1 (Y1) and Year 3 (Y3) of this study. Finally, I provide some interview data to shed light on the changes observed between Y1 and Y3. Following the presentation of findings and discussion, I close with some concluding thoughts.

Matt’s Early Experiences with Proof

Matt did not follow a traditional path through mathematics in high school. He completed geometry in the 8th grade (age 13-14) as an independent study which was two years earlier than most students in the United States. Matt said that he was never asked to develop a proof during his school mathematics experience, and he did not recall even being shown a proof in high school. As a mathematics major in college, however, Matt said:

I was immediately asked to do all sorts of proofs, which now, looking back at it, I can see as not being so bad, but at the time I’m like, this is a joke. I’m like, this is impossible. You know, you can’t do this? (Interview, 6/21/06)

The difficult transition that Matt experienced from school to undergraduate mathematics is not uncommon. The paucity of proof in school mathematics coupled with the fact that even in the lower-level university courses, few, if any, proofs are required of students (Moore, 1994) helps us understand why Matt felt that developing proofs was “impossible.” During Y1, Matt compared the challenge of doing his first proof (as a student) to walking through a wall. This, he said, caused him to rethink his major in mathematics. These comments may seem surprising given that Matt was clearly above-average in school mathematics, evidenced by (among other things) his being two years ahead in his studies prior to graduating from high school. As Moore (1994) explained, however, “This abrupt transition to proof is a source of difficulty for many students, even for those who have done superior work with ease in their lower-level mathematics courses” (p. 249). Matt said that even though he did not take any sort of an introductory proof course, eventually he was “able to understand or

believe that [proof] was something that [he] could do” (Interview, 6/21/06). The experiences described here caused Matt to begin to think about mathematics in ways that were different from his conception of mathematics prior to his university coursework. Next, I describe the ways Matt introduced proof to his own students.

Introducing Proof to His Students

Year 1. When presenting the first proof to his students in Y1, Matt told a story about Pokémon (an anime series, film, and video game from the U.S. and Japan):

If [the line segments] have the same length, then they have to be congruent. So, the definition of congruence, I choose you...Nobody in here watches Pokémon? Ever? Are you kidding me? Are you serious, nobody watches Pokémon?...We’re gonna have to rent it. Alright? Pichachu, I choose you. Right? That’s how you wanna think about this. I remember this. In college, my roommate one time, he was a good friend of mine....And we’re sitting there and one of his internet browsers wasn’t working, so he totally decides to switch his internet browser, and of a sudden we’re sitting there working and he goes “Minsky, I choose you.”...That was really funny. But I remembered that last night. That’s the way you want to think about this, right? Definition of congruence. Go, right? Symmetric property. Go. Definition of congruence. Go. Now I’m done, right? That’s how we proved this. Okay. (Y1, 9/30/05)

In this example, in the absence of tools to introduce proof, Matt attempted to connect with the students by referencing Pokémon. Even after realizing that the students did not understand the reference, Matt continued to connect to Pokémon, saying “Symmetric property. Go. Definition of congruence. Go.” Also, in Y1, the students were not given very many opportunities to participate in the development of proofs.

Year 3. Three specific changes were observed in the way that Matt introduced proofs in Y3: (a) what Matt wanted students to do before they started writing their proofs; and (b) the flexibility Matt stressed related to the *form* of the proof (c) the confidence shown in the way that he spoke about proof which did not involve seemingly random analogies. Rather than using the example proofs from his textbook (as he did in Y1), in Y3, Matt wanted to start with a proof that was “more interesting.” Matt began the lesson by talking about what the students should do before they write a proof:

Before you ever write a proof, you want to make sure that you can convince yourself that it’s true, okay? No one learns anything by writing a proof. They just write down what they already know has to be true. So look at number 12, here. Look at that problem for 25 seconds. See if you can convince yourself that it has to be true. (Y3, 9/21/07)

Rather than Matt immediately demonstrating proof as he did in Y1, Matt gave the students time to think about the proof. Matt attempted to involve students by giving them this time and then *asking* them if they were convinced of the truth of the proposition that they were supposed to prove. He then asked the students *why* the statement was true, and then he called on a student to provide an explanation. Matt

the proceeded to tap into the students' thinking as he simultaneously led them through a two-column and a flow proof of the theorem.

Exploring Observed Changes through Interviews

During an interview at the end of Y1, Matt discussed how students either see or do not see how a particular proposition can be proved.

To do a proof in a real mathematical way is very, it's very isolating. You can't teach somebody how to do a proof....I mean if a student's really gonna do a mathematical proof, you look at the problem and you either see how you do it or you don't. After that, the writing it down, although an important exercise in communication really is sort of pointless. I mean it's not pointless, but it's trivial. You know. If you can see how to prove something, then you can see how to explain it to somebody else and the seeing or not seeing it is nothing that I can teach you. (Interview, 6/21/06)

After hearing Matt say that "seeing it is nothing that I can teach you," I asked him if there was anything that he could do, as a teacher, to provide students access so that they could progress at the pace that was dictated by the demands of the school context. To this question, Matt replied:

I mean you don't want to go so far as to say it doesn't matter what I do, but the reality is that I can't prove it for them. You know, simply showing somebody how to do a proof will help, but only up to a certain point. Only until they understand...the way in which a proof becomes a proof. (Interview, 6/21/06)

Here, Matt expressed what he saw as a limitation for him as the teacher. After teaching proof for the second time, I, again, asked Matt about the comment, "seeing it is nothing that I can teach you." I was curious as to whether Matt still believed that there was nothing or even very little that he could do to help students learn to prove. I was interested in his answer to this question because, at that point, I had observed Matt teach proof for the second time, and he had made changes that I thought might be designed to help his students "see it," whereas the previous year he said that there was very little that he could do. I was trying to understand if there was a shift in his thinking. After discussing the analogy of teacher as coach, which did not seem to resonate with him, Matt initiated a new analogy:

I'm like a Sherpa. Okay? That's the word I'm looking for. So...you know, I've been up and down the mountain 50 times. And if you didn't have me, you could make it to the top of the mountain. 'Cause I'm not a requirement, right? But it'll probably be a lot uglier and take a lot longer. And, there's a good possibility that you would freeze to death and never get to the top. Right? So. Yeah, I'm like the Sherpa guide who like, you know, just walks with you up the mountain, but then at base camp I just, I go off and meditate somewhere else and I really don't pay attention to what you're doing. Right?....And I don't just have one person, right? I'm trying to herd like 30 people to the top of the mountain before next Friday. (Interview, 4/19/07)

So, although, Matt could not climb the mountain for his students in the same way that he could not “see it” or “prove it for them.” He seemed to view his role as one of being there and knowing (or believing) from experience that it was possible to get to the top of the mountain. He also noted the reality of the classroom when he said that he had to herd 30 people to the top of the mountain “before next Friday.”

Significance

Although there is widespread agreement that novice teachers lack a number of important skills, only a few researchers have sought to understand how beginning teachers develop their knowledge of and for teaching (Brown, 1993). Researchers in the area of science education are beginning to explore the challenges that new science teachers face as they begin their teaching careers (Luft, 2007). Similar to Luft’s (2007) work with new science teachers, studies such as this one are important because they reveal the complexity of being a beginning mathematics teacher in the context and setting in which the new teacher works. In this study, data were presented to illustrate the ways in which CK is not necessarily sufficient preparation to teach proof. Even with a strong mathematics background, Matt still struggled to develop tools to support his students through the discovering phase of doing proofs. This study illustrates the need for additional studies that seek to observe teachers introducing and cultivating proof. It could be helpful to understand what successful, experienced teachers do to scaffold proof-development practices in their classrooms. In practice, more support should be provided to beginning teachers in their preparation to help their students develop proofs.

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