BOUNDARY INTEGRAL METHODS IN LOW FREQUENCY ACOUSTICS

George C. Hsiao *
Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716, USA

Wolfgang L. Wendland
Mathematisches Institut A
Universität Stuttgart
Pfaffenwaldring 57
D-70569 Stuttgart, FRG

Key Words: boundary integral equations, the Calderon projector low frequency acoustics, asymptotic expansion.

Abstract

This expository paper is concerned with the direct integral formulations for boundary value problems of the Helmholtz equation. We discuss unique solvability for the corresponding boundary integral equations and its relations to the interior eigenvalue value problems of the Laplacian. Based on the integral representations, we study the asymptotic behaviors of the solutions to the boundary value problems when the wave number tends to zero. We arrive at the asymptotic expansions for the solutions, and show that in all the cases, the leading terms in the expansions are always the corresponding potentials for the Laplacian. Our integral equation procedures developed here are general enough and can be adapted for treating similar low frequency scattering problems.

1 Boundary value problems

Let Ω be a bounded domain in $I\!R^n, n = 2, 3$ with a smooth boundary Γ and $\Omega^c := I\!R^n \setminus \overline{\Omega}$ be its exterior domain. We begin with the Helmholtz equation

$$\Delta u + k^2 u = 0 \;\text{ in }\; \Omega \;(\text{or } \Omega^c)\;, \tag{1}$$

*Correspondence addressee

where $\Delta = \nabla^2$ denotes the standard Laplace operator in $I\!R^2, n = 2, 3$. This equation arises in connection with the wave propagation. In acoustics, $k := \omega/c$ denotes the complex wave number, and u corresponds to the acoustic pressure field. Here ω and c are the frequency and the speed of sound. In order to avoid resonance states, we assume that $Im\, k \geq 0$. We are interested in the solution of (1), when the wave number k is small, the low frequency acoustics. We present here some results concerning the low frequency behavior of the solution of (1) developed in [9] and in the recent work [10]. These results are obtained by using boundary integral equation methods. The solution behaviors depend on the specific boundary conditions to be considered : the *Dirichlet boundary condition*

$$u|_\Gamma = \varphi, \tag{2}$$

or the *Neumann boundary condition*:

$$\frac{\partial u}{\partial n}|_\Gamma = \psi. \tag{3}$$

Here and in the sequel, $\partial/\partial n$ always denotes the normal derivative with respect to the unit outward normal to Γ. The functions φ and ψ are given data. In acoustic scattering, (2) and (3) are the conditions for modeling the situations for *soft* and *hard* scatterers, respectively.

For the equation (1) in the exterior domain Ω^c, one requires the so–called *Sommerfeld radiation conditions*,

$$u(x) = \mathrm{O}(|x|^{-(n-1)/2}) \text{ and } \frac{\partial u}{\partial\,|x|}(x) - iku(x) = o\left(|x|^{-(n-1)/2}\right), \tag{4}$$

where i is the imaginary unit.(see, e.g., [5]). These conditions select the outgoing waves; they are needed for uniqueness of the exterior Dirichlet and Neumann problems. The pointwise condition (4) can be replaced by a more appropriate and weaker version of the radiation condition given by F.Rellich,

$$\lim_{R\to\infty} \int\limits_{|x|=R} |\frac{\partial u}{\partial n}(x) - iku|^2 ds = 0\,. \tag{5}$$

This form is to be used in the variational formulation of exterior boundary value problems.

In this paper we are confined to the following four classes of boundary value problems:

The interior Dirichlet problem (IDP), (1) in Ω, (2),
The exterior Dirichlet problem (EDP), (1) in Ω^c, (2), (4),
The interior Neumann problem (INP), (1) in Ω, (3),
The exterior Neumann problem (ENP), (1) in Ω^c, (3), (4).

In the next section, we consider the boundary potentials associated with the equation (1) and discuss the reductions of the these boundary value problems to the boundary integral equations. Section 3 contains four basic boundary integral operators. We introduce the Calderon projectors and give the basic mapping properties. The solvabilities of the boundary integral equations will also be discussed. These are essential and can be served as the mathematical foundations for the boundary element methods. Our main results concerning the low frequency behaviors of the solutions to the boundary value problems are presented in Section 4.

2 Boundary integral operators

To reduce the boundary value problems to boundary integral equations, we begin with the Green representation for the solution of (1)

$$\begin{aligned} u(x) &= \pm\left\{\int_\Gamma E_k(x,y)\frac{\partial u}{\partial n}(y)ds_y - \int_\Gamma u(y)\frac{\partial E_k(x,y)}{\partial n_y}ds_y\right\} \\ &:= \pm\left\{\mathcal{V}_k\frac{\partial u}{\partial n}(x) - \mathcal{W}_k u(x)\right\} \quad \text{for all } \; x \in \begin{cases}\Omega, \\ \Omega^c,\end{cases} \end{aligned} \tag{6}$$

where the $\pm$ sign corresponds to the interior and the exterior domain, respectively. Here, $\mathcal{V}_k$ and $\mathcal{W}_k$ are referred to as the *single-* and *double- layer potentials*, and $E_k(x,y)$ is the fundamental solution of the Helmholtz equation defined by

$$E_k(x,y) = \begin{cases} \dfrac{i}{4}H_0^{(1)}(k|x-y|) & \text{in } I\!\!R^2, \\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|} & \text{in } I\!\!R^3, \end{cases} \tag{7}$$

where $H_0^{(1)}$ denotes the modified Bessel function of the first kind. We note that for $n=2$, $E_k(x,y)$ has a branch point for $\mathbb{C} \ni k \to 0$. In the representation formula (6), the traces $\mu^\pm := u^\pm|_\Gamma$ and $\sigma^\pm := \frac{\partial u}{\partial n}^\pm|_\Gamma$ are the *Cauchy data* of the solution u on Γ. We have denoted by v^+ and v^- the restriction $v|_\Gamma$ from Ω and Ω^c, respectively. These Cauchy data are related by the boundary integral equations:

$$\begin{pmatrix} \mu^\pm \\ \sigma^\pm \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I \mp K_k & \pm V_k \\ \pm D_k & \frac{1}{2}I \pm K_k^{'} \end{pmatrix} \begin{pmatrix} \mu^\pm \\ \sigma^\pm \end{pmatrix} \quad \text{on} \quad \Gamma. \tag{8}$$

Here V_k, K_k, K_k, D_k are the four basic boundary integral operators defined by

$$V_k\sigma(x) := \int_\Gamma E_k(x,y)\sigma(y)ds_y \quad , \quad K_k\mu(x) := \int_\Gamma \frac{\partial}{\partial n_y}E_k(x,y)\mu(y)ds_y,$$

$$K_k'\sigma(x) := \int_\Gamma \frac{\partial}{\partial n_x}E_k(x,y)\sigma(y)ds_y \quad , \quad D_k\mu(x) := -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}E_k(x,y)\mu(y)ds_y,$$

and these are the *boundary potentials*. The matrices of boundary integral operators

$$\mathcal{C}_{\pm} := \begin{pmatrix} \frac{1}{2}I \mp K_k & \pm V_k \\ \pm D_k & \frac{1}{2}I \pm K_k^{'} \end{pmatrix} \tag{9}$$

are referred to as the *Calderon projectors* with respect to the domain Ω and Ω^c respectively. The Calderon projector maps the Cauchy data into itself. We note that the solution u in the domain Ω (or Ω^c) is completely determined from the representation (6), provided one knows its Cauchy data on the boundary Γ. In the classical Hölder function spaces, the boundary integral operators in (9) have the mapping properties as follows.

Theorem 2.1 *Let $\Gamma \in C^2$ and $0 < \alpha < 1$, a fixed constant. Then the boundary integral operators in* (9) *define continuous mappings in the following spaces,*

$$\begin{array}{rcl} V_k & : & C^{\alpha}(\Gamma) \longrightarrow C^{1+\alpha}(\Gamma), \\ K_k & : & C^{1+\alpha}(\Gamma) \longrightarrow C^{2+\alpha}(\Gamma), \\ K_k' & : & C^{\alpha}(\Gamma) \longrightarrow C^{1+\alpha}(\Gamma), \\ D_k & : & C^{1+\alpha}(\Gamma) \longrightarrow C^{\alpha}(\Gamma). \end{array}$$

Similar mapping properties are also available in the Sobolev spaces (see, e.g., [10]).

Theorem 2.2 *For $\Gamma \in C^2$, the following operators are continuous for $|s| \leq 1/2$:*

$$\begin{array}{rcl} V_k & : & H^{-1/2-s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \\ K_k & : & H^{1/2+s}(\Gamma) \longrightarrow H^{3/2+s}(\Gamma), \\ K_k' & : & H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \\ D_k & : & H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma). \end{array}$$

We remark that for smooth boundary, $\Gamma \in C^{\infty}$, the above theorem remains valid for $s \in I\!R$, while for the Lipschitz domain, $\Gamma \in C^{0,1}$, again $|s| \leq 1/2$, but K_k and K_k' are only continuous as the mappings $K_k : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma)$, and $K_k' : H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma)$ (see, e.g., [6]).

3 Exceptional or irregular frequencies

From (8) we see that the Cauchy data of a solution of (1) in Ω (or Ω^c) are related to each other by two boundary integral equations. As is well known, for regular elliptic boundary value problems only half of the Cauchy data on Γ is given. For the remaining part, the two equations from (8) define an over determined system of boundary integral equations which may be used for determining the complete Cauchy data. In general any combination of them can serve as a boundary integral equation for the missing part of the Cauchy data. Hence the reduction from a boundary value problem to a boundary integral equation is by no means a unique process. However, the so-called direct approach for formulating boundary integral equations becomes rather simple, if one considers the Dirichlet or the Neumann problem. Each one of the boundary integral equations in (8) can be employed for these problems.

For the interior and exterior *Dirichlet problems*, (IDP) and (EDP),

$$\mu = \varphi = u_{|\Gamma} \quad \text{on } \Gamma \text{ is given} .$$

Here the missing Cauchy datum on Γ is $\sigma = \frac{\partial u}{\partial n}_{|\Gamma}$. Thus, for instance, we may use either the boundary integral equation of the first kind

$$V_k\sigma(x) = \frac{1}{2}\varphi(x) + K_k\varphi(x), \quad x \in \Gamma\,, \tag{10}$$

or boundary integral equation of the second kind

$$(\frac{1}{2}I - K_k')\sigma(x) = D_k\varphi(x), \quad x \in \Gamma \tag{11}$$

for the unknown σ in case of (IDP). On the other hand, for the Neumann Problems, (INP) and (ENP),

$$\sigma = \psi = \frac{\partial u}{\partial n}|_\Gamma \quad \text{on } \Gamma \text{ is given} .$$

Here the missing Cauchy datum on Γ is $\mu = u|_\Gamma$. Then, for the (INP), the corresponding boundary integral equations for the unknown μ now read

$$D_k\mu(x) = (\frac{1}{2}I - K_k')\psi(x), \quad x \in \Gamma, \tag{12}$$

$$(\frac{1}{2}I + K_k)\mu(x) = V_k\psi(x), \quad x \in \Gamma. \tag{13}$$

We have collected all these formulations in the second column in Table 1.

The unique solvability for these boundary integral equations are important. In particular, for $k \neq 0$ and for given $\varphi \in C^{1+\alpha}(\Gamma)$, (10) is uniquely solvable with $\sigma \in C^\alpha(\Gamma)$, except for certain values of $k \in \mathbb{C}$ which are the so–called *exceptional* or *irregular frequencies* of the boundary integral operator V_k. For any irregular frequency k_0, the operator V_{k_0} has a nontrivial null space $ker\ V_{k_0} = span\ \{\sigma_{0j}\}$. The eigensolutions σ_{0j} are related to the eigensolutions u_{0j} of the *interior Dirichlet problem for the Laplacian*,

$$\begin{array}{rcl} -\Delta u_0 &=& k_0^2 u_0 \text{ in } \Omega\,, \\ u_{0|\Gamma} &=& 0 \text{ on } \Gamma\,, \end{array} \tag{14}$$

according to

$$\sigma_{0j} = \frac{\partial u_{0j}}{\partial n}|_\Gamma\,.$$

Moreover, the solutions are real–valued and

$$\dim kerV_{k_0} = \text{ dimension of the eigenspace of (14).}$$

As is known, the eigenvalue problem (14) admits denumerable infinitely many eigenvalues k_{0l}^2. They are all real and have at most finite multiplicity. Moreover, they can be ordered according to size $0 < k_{01}^2 < k_{02}^2 < \cdots$ and have $+\infty$ as their only limit point. When k_0 is an eigenvalue, (10) admits solutions in $C^\alpha(\Gamma)$ if and only if the given boundary values $\varphi \in C^{1+\alpha}(\Gamma)$ satisfy the *orthogonality conditions*

$$\int_\Gamma \varphi\sigma_0 ds = \int_\Gamma \varphi \frac{\partial u_0}{\partial n} ds = 0 \quad \text{for all } \sigma_0 \in kerV_{k_0} \,. \tag{15}$$

Correspondingly, for $\varphi \in C^{1+\alpha}(\Gamma)$, the boundary integral equation (10) has solutions $\sigma \in C^\alpha(\Gamma)$ if and only if (15) is satisfied.

For the *exterior Dirichlet problem* (IDP), from (8) again we obtain a boundary integral equation of the first kind,

$$V_k\sigma(x) = -\frac{1}{2}\varphi(x) + K_k\varphi(x)\,,\ x \in \Gamma\,, \tag{16}$$

which differs from (10) only by a sign in the right–hand side. Hence, the exceptional values k_0 are the same as for the interior Dirichlet problem, namely the eigenvalues of (14). If $k \neq k_0$, (16) is always uniquely solvable for $\sigma \in C^\alpha(\Gamma)$, provided $\varphi \in C^{1+\alpha}(\Gamma)$. For $k = k_0$, in contrast to (IDP), (EDP) remains uniquely solvable. However, (16) now has eigensolutions, and the right–hand side always satisfies the orthogonality conditions

$$\begin{aligned}\int_{x\in\Gamma} &\left(-\frac{1}{2}\varphi(x) + K_{k_0}\varphi(x)\right)\sigma_0(x)ds_x \\ &= \int_{x\in\Gamma} \varphi(x)\left\{-\frac{1}{2}\sigma_0(x) + K'_{k_0}\sigma_0(x)\right\} ds_x = 0 \ \text{ for all } \sigma_0 \in \ker\, V_{k_0}\,,\end{aligned}$$

since σ_0 is real valued and the single layer potential $\mathcal{V}_{k_0}\sigma_0(x)$ vanishes identically for $x \in \Omega^c$. The latter implies

$$\frac{\partial}{\partial n_x}\mathcal{V}_{k_0}\sigma_0(x) = -\frac{1}{2}\sigma_0(x) + K'_{k_0}\sigma_0(x) = 0 \ \text{ for } x \in \Gamma\,.$$

Accordingly, the representation formula (6) in Ω^c with $u_{|\Gamma} = \varphi$ and $\frac{\partial u}{\partial n}_{|\Gamma} = \sigma$ will generate a *unique* solution for any σ solving (16).

The relations between the eigensolutions of the BIEs and the interior eigenvalue problems of the Laplacian are given explicitly in column three of Table 1. We observe that for the exterior boundary value problems the exceptional values k_0 and k_1 of the

BVP	BIE	Eigensolutions u_0 or u_1 for BVP and Exceptional values k_0, k_1	Eigensolutions for BIE, σ_0, σ_1 or μ_0, μ_1	Solvability Conditions for given φ, ψ
IDP	(1) $V_k\sigma = (\frac{1}{2}I + K_k)\varphi$	(D_0): $\Delta u_0 + K_0^2 u_0 = 0$ in Ω, $u_{0\vert\Gamma} = 0$ on Γ	$\sigma_0 = \frac{\partial u_0}{\partial n}_{\vert\Gamma}$	$\int_\Gamma \sigma_0 \varphi ds = 0$
	(2) $(\frac{1}{2}I - K'_k)\sigma = D_k\varphi$			
EDP	(1) $V_k\sigma = (-\frac{1}{2}I + K_k)\varphi$			None
	(2) $(\frac{1}{2}I + K'_k)\sigma = -D_k\varphi$	(N_0): $\Delta u_1 + k_1 u_1 = 0$ in Ω $\frac{\partial u_1}{\partial n}_{\vert\Gamma} = 0$ on Γ	$V_{k_1}\sigma_1 = u_1$ on Γ	
INP	(1) $D_k\mu = (\frac{1}{2}I - K'_k)\psi$		$\mu_1 = u_{1\vert\Gamma}$ on Γ	$\int_\Gamma \mu_1 \psi ds = 0$
	(2) $(\frac{1}{2}I + K_k)\mu = V_k\psi$			
ENP	(1) $D_k\mu = -(\frac{1}{2}I + K'_k)\psi$			None
	(2) $(\frac{1}{2}I - K_k)\mu = -V_k\psi$	(D_0)	$D_{k_0}\mu_0 = \frac{\partial u_0}{\partial n}_{\vert\Gamma}$	

Table 1: Summary of the boundary integral equations for the Helmholtz equation and the related eigenvalue problems

corresponding boundary integral operators depend on the type of boundary integral equations derived by the direct formulation. For instance, we see that for (EDP), k_0 are the exceptional values for V_k whereas k_1 are those for $(\frac{1}{2}I + K'_k)$.

It is worthy mentioning that for the *exterior* boundary value problems, the solvability conditions of the corresponding (BIE) at the exceptional values are always satisfied due to the special forms of the corresponding right-hand sides. For the indirect approach, this is not the case anymore (see, e.g.,[5]). There are various ways to modify the boundary integral equations so that some of the exceptional values will not belong to the spectrum of the boundary integral operator anymore. In this connection, we refer to [1], [2] and the recent work [3], [4], to name a few.

4 Low frequency behavior

Of particular interest is the case $k \to 0$ which corresponds to the low–frequency behavior. This case also determines the large–time behavior of the solution to time–dependent problems if (1) is obtained from the wave equation by the Fourier–Laplace transformation. As will be seen, some of the boundary value problems will exhibit a singular behavior for $k \to 0$.

The singular behavior can be obtained from the explicit asymptotic expansions of the boundary integral equations in Table 1. The latter then follows directly from the series development of the fundamental solutions and their derivatives. To illustrate the idea, let us consider the fundamental solution $E_k(x, y)$ in (7) for $n = 2$. We see that for small kr

$$E_k(x, y) = \frac{i}{4} H_0^{(1)}(kr) = E(x, y) - \frac{1}{2\pi}(\log k + \gamma_0) + S_k(x, y)\,. \tag{17}$$

Here

$$E(x,y) = -\frac{1}{2\pi} log|x-y|$$

denotes the fundamental solution for the 2-dimensional Laplace equation,

$$\gamma_0 = c_0 - \log 2 - i\frac{\pi}{2} \text{ with } c_0 \approx 0.5772\,, \text{ Euler's constant },$$

and

$$\begin{aligned} S_k(x,y) &= \frac{i}{4}H_0^{(1)}(kr) + \frac{1}{2\pi}(\log(kr)+\gamma_0) \\ &= -\frac{1}{2\pi}\{\log(kr)\sum_{m=1}^{\infty} a_m(kr)^{2m} + \sum_{m=1}^{\infty} b_m(kr)^{2m}\}\,, \\ a_m &= \frac{(-1)^m}{2^{2m}(m!)^2}, \qquad b_m = (\gamma_0 - 1 - \frac{1}{2} - \cdots - \frac{1}{m})a_m\,. \end{aligned}$$

As can be seen from the above expansions, the term $\log k$ appears in (17) explicitly which shows that V_k is a singular perturbation of V (the corresponding boundary integral operator V_k with E_k replaced by E). We have shown in [10] that the other boundary integral operators are regular perturbations of the corresponding operators of the Laplacian.

In the following let us consider the analysis for the integral equation of the first kind (10) for (IDP). From (17), we see that

$$V\sigma + \omega + S_k\sigma = \frac{1}{2}\varphi + K\varphi + R_k\varphi \tag{18}$$

with

$$\omega = -\frac{1}{2\pi}(\log k + \gamma_0)\int_\Gamma \sigma ds\,. \tag{19}$$

Here

$$K\varphi = \int_\Gamma \frac{\partial}{\partial n_y}E(x,y)\varphi(y)ds_y$$

is the corresponding double-layer boundary integral operator for the Laplacian and $R_k\varphi := (K_k - K)\varphi$. The right hand side of (18) is bounded. This suggests that the solution of (18), (19) can be decomposed in the form of an asymptotic expansion,

$$\begin{aligned} \sigma &= \widetilde{\sigma} + \alpha_1(k)\widetilde{\sigma}_1 + \sigma_R\,, \\ \omega &= \widetilde{\omega} + \alpha_1(k)\widetilde{\omega}_1 + \omega_R\,, \end{aligned} \tag{20}$$

where the leading terms $\widetilde{\sigma}, \widetilde{\omega}$ correspond to boundary densities from the Laplacian and satisfy the system [7], [8],

$$V\widetilde{\sigma} + \widetilde{\omega} = \frac{1}{2}\varphi + K\varphi \text{ and } \int_\Gamma \widetilde{\sigma} ds = 0 \tag{21}$$

with

$$\widetilde{\omega} = 0\,.$$

The first perturbation terms $\widetilde{\sigma}_1, \widetilde{\omega}_1$ are independent of k with the coefficient $\alpha_1(k) = o(1)$ as $k \to 0$. The remainders σ_R, ω_R are of order $o(\alpha_1(k))$. To construct $\widetilde{\sigma}_1$ and $\widetilde{\omega}_1$, we employ equation (18) with (19) and (20). As $k \to 0$ we arrive at

$$\begin{aligned} V\widetilde{\sigma}_1 + \widetilde{\omega}_1 &= 0\,, \\ \int\limits_\Gamma \widetilde{\sigma}_1 ds &= 1\,, \end{aligned} \tag{22}$$

where we appended the last normalizing condition for $\widetilde{\sigma}_1$ in order to obtain nontrivial solution pair $\widetilde{\sigma}_1, \widetilde{\omega}_1$. However if we insert (20) into (19) with $\widetilde{\omega} = 0$, then from $\int_\Gamma \widetilde{\sigma} ds = 0$, and $\int_\Gamma \widetilde{\sigma}_1 ds = 1$, we see that

$$\alpha_1(k) = \left\{ \frac{-1}{1 + 2\pi\widetilde{\omega_1}(logk + \gamma_0)^{-1}} \right\} (\int_\Gamma \sigma_R\, ds - \omega_R) = O(\sigma_R).$$

Hence, without loss of generality, we may set $\alpha_1(k) = 0$ in (20). Now from (18) and (19) with (20) this leads to the equations for the remainder terms σ_R, ω_R:

$$\begin{aligned} V\sigma_R + \omega_R + S_k\sigma_R &= R_k\varphi - S_k\widetilde{\sigma}\,, \\ \int\limits_\Gamma \sigma_R ds + 2\pi(\log k + \gamma_0)^{-1}\, \omega_R &= 0\,, \end{aligned} \tag{23}$$

which can be solved by the regular perturbation techniques.

By substituting the boundary densities into the representation formula (6), we obtain the asymptotic behavior of the solutions to the BVPs for small k. In all the cases, we arrive at the following asymptotic expression

$$u(x) = \pm[\mathcal{V}\widetilde{\sigma}(x) - \mathcal{W}\widetilde{u}(x)] + C(x;k) + R(x;k), \tag{24}$$

where the $\pm$ sign corresponds to the interior and exterior domain and $x \in \Omega$ or Ω^c as in (6). For the Dirichlet problems, $\widetilde{u}_{|\Gamma} = \varphi$ and for the Neumann problems, $\widetilde{\sigma}_{|\Gamma} = \psi$ on Γ are the given boundary data, respectively, whereas the missing densities are the solutions of the corresponding BIEs presented above. In Formula (24), $C(x;k)$ denotes the lowest order of perturbation terms in Ω or Ω^c, whereas $R(x;k)$ denotes the remaining boundary potentials. The behavior of both for $k \to 0$ is summarized in Table 2 below. The remainders $R(x;k)$ are of the orders as shown in the table, uniformly in $x \in \Omega$ for the interior problems and in compact subsets of $\overline{\Omega^c}$ only, for the exterior problems. It should be mentioned that not all our results presented in Table 2 are new. Some similar results obtained by other methods are also available. In this connection, we refer in particular to the work [11], [12].

BVP	$C(x;k)$	$R(x;k)$	n
IDP	0	$O((k\log k)^2)$ $O(k^2)$	$n=2$ $n=3$
EDP	$-\widetilde{\omega}$	$O((k\log k)^2)$	$n=2$
	$-k\{V\widetilde{\sigma}_1(x)+\frac{i}{4\pi}\int\limits_{\Gamma}\widetilde{\sigma}ds\}$	$O(k^2)$	$n=3$
INP	$-\{\frac{1}{k^2}-\frac{1}{2\pi}\int\limits_{\Omega}\log\|x-y\|dy\}\frac{1}{\|\Omega\|}\int\limits_{\Gamma}\psi ds$	$O(k^2\log k)$	$n=2$
	$-\{\frac{1}{k^2}+\frac{1}{4\pi}\int\limits_{\Omega}\frac{1}{\|x-y\|}dy\}\frac{1}{\|\Omega\|}\int\limits_{\Gamma}\psi ds$	$O(k^2)$	$n=3$
ENP	$\frac{1}{2\pi}(\log k+\gamma_0)\int\limits_{\Gamma}\psi ds$	$O((k\log k)^2)$	$n=2$
	$-\frac{ik}{4\pi}\int\limits_{\Gamma}\psi ds$	$O(k^2)$	$n=3$

Table 2: Low frequency characteristics

Acknowledgment

This work was supported by the Air Force Office of Scientific Research, Air Force Material Command, USAF, under Grant F9620-96-1-10039. The views and conclusions contained herein are those of the author and should not be interpreted as necessarily representing the office policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the US government.

References

[1] Brakhage, H. and P. Werner, "Über das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung," *Arch. Math.*, Vol. 16, pp. 325–329 (1965).

[2] Burton, A.J. and G.F. Miller, "The application of integral equation methods to numerical solution of some exterior boundary-value problems," *Proc. Royal Soc. London*, Vol. A323, pp. 201–220 (1971).

[3] Chen, J.T., "On fictitious frequencies using dual series representation," *Mechanics Research Communications*, Vol. 25, No.5, pp. 529-534 (1998).

[4] Chen, J.T. and S.R. Kuo, "On fictitious frequencies using circulants for radiation problems of a cylinder," *Mechanics Research Communications*, Vol. 27, No.1, pp. 1-6 (2000).

[5] Colton, D. and R. Kress, Integral equation methods in scattering theory, John Wiley and Sons Inc. (1983).

[6] Costabel, M. " Boundary integral operators on Lipschitz domains: Elementary results," *SIAM J. Math. Anal.,* Vol. 19, pp. 613-626 (1988).

[7] Hsiao, G.C. and R.C. MacCamy, "Solution of boundary value problems by integral equations of the first kind," *SIAM Rev.,* Vol. 15, pp. 687-705 (1973).

[8] Hsiao, G.C. and W.L. Wendland, "A finite element method for some integral equations of the first kind," *J. Math. Anal. Appl.,* Vol. 58, pp. 449-481 (1977).

[9] Hsiao, G.C. and W.L. Wendland, "On the low frequency asymptotics of exterior Dirichlet problems in dynamic elasticity," *Inverse and Ill-posed problems*, Engl and Groetsch (eds), Academic Press, pp. 461-482 (1987).

[10] Hsiao, G.C. and W.L. Wendland, Boundary Integral Equations, Springer Verlag, in preparation.

[11] MacCamy, R.C., " Low frequency acoustic oscillations," *Quart. Appl. Math.,* Vol. 23, pp.247-255 (1965).

[12] MacCamy, R.C., " Low frequency expansions for two-dimensional interface scattering problems," *SIAM J. Appl. Math.,* Vol. 57, No.6, pp. 1687-1701 (1997).