TOWARD THE DIRECT ANALYTICAL DETERMINATION OF THE PARETO OPTIMA OF A DIFFERENTIABLE MAPPING, I: DOMAINS IN FINITE-DIMENSIONAL SPACES

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Toward the Direct Analytical Determination of the Pareto Optima of a Differentiable Mapping, I: Domains in Finite-Dimensional Spaces*

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Abstract. The problem of locating the Pareto-optimal points of a differentiable mapping $F : \mathcal{M}^N \to \mathbb{R}^n$ is studied, with the domain $\mathcal{M}^N$ a differentiable N-dimensional submanifold-without-boundary in a euclidean space $\mathbb{R}^{N_0}$ and $N_0 \geq N \geq n$. The case in which the domain is the closure of a bounded, regular, open subset of $\mathbb{R}^N$ is also discussed. The search is initiated from these observations: for a manifold-domain, (1) the image of any Pareto optimum lies in the boundary of the range of F; (2) a point of the boundary of the range of F that also lies in the range must be the image of a singular point of F, *i.e.*, must appear amongst the singular values of the map. Further conditions are then needed to distinguish which of the singular values should be discarded because they belong to the interior of the range; local tests of this sort are given for the bicriterial case ($n = 2$). A search procedure based on the present developments can systematically determine all of the Pareto optima for sufficiently simple F. The conditions established here may be regarded as analogues of the classical ones for the determination of the global extrema of a real-valued differentiable function. The results proven are illustrated with simple examples, including plots of the ranges, singular points, and singular values.

Contents

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0. Introduction.

We study the problem of locating the *Pareto-optimal points* for a differentiable map $F : \mathcal{D}_F \to \mathbb{R}^n$. For the most part, the domain $\mathcal{D}_F$ is assumed to be a differentiable N-dimensional submanifold-without-boundary $\mathcal{M}^N$ in a euclidean space $\mathbb{R}^{N_0}$, with $N_0 \geq N \geq n \geq 2$ (which includes the special case in which $\mathcal{M}^N$ is an open subset of some $\mathbb{R}^N$ with $N \geq 2$). Some of the results that we establish can be extended to the case in which $\mathcal{D}_F$ is a differentiable N-dimensional manifold-without-boundary in a euclidean space $\mathbb{R}^{N_0}$ (again, with $N_0 \geq N \geq n \geq 2$). However, we prefer to maintain throughout the submanifold-hypothesis, since the underlying machinery—tangent spaces, differentiable maps, *etc.*—is much more accessible (*cf.*, *e.g.*, Appendix B here), while the analysis is correspondingly more tractable for that case. One can also consider the domain as contained in an infinite-dimensional Hilbert space; we intend to examine in a separate note the analogous developments in that setting.

Another situation of frequent occurrence in applications is that in which $\mathcal{D}_F$ is the closure $\mathcal{K}$ of a bounded, open, regular subset of $\mathbb{R}^N$; such a case we propose to treat by a "reduction" to several applications of the submanifold-domain case, through dissection of the domain $\mathcal{K}$ into its interior and boundary submanifold-pieces. For the sake of definiteness in these introductory remarks, we are thinking of the submanifold-domain $\mathcal{D}_F = \mathcal{M}^N$ with $N_0 \geq N \geq n \geq 2$, unless otherwise noted.

We want to exploit the most elementary observation concerning the Pareto optima, *viz.*, that if one has a sufficiently good "atlas" of the range $\mathcal{R}_F$ of F in $\mathbb{R}^n$, then the images of the Pareto points of F can be picked out, owing to the characteristic locations of those images on the boundary of the range of F, whence the Pareto points themselves can be identified. Specifically, for the implementation of such an approach one would wish to know (*i*) the set $\partial^\star\mathcal{R}_F := \partial\mathcal{R}_F \cap \mathcal{R}_F$, comprising the points of the boundary $\partial\mathcal{R}_F$ of the range of F that also lie in the range $\mathcal{R}_F$ itself, along with (*ii*) a vector indicating the "local mapping direction" at each point of $\partial^\star\mathcal{R}_F$; at "regular" points of the boundary, more specifically, where the boundary is sufficiently smooth and the interior of the range is locally nonvoid, the interior normal to $\partial\mathcal{R}_F$ will serve to indicate the local mapping direction. This motivates the search for an analytical means of identifying the set $\partial^\star\mathcal{R}_F$, which we call the *included boundary of the range of F*. In fact, as we show here in Section 2, when F is differentiable (in the submanifold-domain setting) there is readily at hand a useful condition that is necessary for the inclusion of a value $F(\xi)$ in $\partial\mathcal{R}_F$: ξ must be a *singular point of F*, *i.e.*, a point for which the rank of the differential $\mathrm{D}F(\xi)$ has less than the maximal value n. Thus, the included boundary of the range of F is to be found within the set of *singular values of F* and the Pareto minima of F must lie amongst the singular points of F. This fact is completely elementary and follows directly from the inverse-function theorem of classical analysis, so it is surprising that such a simple connection has apparently not been remarked and pursued before.

Unfortunately, however, it is generally *not* the case that *every* singular value belongs to the boundary of the range, a circumstance greatly complicating any search for $\partial^\star\mathcal{R}_F$ that is based on the singular-value necessary condition. Indeed, it is very common to find that F maps many of its singular points into the interior ${\mathcal{R}_F}^\circ$ of its range, so one is forced to seek additional tests aimed at determining how a given singular value is situated relative to $\mathcal{R}_F$. This leads naturally to interesting excursions into the differential geometry and topology of maps between euclidean spaces and the study of singularities of smooth maps.

Of course, the ultimate aim should be the formulation of a systematic and reliable procedure for the location of all of the Pareto optima of a given mapping F. This is an ambitious project, not only because of the many cases and pathologies that arise to be handled, but especially because of the numerical difficulties (entailing the solution of nonlinear systems of equations) inherent in the global searching that will always be in some measure unavoidable. In any event, we can claim only that the results reported here provide just a start on such a program. On the other hand, the procedures developed in this article will in fact serve to locate all of the Pareto optima for smooth maps F that are sufficiently simple to permit reliable analytical global searching for preimages. In more complex examples, such searching must be carried out discretely and, consequently, will be much less reliable.

After recalling the basic definitions in Section 1, we establish simple first results about the location of the images of Pareto minima, identify the cases to be studied, and introduce some standard notations. Section 1 ends with some observations on the special case in which F is a continuous injection of a subset of $\mathbb{R}^N$ into itself. There is a fundamental difference between the case in which the dimensions of the domain-manifold and the codomain satisfy $1 \leq N < n$ and that in which $N \geq n \geq 2$; throughout, we shall restrict attention to the latter setting, which would appear to be the more common one for applications. Under this hypothesis, the fundamental condition that is necessary for the inclusion of an image-point in the boundary of the range is developed in Theorem 2.2 of Section 2. In the second part of that section we provide some information about the general structure of the singular points and singular values of a differentiable mapping. In Section 3 we block out the major steps in proposed "systematic" procedures for locating the Pareto minima of a sufficiently smooth mapping F, first for the manifold-domain case, then for the compact-domain case. The procedures are at present tentative in some aspects, since we leave certain eventualities to be dealt with on an *ad hoc* basis. One can regard Section 3 as placing in context the statements proven in Sections 2, 4, and 5, in particular, Theorems 2.2, 4.1, and 5.1, which are the main results.

Following the fairly general settings of Sections 1, 2, and 3, for the development of further discriminatory tests we restrict attention to only the simplest of the many cases to be studied. That is, in Sections 4 and 5 we examine only the *bicriterial* setting, in which $n = 2$. Already in that case we find the formulation and proof of definitive statements to be nontrivial. In particular, Theorems 4.1 and 5.1, presented and proven in Sections 4 and 5, respectively, apply only in the bicriterial case. Theorem 4.1 pertains to the special case in which $\mathcal{D}_F$ is an open set in $\mathbb{R}^N$, while Theorem 5.1 is concerned with the more general setting, where $\mathcal{D}_F$ is a sufficiently smooth N-submanifold in some $\mathbb{R}^{N_0}$. Each statement provides conditions of a *local* nature sufficient to decide whether a given singular value belongs to the interior of the range of F or to the "local boundary" of the range of F (which may be more familiarly known as the set of points at which the range is "folded"; *cf*., the definitions given in Section 3). The theorems also show how one can compute the "local mapping direction" at a point of the included boundary of the range, when the appropriate conditions are met, a calculation essential for the final determination of Pareto-optimal points. Section 6 contains a number of elementary but pertinent examples included to demonstrate in the simplest manner the general considerations covered by the theorems proven in the preceding sections. The examples are not provided with the intent of establishing the viability of the search procedure outlined in Section 3; instead, the illustrations are chosen primarily for their aptness in capturing the ideas and observations developed in Sections 2 through 5. Strictly speaking, a full understanding of all of Section 6 requires every result of the paper, but one can browse at any stage with profit the examples presented there. Some concluding remarks are made in Section 7.

Two appendices are included. In Appendix A we develop some needed facts concerning "biquadratic maps," in particular, a few properties of the range of such maps. Appendix B contains a brief review of the simplest definitions and results about (differentiable) submanifolds of euclidean spaces. The material there consists of basic definitions and results taken primarily from the presentation of FLEMING [9] and augmented with the required definitions and some elementary facts about differentiable mappings from one submanifold into another.

It is possible to regard several of the statements proven here as analogues of the classical necessary conditions and sufficient conditions employed in the search for the local extrema of a real-valued differentiable function $f : \mathcal{D}_f \to \mathbb{R}$ (*i.e.*, for $n = 1$), when $\mathcal{D}_f$ is, say, an open subset of some $\mathbb{R}^{N_0}$. Thus, the necessary condition here in Theorem 2.2 is the analogue of the "first-derivative test"—the assertion that f can take an extreme value only at a critical point, so that all extreme values are to be found amongst the critical values. The statements of Theorems 4.1 and 5.1, requiring more smoothness of the mapping, involve conditions on the definiteness properties of a certain quadratic form constructed from the second quadratic differential of F acting on the nullspace of its first differential; these conditions correspond to a "second-derivative test," a counterpart of the Hessian-matrix criteria appropriate in the case of a twice-differentiable real-valued f. Here also (the present form of) our statements have an "indeterminate" case, in which the quadratic form is semidefinite and the test fails to give information (and so should be extended to some condition on higher-order differentials).

The present "direct analytical" approach to the search for Pareto optima is just the pursuit of an obvious strategy that would surely have been undertaken long ago if the basic necessary condition of Theorem 2.2 had been recognized. Instead, previous searches for Pareto optima have proceeded by exploiting devices such as "scalarization," "brute-force but judicious" plotting of points of the range, and/or restrictions to circumstances involving special features such as convexity; *cf.*, *e.g.*, [4], [5], [10], [16], [17], and [18]. By "scalarization" is meant the replacement of the original "vector" problem, involving an $\mathbb{R}^n$-valued map F, by a parametric family of scalar optimization problems involving real-valued functions constructed by forming linear combinations of the components of F, with the coefficients playing the rôle of the parameters generating the family. For example, AUBIN [4] shows that if the domain $\mathcal{D}_F$ is a convex set and the components of F are convex functions, then the scalarization device based upon the formation of *convex* linear combinations of the components of F produces a family of minimization problems whose solutions yield all of the weak Pareto minima of F. We anticipate that the usual sort of situation will evolve, *viz.*, that each approach, including that proposed here, will perform well on its own class of problems, with no one method serving to handle the entire universe of mappings whose Pareto-optimal points are wanted.

Throughout, we consider the location of Pareto *minima*, as we may, without loss of generality.

1. Pareto minima.

Recall that the Pareto minima are of central interest in settings such as the design of systems or processes and the selection of strategies in multi-person coöperative games, where one would like to adjust the operating conditions so as to "make simultaneously as small as possible" a number n (≥ 2) of *costs*, or *losses*. The Pareto-optimization approach is of particular importance when the costs are "conflicting," in the sense that working to decrease some of them may have the effect of increasing others.

For definiteness, let us describe the basic ideas within the context of an optimal-control problem. Suppose that the state of a system under study is completely fixed by assigning values to the N_0 real (control) variables ξ_1, ξ_2, ..., ξ_{N_0}, with the provision that the control-point $\xi := (\xi_1, \ldots, \xi_{N_0})$ be constrained to lie in an *admissible set* $\mathcal{D}_F \subset \mathbb{R}^{N_0}$. Assume further that we have identified n costs $F_1(\xi), \ldots, F_n(\xi)$ associated with operating the system at each control point ξ, so that we have n real-valued functions $F_k : \mathcal{D}_F \to \mathbb{R}$, $k = 1, \ldots, n$, defined on the set of admissible controls; this produces a map $F \equiv \big(F_1, \ldots, F_n\big) : \mathcal{D}_F \to \mathbb{R}^n$, with $F(\xi) := \big(F_1(\xi), \ldots, F_n(\xi)\big)$ for $\xi \in \mathcal{D}_F$. We would like to control the system to "operate at minimal cost," in some useful sense. Of course, it will not in general happen that the costs reach global—or even local—minimum values at some single control point (although the analysis should subsume that possibility, in particular, by automatically notifying us of any such point(s)). In such a situation, it is reasonable to seek the control-points ξ^0 enjoying the property that, relative to ξ^0, *no other control point* in $\mathcal{D}_F$ (1) realizes *at least one strictly lower cost* while (2) *increasing none* of the costs. Such a point ξ^0 is called "a strong global Pareto minimum of F"; in general, many such points will exist, so that, even after all have been found, there will remain the question of selecting one of them as the "best," or "least-cost," control point of the system, in a trade-off decision, perhaps based on some other criteria. That is, the identification of the Pareto minima will afford the opportunity to effect a choice of control point through some subsequent additional weighting of the costs. These ideas underlie the

Definitions. Let $F \equiv \big(F_1, \ldots, F_n\big) : \big\{\mathcal{D}_F \subset \mathbb{R}^{N_0}\big\} \to \mathbb{R}^n$. Then *$F$ has a strong local Pareto minimum at* $\xi^0 \in \mathcal{D}_F$ iff there is a neighborhood $W_{F(\xi^0)}$ of $F(\xi^0)$ in $\mathbb{R}^n$ such that the condition

$$F_k(\xi) \leq F_k(\xi^0) \quad \text{for } k = 1, \ldots, n \qquad \text{and} \qquad F_j(\xi) < F_j(\xi^0) \quad \text{for some } j \in \{1, \ldots, n\}, \tag{1.1.1}$$

or, equivalently,

$$F_k(\xi) \leq F_k(\xi^0) \quad \text{for } k = 1, \ldots, n \qquad \text{and} \qquad F(\xi) \neq F(\xi^0), \tag{1.1.2}$$

holds for *no* $\xi \in \mathcal{D}_F$ with $F(\xi) \in W_{F(\xi^0)}$. *F has a strong global Pareto minimum at* $\xi^0 \in \mathcal{D}_F$ iff the inequalities in (1.1.1) hold for *no* $\xi \in \mathcal{D}_F$. (Obviously, a strong global Pareto minimum is also a strong local Pareto minimum.) We denote the set of strong local [global] Pareto minima of F by $\Pi_{\mathrm{loc}} F$ [by ΠF].

Remark. It is important to note that the definition of a strong Pareto minimum, whether local or global, is based upon the behavior of the map on its entire domain, and not merely locally to the point in question. That is, the preceding definition in the local case is *not* constructed as "F has a strong local Pareto minimum at $\xi^0 \in \mathcal{D}_F$ iff there is a neighborhood U_{ξ^0} of ξ^0 such that the inequalities in (1.1) hold for *no* $\xi \in U_{\xi^0} \cap \mathcal{D}_F$" (which is, at least for a continuous F, a necessary but not generally sufficient condition for F to possess a strong local Pareto minimum at ξ^0 according to the actual definition). In fact, it is easy to construct, and common to encounter, points ξ^0 satisfying the latter weaker condition but for which the image $F(\xi^0)$ lies in the *interior* of the range of F, so that such points have no minimum property at all *when this is reckoned relative to the full domain*. This usually happens because F fails to be injective, so that another portion of the domain is mapped to cover a point fulfilling only the weaker condition. The difficulty here captures the essentially *global* nature of the problem of locating the Pareto minima of a given map. Indeed, the identification and exclusion of points satisfying only the cited weaker property is a major issue to which we devote a good deal of effort, but even with the results that we present here, some global searches will still be ultimately required, in general.

The form of the definitions given above is somewhat awkward to use in application. There are alternate characterizations of Pareto minima that both facilitate the analysis and assist in visualization. Thus, for example, it is easy to use the conclusions of the following lemma to sketch the typical local and global situations in the case when $n = 2$. We employ the notation

$$\mathcal{Q}_0^n := \{\, y \in \mathbb{R}^n \,|\, y \neq 0, \quad y_j \leq 0 \text{ for } j = 1, \ldots, n \,\}$$

for the "closed negative 2^nant" in $\mathbb{R}^n$ with the origin removed.

Lemma 1.1. *Let $F \equiv (F_1, \ldots, F_n) : \{\mathcal{D}_F \subset \mathbb{R}^{N_0}\} \to \mathbb{R}^n$. Then F has a strong local Pareto minimum at $\xi^0 \in \mathcal{D}_F$ iff there exists a neighborhood $W_{F(\xi^0)} \subset \mathbb{R}^n$ of $F(\xi^0)$ such that either of these conditions obtains:*

- *(i.) $W_{F(\xi^0)} \cap \mathcal{R}_F \cap \{F(\xi^0) + \mathcal{Q}_0^n\} = \varnothing$, i.e., the translate $F(\xi^0) + \mathcal{Q}_0^n$ does not meet the range of F inside $W_{F(\xi^0)}$;*
- *(ii.) for every $\xi \in \mathcal{D}_F$ such that $F(\xi) \in W_{F(\xi^0)}$, either $F(\xi) = F(\xi^0)$ or $F_j(\xi^0) < F_j(\xi)$ for some $j \in \{1, \ldots, n\}$.*

F has a strong global Pareto minimum at $\xi^0 \in \mathcal{D}_F$ iff either of these conditions obtains:

- *(iii.) $\mathcal{R}_F \cap \{F(\xi^0) + \mathcal{Q}_0^n\} = \varnothing$, i.e., the translate $F(\xi^0) + \mathcal{Q}_0^n$ does not meet the range of F;*
- *(iv.) for every $\xi \in \mathcal{D}_F$, either $F(\xi) = F(\xi^0)$ or $F_j(\xi^0) < F_j(\xi)$ for some $j \in \{1, \ldots, n\}$.*

In particular, it follows that if F has either a strong local or global Pareto minimum at $\xi^0 \in \mathcal{D}_F$, then $F(\xi^0)$ lies in the boundary $\partial\mathcal{R}_F$ of the range of F.

Proof. (i) and (iii). Let $\xi^0 \in \mathcal{D}_F$. The translate $F(\xi^0) + \mathcal{Q}_0^n$ can be written as

$$F(\xi^0) + \mathcal{Q}_0^n := \{\, F(\xi^0) + z \,|\, z \in \mathcal{Q}_0^n \,\} = \{\, y \in \mathbb{R}^n \,|\, y \neq F(\xi^0), \quad y_k \leq F_k(\xi^0) \text{ for } k = 1, \ldots, n \,\}.$$

From the latter characterization it is clear that, if $W_{F(\xi^0)}$ is a neighborhood of $F(\xi^0)$, the set $F(\xi^0) + \mathcal{Q}_0^n$ meets $W_{F(\xi^0)} \cap \mathcal{R}_F$ iff there is some $\xi \in \mathcal{D}_F$ such that $F(\xi) \in W_{F(\xi^0)}$ and (1.1.2) holds, *i.e.*, iff ξ^0 is not a strong local Pareto minimum for F. This proves (i). The proof of (iii) is constructed in the same manner by omitting mention of the neighborhood $W_{F(\xi^0)}$.

(ii) and (iv). Clearly, an equivalent form of the definition of "strong local Pareto minimum" is phrased as "F has a strong local Pareto minimum at $\xi^0 \in \mathcal{D}_F$ iff there is a neighborhood $W_{F(\xi^0)}$ of $F(\xi^0)$ in $\mathbb{R}^n$ such that the *negation* of condition (1.1.2) holds for *every* $\xi \in \mathcal{D}_F$ such that $F(\xi) \in W_{F(\xi^0)}$." Statement (ii) results from this observation by explicitly writing out the negation of condition (1.1.2). The proof of (iv) runs in essentially the same way by omitting the neighborhood $W_{F(\xi^0)}$.

The final assertion of the Lemma follows immediately from (i) and (iii). In fact, let F have a strong local [global] Pareto minimum at $\xi^0 \in \mathcal{D}_F$. Then, by (i) [by (iii)], any sequence contained in $F(\xi^0) + \mathcal{Q}_0^n$ and converging to $F(\xi^0)$ eventually lies [completely lies] outside the range of F, so $F(\xi^0)$ belongs to the boundary $\partial\mathcal{R}_F$. $\square$

Remarks. (1.) From Lemma 1.1, it is easy to check that a point $\xi^0 \in \mathcal{D}_F$ at which some one of the component-functions F_k takes on an absolute minimum value will be a strong local Pareto minimum of F; such a point will be a strong global Pareto minimum if the absolute minimum is taken on only at ξ^0. Apparently, a point of local minimum of one of the F_k may or may not be a local Pareto minimum of F.

(2.) Some investigators introduce a "*weak* global Pareto minimum" of $F \equiv (F_1, \ldots, F_n) : \{\mathcal{D}_F \subset \mathbb{R}^{N_0}\} \to \mathbb{R}^n$ as a point $\xi^0 \in \mathcal{D}_F$ for which the *strict* inequalities $F_k(\xi) < F_k(\xi^0)$ for $k = 1, \ldots, n$ hold for *no* $\xi \in \mathcal{D}_F$; *cf.*, *e.g.*, AUBIN [4]. Equivalently, with the "negative 2^nant" in $\mathbb{R}^n$ denoted by

$$\mathcal{Q}^n := \{\, y \in \mathbb{R}^n \,|\, y_j < 0 \quad \text{for } j = 1, \ldots, n \,\}$$

a weak global Pareto minimum of F is a point $\xi \in \mathcal{D}_F$ for which the intersection $\mathcal{R}_F \cap \{F(\xi) + \mathcal{Q}^n\}$ is void. Weak *local* Pareto minima can be introduced in a similar manner. The reader can make a sketch of the range of F and the translated cones $\mathcal{Q}_0^n$ and $\mathcal{Q}^n$ for the case $n = 2$ to picture the distinction between "strong" and "weak" Pareto minima. In particular, it should be clear that it is more desirable to be able to locate the strong Pareto minima. However, we will not need to make such a distinction in the present developments, so we shall not introduce the weak points, and henceforth (usually) drop the qualifier "strong" in referring to Pareto minima.

With the definitions set, we consider how one might develop a systematic and direct procedure for finding all of the Pareto minima of a given function $F : \{\mathcal{D}_F \subset \mathbb{R}^{N_0}\} \to \mathbb{R}^n$. As we have noted, our strategy is based on the fundamental observation, made in Lemma 1.1, that the images of the Pareto-minimal points, both local and global, lie in the boundary $\partial\mathcal{R}_F$ of the range. Therefore, more precisely, these images belong to the *included boundary* $\partial^\star\mathcal{R}_F := \partial\mathcal{R}_F \cap \mathcal{R}_F$ *of the range of* F, *i.e.*, they belong both to the range and to its boundary. (For any subset A of a topological space, we call $\partial^\star A := A \cap \partial A$ the *included boundary of* A; we may use this term even when we know A to be closed, so that $\partial^\star A = \partial A$.) Indeed, the basic necessary condition, developed in the next section, will assist only in the location of points of the included boundary of the range, so we are taking care now to point out that this is all that is required. Accordingly, the first aim is the identification of $\partial^\star\mathcal{R}_F$ and its preimage $F^{-1}\{\partial^\star\mathcal{R}_F\}$, which is to be followed by a search in the latter set for the Pareto minima. The separation of the *local* and *global* Pareto minima requires a final global study with knowledge of the entire boundary of the range.

Since we are ultimately to infer the locations of the Pareto minima by picking out their *images* from amongst all the points of the included boundary of the range, it is important to develop a test for effecting this discrimination. For this, we can exploit Lemma 1.1.*i* and *iii*, which indicate how the images of Pareto minima occupy characteristic locations within that boundary. We give at this point a simple and convenient sufficiency test, implied by those statements in Lemma 1.1 and applying wherever there is enough local regularity of the boundary of the range of F. Although we do not "officially" restrict attention to the case $N \geq n$ until the next section, this will be the only setting in which the conditions make sense, since we need the interior of the range to be locally nonvoid—a circumstance that will never obtain when $N < n$. The test is conducted by looking for a distinguishing property of the interior normal to the boundary of the range at the image of a prospective Pareto minimum. The statement is not definitive; *e.g.*, we cannot use it to examine those situations in which the range locally has void interior, and even when the regularity conditions are fulfilled there will be exceptional cases that are not covered.

To begin, we need a definition of "regularity" of a set at a point of its boundary that generalizes the one usually stated for an open set. We continue to denote by A° the interior of a set A in a topological space.

Definition. Let Ω be a subset of $\mathbb{R}^n$; here, $n \geq 2$ and Ω need not be open. Let $x \in \partial\Omega$. We say that Ω is *C^k-regular at x* for some positive integer k iff there exists a neighborhood U_x of x and a C^k-function Φ_x on U_x such that $\operatorname{grad}\Phi_x(x) \neq 0$ and $\Omega^\circ \cap U_x = \{\, y \in U_x \mid \Phi_x(y) > 0 \,\}$ while $\partial\Omega \cap U_x = \{\, y \in U_x \mid \Phi_x(y) = 0 \,\}$. We usually say simply "regular at x" in place of "C^1-regular at x."

Remarks. Retain the setting and notation of the preceding definition. Let Ω be regular at $x \in \partial\Omega$.

(1.) Of course, we may suppose that $\operatorname{grad}\Phi_x(y) \neq 0$ for each $y \in U_x$. Further, it is easy to check that $\overline{\Omega} \cap U_x = \overline{\Omega^\circ} \cap U_x$, $\partial\Omega \cap U_x = \partial\{\Omega^\circ\} \cap U_x$, and $\{\mathbb{R}^n \setminus \overline{\Omega}\} \cap U_x = \{\, y \in U_x \mid \Phi_x(y) < 0 \,\}$.

(2.) It follows from the definition that $\partial\Omega \cap U_x$ is an $(n-1)$-submanifold of class C^1, while Ω° "lies on one side of its boundary" in the neighborhood U_x of x. Thus, there is a unique unit-normal $\hat{\nu}$ to $\partial\Omega$ at x such that $x + s\hat{\nu}$ lies in Ω° for all sufficiently small positive s; we call $\hat{\nu}$ *the interior unit-normal to $\partial\Omega$ at x* and any positive multiple of $\hat{\nu}$ *an interior normal at x*. In the present setting, of course, an interior normal at x is given by $\operatorname{grad}\Phi_x(x)$.

Lemma 1.2. *Let $F : \{\mathcal{D}_F \subset \mathbb{R}^N\} \to \mathbb{R}^n$. Suppose that $\xi \in \mathcal{D}_F$ with $F(\xi) \in \partial\mathcal{R}_F$ and the range $\mathcal{R}_F$ is regular at $F(\xi)$.*

- *(i.) If all of the (Cartesian) components of an interior normal to $\partial\mathcal{R}_F$ at $F(\xi)$ are positive, then ξ is a local Pareto minimum of F.*
- *(ii.) If at least one of the (Cartesian) components of an interior normal to $\partial\mathcal{R}_F$ at $F(\xi)$ is negative, then ξ is not a local Pareto minimum of F.*

Remark. It is easy to construct borderline cases, *i.e.*, with all of the components of an interior normal to $\partial\mathcal{R}_F$ at $F(\xi)$ nonnegative and at least one vanishing, for which ξ is a Pareto minimum and other such cases in which ξ is not a Pareto minimum.

Proof. Let $U_{F(\xi)}$ and $\Phi_{F(\xi)}$ be as in the definition of regularity of $\mathcal{R}_F$ at $F(\xi)$; we may suppose that $U_{F(\xi)}$ is a ball centered at $F(\xi)$.

(i) Suppose that all of the components of an interior normal to $\partial\mathcal{R}_F$ at $F(\xi)$ are positive. Then we may assume that all of the components of $\operatorname{grad}\Phi_{F(\xi)}(z)$ are positive for every $z \in U_{F(\xi)}$. If $y \in U_{F(\xi)} \cap \{F(\xi) + \mathcal{Q}_0^n\}$, there is some y_ξ on the line segment joining $F(\xi)$ and y for which we have

$$\Phi_{F(\xi)}(y) = \Phi_{F(\xi)}(y) - \Phi_{F(\xi)}\big(F(\xi)\big) = \operatorname{grad}\Phi_{F(\xi)}\big(y_\xi\big) \cdot \big(y - F(\xi)\big) < 0,$$

so that $y \notin \overline{\mathcal{R}_F}$. By Lemma 1.1.$i$, we conclude that F has a local Pareto minimum at ξ.

(ii) Now assume that at least one of the components of an interior normal to $\partial\mathcal{R}_F$ at $F(\xi)$ is negative. In this case, it is easy to construct points $y \in U_{F(\xi)} \cap \{F(\xi) + \mathcal{Q}_0^n\}$ such that $\Phi_{F(\xi)}(y) > 0$, *i.e.*, such that $y \in \mathcal{R}_F{}^\circ$, so that ξ cannot be a local Pareto minimum of F. For the simplest example, suppose k is such that $\Phi_{F(\xi)},{}_k\big(F(\xi)\big) < 0$; then we may suppose that $\Phi_{F(\xi)},{}_k(z) < 0$ for every $z \in U_{F(\xi)}$. Construct $y \in \mathbb{R}^n$ by $y_j := F_j(\xi)$ for $j \neq k$ and $y_k := F_k(\xi) - \epsilon$ for a positive ϵ taken sufficiently small so that $y \in U_{F(\xi)}$. Then we have $y \in U_{F(\xi)} \cap \{F(\xi) + \mathcal{Q}_0^n\}$, while the mean-value theorem implies that $\Phi_{F(\xi)}(y) > 0$. $\square$

In Propositions 4.1 and 5.1, when $n = 2$ and the domain $\mathcal{D}_F$ is a submanifold, we show how one can compute a normal to the boundary of the range at an appropriate point; moreover, following the statements of Theorems 4.1 and 5.1 we indicate how one can use those theorems to decide which normals are *interior* normals, all under appropriate regularity hypotheses. Presently, we have no condition on F itself that we know to be sufficient to ensure all of the requisite regularity. For example, we have not shown that the hypotheses of Theorems 4.1 and 5.1 suffice to guarantee that the range of F is regular at a singular value of the sort considered there. Consequently, Lemma 1.2 must now be considered simply as a statement that a certain test will work under the reasonable circumstances that one expects to prevail in most cases.

The nature of the domain $\mathcal{D}_F$ of the mapping F is a crucial element in the analysis. We want to treat two types of domains that commonly occur in the modelling of physical circumstances.

A. $\mathcal{D}_F$ is an N-dimensional differentiable submanifold-without-boundary $\mathcal{M}^N$ in some $\mathbb{R}^{N_0}$, with $N_0 \geq N$.

Of course, the submanifold $\mathcal{M}^N$ here may be compact. This case can arise in practice, *e.g.*, when there are constraint conditions of the form $\Phi_j(\xi_1, \ldots, \xi_{N_0}) = 0$, $j = 1, \ldots, N_0 - N$, with a full-rank condition on the differential of the map $\Phi \equiv \big(\Phi_1, \ldots, \Phi_{N_0 - N}\big)$. Of particular importance is the special situation in which

$\mathbf{A}_0$. $\mathcal{D}_F$ is an open set $\mathcal{U}$ in $\mathbb{R}^N$ (when $N_0 = N$).

Indeed, not only is it easier to follow the arguments and visualize the underlying geometry of case (A_0), it will become clear that the open-set-domain setting must be mastered as a prerequisite to the understanding of all the others.

B. $\mathcal{D}_F$ is a (nonvoid!) compact and regularly closed set $\mathcal{K}$ in $\mathbb{R}^N$ with piecewise-smooth boundary $\partial\mathcal{K}$.

By *regularly closed* we mean, as usual, that $\mathcal{K}$ is the closure of its interior: $\overline{\mathcal{K}^\circ} = \mathcal{K}$. This case can occur, *e.g.*, when there are simple inequality bounds on the permissible variation of the control variables and/or the costs.

However, with one exception our theorems are set only in case (A) (and (A_0)). Roughly speaking, the examination of case (B) is to be reduced to that of successive applications of case (A), by seeking separately the included boundaries of the images $F\big(\mathcal{K}^\circ\big)$ and $F\big(\partial\mathcal{K}\big)$ of the interior and boundary of $\mathcal{K}$. The legitimacy of this reduction may not be immediately evident, so some explanation is in order. Accordingly, in Section 3 we provide some analysis of case (B). In particular, we show there that none of the Pareto minima of case (B) is lost by effecting this splitting into two subproblems, and indicate how the three boundaries $\partial F(\mathcal{K})$, $\partial F(\mathcal{K}^\circ)$, and $\partial F(\partial\mathcal{K})$ are relatively situated in general.

Thus, except for the indicated discussion in Section 3, a brief observation at the end of this section, and an example in Section 6, we shall be concerned here with case (A). Therefore, unless otherwise indicated, F will generally denote a mapping $F : \mathcal{M}^N \to \mathbb{R}^n$ of the N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ into the euclidean space $\mathbb{R}^n$, with $N_0 \geq N \geq n \geq 2$; when $N = N_0$, we consistently denote the common value by N. Moreover, in the latter special setting, we may use the notation $\mathcal{U}$ in place of $\mathcal{M}^N$ to remind ourselves that the domain is an open subset of $\mathbb{R}^N$.

Other sorts of domains can occur in applications. For example, natural constraints in models of physical problems may lead to a domain which is the closure of an N-dimensional manifold in

$\mathbb{R}^{N_0}$ with $N < N_0$. Such problems can be attacked by proceeding along the lines of the approach indicated here for Case B.

As we have done to this point, we shall always denote the components of the function F by F_j, $j = 1, \dots, n$, so that $F \equiv (F_1, \dots, F_n)$. The euclidean norm of $x \in \mathbb{R}^k$ we denote by $|x|_k$; the euclidean inner product of x, $y \in \mathbb{R}^k$ we shall indicate simply by $x \cdot y$, leaving the dimension k to be inferred from the context. The open ball in $\mathbb{R}^k$ of radius $r > 0$ and centered at x is denoted $\mathrm{B}^k_r(x)$, while its boundary is written as $\mathrm{S}^k_r(x)$. The standard unit basis-vectors of $\mathbb{R}^{N_0}$ are written $\mathrm{e}^{(j)}$, $j = 1, \dots, N_0$, while those of $\mathbb{R}^2$ are indicated by $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$. For brevity and convenience, we sometimes abuse language and notation by referring to an $\mathbb{R}^k$-vector as a "point" of $\mathbb{R}^k$ and *vice versa*; the intended meaning here is clear, and the practice should cause no confusion. We are lax in distinguishing the terms "differential" and "derivative"; the terminology is not standardized in the literature. Again, the context will eliminate any possible ambiguity. Finally, the term "manifold" will always mean "manifold-without-boundary," since we consider only the latter structures; a similar restriction applies to the term "submanifold."

By way of orientation, we close this Section 1 with observations concerning a special situation in the case with $N_0 = N = n$, *i.e.*, with $F : \{\mathcal{D}_F \subset \mathbb{R}^N\} \to \mathbb{R}^N$; here, either case ($\mathrm{A}_0$) (open-set domain) or case (B) (compact, regularly closed domain) is permitted. Sometimes in the applications one encounters an F in such a setting that is injective. The well-known Brouwer theorem on invariance of domain then permits one to qualitatively describe the situation regarding Pareto minima—and under merely the hypothesis of continuity for F.

Theorem (BROUWER). *Let $F : \{\mathcal{U} \subset \mathbb{R}^N\} \to \mathbb{R}^N$ be a continuous injection of the open subset $\mathcal{U}$ of $\mathbb{R}^N$ into $\mathbb{R}^N$. Then F is an open mapping; in particular, $F(\mathcal{U})$ is open in $\mathbb{R}^N$.*

Proof. One can consult *e.g.*, [7] for references to works containing proofs of the theorem. □

Example 1.1. Let $F : \{\mathcal{U} \subset \mathbb{R}^N\} \to \mathbb{R}^N$ be exactly as in the Brouwer theorem. Since the range $\mathcal{R}_F = F(\mathcal{U})$ is open, the included boundary $\partial^\star \mathcal{R}_F$ of the range is void. Therefore, such an F has no Pareto-optimal points.

Remark. Consider the case $N = n = 2$ and suppose that the mapping $F : \{\mathcal{U} \subset \mathbb{R}^2\} \to \mathbb{R}^2$, with $\mathcal{U}$ open, is holomorphic when regarded as a complex-valued function on the complex domain $\mathcal{U}$. As pointed out by A. E. LIVINGSTON [13], in this setting the analyticity of F alone implies that it is an open mapping, and so, just as in Example 1.1 (which relied on the assumed injectivity of F and the Brouwer Theorem), again F can have no Pareto-optimal points, since the included boundary of its range is void. Of course, this conclusion is completely independent of any hypothesis of injectivity for F.

Example 1.2. Again with $F : \{\mathcal{U} \subset \mathbb{R}^N\} \to \mathbb{R}^N$ and $\mathcal{U}$ open, now replace the hypotheses of continuity and injectivity of F with the conditions that F is differentiable and has no singular points. The inverse-function theorem now implies that F is an open mapping, just as under the Brouwer hypotheses, so F is devoid of Pareto-optimal points in this setting, as well. It is illuminating to review this remark in light of the developments of the next section (where the definition of "singular point" is recalled and a statement of the inverse-function theorem is given).

Example 1.3. Let $F : \{\mathcal{K} \subset \mathbb{R}^N\} \to \mathbb{R}^N$ be a continuous injection, where $\mathcal{K}$ is compact and regularly closed. (Thus, F is a homeomorphism of $\mathcal{K}$ onto $F(\mathcal{K})$.) We shall check that F here maps the boundary $\partial\mathcal{K}$ of its domain onto the boundary $\partial F(\mathcal{K})$ of its range; from this, it will follow immediately that any Pareto-optimal point of F must lie on $\partial\mathcal{K}$. To prove that $F(\partial\mathcal{K}) = \partial F(\mathcal{K})$, we show first that $F(\mathcal{K}^\circ) = F(\mathcal{K})^\circ$: the Brouwer theorem says that $F(\mathcal{K}^\circ)$ is open in $\mathbb{R}^N$, which implies that $F(\mathcal{K}^\circ) \subset F(\mathcal{K})^\circ$; the reversed inclusion just follows from the continuity of F, which gives $F^{-1}(F(\mathcal{K})^\circ) \subset (F^{-1}(F(\mathcal{K})))^\circ = \mathcal{K}^\circ$. From the partition $\mathcal{K} = \mathcal{K}^\circ \cup \partial\mathcal{K}$ of $\mathcal{K}$ and the injectivity of F we get the partition $F(\mathcal{K}) = F(\mathcal{K}^\circ) \cup F(\partial\mathcal{K})$ of $F(\mathcal{K})$, which we can now rewrite as $F(\mathcal{K}) = F(\mathcal{K})^\circ \cup F(\partial\mathcal{K})$. On the other hand, we also have the partition $F(\mathcal{K}) = F(\mathcal{K})^\circ \cup \partial F(\mathcal{K})$, following just from the fact that $F(\mathcal{K})$ is closed. From these observations, we can conclude that $F(\partial\mathcal{K}) = \partial F(\mathcal{K})$, as claimed.

Example 1.4. Carmichael [5] discusses an example from mechanics, involving a fixed loading of a simple symmetric five-member truss comprising bars of just two cross-sections. The design problem consists in choosing the two cross-sectional areas of the members of the truss so as to Pareto-minimize the weight and a certain sum of deflections of two chosen nodes of the truss. There are constraints imposed: neither the cross-sectional areas nor the individual deflections of the two selected nodes are permitted to exceed certain given values. The underlying mapping, carrying each admissible pair of cross-sectional areas into a pair consisting of a corresponding weight and node-deflection sum, is of the form $F : \mathcal{K} \to \mathbb{R}^2$, with $\mathcal{K}$ a curvilinear quadrilateral in $\mathbb{R}^2$. It is easy to determine the explicit rule of the mapping F, close examination of which reveals its injectivity. Consequently, the situation is just as in the preceding Example 1.3 (with $N = 2$). The Pareto-minimal points of F can therefore be determined by searching along the boundary of $\mathcal{K}$, using an obvious strategy for computation of the interior normal at an image-point on the boundary of the range.

2. The fundamental necessary condition for the included boundary of the range.

We pointed out in Lemma 1.1 that $F(\xi)$ lies in the included boundary $\partial^\star\mathcal{R}_F := \partial\mathcal{R}_F \cap \mathcal{R}_F$ of the range of the mapping F whenever $\xi \in \mathcal{D}_F$ is a Pareto minimum (global or local) of F. We can exploit this observation to narrow the search for Pareto-minimal points. Indeed, our first main result is simply a condition necessary for the inclusion $F(\xi) \in \partial^\star\mathcal{R}_F$ to obtain, when $N_0 \geq N \geq n \geq 2$ and the domain $\mathcal{D}_F$ is an N-dimensional submanifold $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ of class C^1 and $\xi \in \mathcal{M}^N$. We have the partition of the range given by

$$\mathcal{R}_F = \mathcal{R}_F{}^\circ \cup \{\partial\mathcal{R}_F \cap \mathcal{R}_F\} = \mathcal{R}_F{}^\circ \cup \partial^\star\mathcal{R}_F, \tag{2.1}$$

so the negation of any condition *sufficient* to ensure that a value $F(\xi)$ lie in $\mathcal{R}_F{}^\circ$ will yield a condition *necessary* for $F(\xi)$ to lie in $\partial^\star\mathcal{R}_F$. A little reflection reveals that the well-known inverse-function theorem of analysis affords a condition of the former sort. Accordingly, we begin by recalling a statement of this theorem:

Theorem 2.1 (Inverse-Function Theorem). *Let $\mathcal{U} \subset \mathbb{R}^N$ be an open set and $F : \mathcal{U} \to \mathbb{R}^N$ a mapping of class C^q $(q \geq 1)$. Let $\xi \in \mathcal{U}$ and suppose that the Jacobian determinant $\mathcal{J}F(\xi)$ is nonzero. Then there exists an open neighborhood $\mathcal{U}_\xi$ of ξ contained in $\mathcal{U}$ for which the image $F(\mathcal{U}_\xi)$ is open in $\mathbb{R}^N$, while the restriction $F\,\big|\,\mathcal{U}_\xi : \mathcal{U}_\xi \to F(\mathcal{U}_\xi)$ is a C^q-diffeomorphism.*

Proof. *Cf., e.g.*, Fleming [9] or Apostol [2]. □

From this statement, in the special setting $F : \{\mathcal{U} \subset \mathbb{R}^N\} \to \mathbb{R}^N$ it is immediate that the equality $\mathcal{J}F(\xi) = 0$ for some $\xi \in \mathcal{U}$, *i.e.*, the singularity of the differential $\mathrm{D}F(\xi) : \mathbb{R}^N \to \mathbb{R}^N$, is *necessary* for the inclusion $F(\xi) \in \partial^\star\mathcal{R}_F$. It requires only a bit more work to extend this result to a more general case.

To that end, we recall some useful ideas. If $\mathcal{U} \subset \mathbb{R}^N$ is open and $F : \mathcal{U} \to \mathbb{R}^n$ is a mapping of class C^1, then the *rank of F at $\xi \in \mathcal{U}$* is the rank of the differential $\mathrm{D}F(\xi) : \mathbb{R}^N \to \mathbb{R}^n$. More generally, if $\mathcal{M}^N$ is an N-dimensional submanifold of class C^1 of a euclidean space $\mathbb{R}^{N_0}$ and $F : \mathcal{M}^N \to \mathbb{R}^n$ is of class C^1, then the *rank of F at $\xi \in \mathcal{M}^N$* is the rank of the differential $\mathrm{D}F(\xi) : \mathsf{T}_\xi\mathcal{M}^N \to \mathbb{R}^n$; *cf., e.g.*, Appendix B. Of course, the second case here subsumes the first, since an N-submanifold of $\mathbb{R}^N$ is, by definition, simply an open set.

Definitions. Let $1 \leq N \leq N_0$. Let $F : \mathcal{M}^N \to \mathbb{R}^n$ be a map of class C^1 on the N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ of class C^1, and let $\xi \in \mathcal{M}^N$: ξ is a *singular point of F* iff the rank of F at ξ is less than n; ξ is a *regular point of F* iff it is not a singular point of F, *i.e.*, iff the rank of F at ξ is n, so that the differential $\mathrm{D}F(\xi)$ is surjective. We denote by ΣF the set of singular points of F. A point $x \in \mathbb{R}^n$ is a *singular value of F* iff the inverse image $F^{-1}\{x\}$ contains a singular point of F, *i.e.*, iff x is the image of some singular point; otherwise (including the case in which $F^{-1}\{x\}$ is void), x is a *regular value of F*.

Remark. The terms introduced here are not fully standardized in the literature. Thus, some authors define a "singular point" to be a point at which the rank of F is less than its maximal value $\min\{N, n\}$; when $N \geq n$, which is the case of present interest, the two definitions yield the same points. Moreover, the terms "critical point" and "critical value" are sometimes used in place of "singular point" and "singular value," respectively, while other writers reserve those words to apply only in the case in which F is real-valued, *i.e.*, when $n = 1$.

Within the present context of the search for the boundary of the range, confusion may arise from the fact that a singular value may be the image of a regular point, as well as of a singular point. Consequently, we introduce additional terms allowing us to discriminate more precisely for present purposes.

Definition. We say that a singular point $\xi \in \mathcal{M}^N$ of a class-C^1 map $F : \mathcal{M}^N \to \mathbb{R}^n$ on the N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ of class C^1 is a *completely singular point* iff $F(\xi)$ has no preimage that is a regular point of F; naturally, in that case, we refer to $F(\xi)$ as a *completely singular value*. The set of completely singular points of F we denote by $\Sigma^\star F$:

$$\Sigma^\star F := \big\{\, \xi \in \Sigma F \,\big|\, F^{-1}\{F(\xi)\} \quad \text{contains no regular point of } F \,\big\}.$$

Still considering $F : \mathcal{M}^N \to \mathbb{R}^n$, suppose that $N < n$: then F will map a full neighborhood of any regular point in $\mathcal{M}^N$ to an N-dimensional smooth submanifold of $\mathbb{R}^n$; *cf., e.g.*, CHILLINGWORTH [6]. That is, roughly, the range of F is of "smaller dimension" than the codomain $\mathbb{R}^n$. In the contrary case, when $N \geq n$, F will map appropriate relatively open neighborhoods of regular points to open sets in $\mathbb{R}^n$. This points up the distinction between the two cases $N < n$ and $N \geq n$, which are essentially different geometrically with regard to the topological location in the range of the images of Pareto minima, as simple schematic examples will show. Henceforth, we shall suppose that we are in the latter setting (which seems to be the more common one for applications). Under this restriction, we can establish

Theorem 2.2. *Let $N_0 \geq N \geq n \geq 1$ and suppose that $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ is an N-dimensional submanifold-without-boundary of class C^1. Let $F : \mathcal{M}^N \to \mathbb{R}^n$ be of class C^1.*

- *(i.)* *If $\xi^0 \in \mathcal{M}^N$ is a regular point of F, then $F(\xi^0)$ belongs to the interior $\mathcal{R}_F{}^\circ$ of the range of F.*
- *(ii.)* **(Fundamental necessary condition.)** *If $\xi^0 \in \mathcal{M}^N$ and $F(\xi^0) \in \partial\mathcal{R}_F$, then ξ^0 is a completely singular point of F. Thus, the preimages of the points of the included boundary of the range of F, and hence, in particular, all local Pareto minima of F, lie amongst the completely singular points of F:*

$$\Pi_{\text{loc}} F \subset F^{-1}\big(\partial^\star \mathcal{R}_F\big) \subset \Sigma^\star F. \tag{2.2}$$

Of the several arguments available for the proof of Theorem 2.2, we shall present one that readily admits of generalization to the case in which the domain of F is contained in an infinite-dimensional Hilbert space. To that end, we begin with a special case of the "constant-rank theorem," a full version of which is given as Theorem 2.4.4 of CONLON [7]. As pointed out in [7], although the constant-rank theorem is usually proven by an appeal to the inverse-function theorem, the two statements are equivalent. For the particular case that is needed presently, we provide a variant of the proof given in [7] that can be generalized to accommodate an infinite-dimensional domain.

Lemma 2.1. *Let $N \geq n \geq 1$. Let $\mathcal{U}$ be open in $\mathbb{R}^N$ and $F : \mathcal{U} \to \mathbb{R}^n$ be a mapping of class C^1. Suppose that F has (maximal) rank n at $\xi^0 \in \mathcal{U}$. Then there are an open set $\widetilde{U} \subset \mathbb{R}^N$ and a diffeomorphism $\Phi : \widetilde{U} \to U_{\xi^0}$ of $\widetilde{U}$ onto an open neighborhood $U_{\xi^0} \subset \mathcal{U}$ of ξ^0 such that*

$$F \circ \Phi(\xi_1, \ldots, \xi_N) = (\xi_1, \ldots, \xi_n) \qquad \textit{for every} \quad \xi \equiv (\xi_1, \ldots, \xi_N) \in \widetilde{U}.$$

Proof. Since the differential $\mathrm{D}F(\xi^0) : \mathbb{R}^N \to \mathbb{R}^n$ is surjective, we can find $h^1, \ldots, h^n$ in $\mathbb{R}^N$ such that the set $\{\mathrm{D}F(\xi^0)h^1, \ldots, \mathrm{D}F(\xi^0)h^n\}$ of images is linearly independent, and so spans $\mathbb{R}^n$. The collection $\{h^1, \ldots, h^n\}$ must also then be linearly independent, so that the span $H_n^N := \operatorname{sp}\{h^1, \ldots, h^n\}$ is an n-dimensional subspace of $\mathbb{R}^N$, while the restriction $\mathrm{D}F(\xi^0) \,|\, H_n^N$ is a bijection of H_n^N onto

$\mathbb{R}^n$. With $\{\iota^{(k)}\}_{k=1}^n$ denoting an orthonormal basis chosen for H_n^N, if $N > n$ we adjoin elements to form an orthonormal basis $\{\iota^{(k)}\}_{k=1}^N$ for $\mathbb{R}^N$.

Let the C^1-mapping $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ be defined by setting

$$\Psi(\xi) := \xi + \sum_{k=1}^{n} \Big\{ F_k(\xi) - \xi \cdot \iota^{(k)} \Big\} \iota^{(k)} = \sum_{k=1}^{n} F_k(\xi)\iota^{(k)} + \sum_{k=n+1}^{N} \big(\xi \cdot \iota^{(k)}\big)\iota^{(k)}, \qquad \text{for} \quad \xi \in \mathbb{R}^N;$$

if $N = n$, of course the second sum in the latter form is absent. For the differential $\mathrm{D}\Psi(\xi)$ at any $\xi \in \mathbb{R}^N$ we readily find

$$\mathrm{D}\Psi(\xi)h = h + \sum_{k=1}^{n} \Big\{ \mathrm{d}F_k(\xi)h - h \cdot \iota^{(k)} \Big\} \iota^{(k)} \qquad \text{for} \quad h \in \mathbb{R}^N.$$

Now, observe that $\mathrm{D}\Psi(\xi^0) : \mathbb{R}^N \to \mathbb{R}^N$ is bijective. In fact, if $h \in \mathbb{R}^N$ and $\mathrm{D}\Psi(\xi^0)h = 0$, we find that $\mathrm{d}F_k(\xi^0)h = 0$ for $k = 1, \ldots, n$ and $h \cdot \iota^{(k)} = 0$ for $k = n+1, \ldots, N$ if $N > n$. The latter set of equalities says that $h \in H_n^N$, while the former set implies that $\mathrm{D}F(\xi^0)h = 0$; upon recalling that $\mathrm{D}F(\xi^0) \mid H_n^N$ is bijective, we see that $h = 0$, which establishes the claim. With this, we can apply the inverse-function theorem (Theorem 2.1) to Ψ to conclude that there is an $\mathbb{R}^N$-neighborhood U_{ξ^0} of ξ^0 such that the restriction $\widetilde{\Psi} := \Psi \mid U_{\xi^0}$ is a C^1-diffeomorphism of U_{ξ^0} onto the open set $\Psi\big(U_{\xi^0}\big)$; we may suppose that $U_{\xi^0} \subset \mathcal{U}$, by replacing U_{ξ^0} with the intersection $\mathcal{U} \cap U_{\xi^0}$, if necessary.

Denoting the standard unit-basis vectors for $\mathbb{R}^n$ by $\varepsilon^{(k)}$, $k = 1, \ldots, n$, it is easy to see that

$$F \circ \widetilde{\Psi}^{-1}(\zeta) = \sum_{k=1}^{n} \{\zeta \cdot \iota^{(k)}\} \varepsilon^{(k)} \qquad \text{for every} \quad \zeta \in \Psi\big(U_{\xi^0}\big).$$

In fact, for $\zeta = \Psi\big(\xi^\zeta\big)$ with $\xi^\zeta \in U_{\xi^0}$, so that $F_k\big(\xi^\zeta\big) = \zeta \cdot \iota^{(k)}$, $k = 1, \ldots, n$, we get $F \circ \widetilde{\Psi}^{-1}(\zeta) = F\big(\xi^\zeta\big) = \sum_{k=1}^n F_k\big(\xi^\zeta\big)\varepsilon^{(k)}$, whence the result follows. Finally, let $\mathcal{I} : \mathbb{R}^N \to \mathbb{R}^N$ denote the isometric isomorphism such that $\mathcal{I}\iota^{(k)} = \mathrm{e}^{(k)}$, the standard unit-basis vector for $\mathbb{R}^N$, for $k = 1, \ldots, N$. As one can check, the proof of the lemma is now completed by setting $\widetilde{U} := \mathcal{I}^{-1}\big(\Psi(U_{\xi^0})\big)$ and $\Phi := \widetilde{\Psi}^{-1} \circ \mathcal{I} : \widetilde{U} \to U_{\xi^0}$. $\square$

Proof of Theorem 2.2. Suppose that statement (i) has been proven. Then, if $\xi^0 \in \mathcal{M}^N$ and $F(\xi^0) \in \partial\mathcal{R}_F$, (i) and the partition of $\mathcal{R}_F$ given in (2.1) combine to imply that $F(\xi^0)$ can have no regular preimage, *i.e.*, that ξ^0 is completely singular. Thus, statement (ii) will follow as soon as (i) is established, so we restrict attention to the latter from this point.

Suppose first that $N_0 = N$, so the domain of F is an open set $\mathcal{M}^N = \mathcal{U} \subset \mathbb{R}^N$. Let $\xi^0 \in \mathcal{U}$ be a regular point of F, so $\mathrm{D}F(\xi^0)$ has (maximal) rank n. Then we can apply Lemma 2.1, from which it follows directly that F maps the neighborhood U_{ξ^0} of ξ^0 onto an open neighborhood of $F(\xi^0)$ in $\mathbb{R}^n$. Indeed, the projection operator $\xi \mapsto (\xi_1, \ldots, \xi_n)$, carrying $\mathbb{R}^N$ onto $\mathbb{R}^n$, is an open mapping, and, continuing to employ the notation of Lemma 2.1, $F\big(U_{\xi^0}\big)$ coincides with the image of the open set $\widetilde{U}$ under this projection. In particular, we can assert that $F(\xi^0)$ belongs to the interior $\mathcal{R}_F{}^\circ$ of the range of F.

We consider next the setting $F : \mathcal{M}^N \to \mathbb{R}^n$, in which the domain of F is an N-dimensional submanifold-without-boundary $\mathcal{M}^N$ of class C^1 in $\mathbb{R}^{N_0}$ and $N < N_0$. This case is reduced to the previous one by the use of coördinate systems for $\mathcal{M}^N$. Again, with $\xi^0 \in \mathcal{M}^N$ a regular point of F, we must show that F maps ξ^0 into the interior $\mathcal{R}_F{}^\circ$. For this, we choose any coördinate system (U, g) for $\mathcal{M}^N$ with ξ^0 in the coördinate patch U; since every value of the composition $F \circ g^{-1} : \{g(U) \subset \mathbb{R}^N\} \to \mathbb{R}^n$ is a value of F itself, it suffices to show that $F \circ g^{-1}$ carries some open neighborhood of $g(\xi^0)$ in the open subset $g(U) \subset \mathbb{R}^N$ onto an open set in $\mathbb{R}^n$. But this follows immediately from the first part of the proof and the assumption that ξ^0 is a regular point of F. That is, $F \circ g^{-1}$ has (maximal) rank n at $g(\xi^0)$, since F has rank n at ξ^0 (*cf., e.g.*, Appendix B). (Since g is a homeomorphism, it is also clear that F maps an open neighborhood of ξ^0 in $\mathcal{M}^N$ onto an open neighborhood of $F(\xi^0)$ in $\mathbb{R}^n$.) Consequently, statement (i) is now proven. As noted, this effectively completes the proof of the theorem. □

Remark. Consider the case $n = 1$ of Theorem 2.2, *i.e.*, when F is $\mathbb{R}$-valued. Then, it is clear that the included boundary of the range in $\mathbb{R}$ will lie within the set of extreme values that F takes, if any, on the various components of its domain; we already know that these extreme values are amongst the images of the critical points of F—which is in agreement with the theorem.

As a corollary of the proofs of Lemma 2.1 and Theorem 2.2, we point out an extension of the well-known open-mapping theorem for a transformation from an open subset of $\mathbb{R}^N$ into $\mathbb{R}^N$ and having no singular points; *cf., e.g.*, Apostol [2]. This is inserted in passing just because it does not seem to be explicitly enunciated in the basic texts. First, we verify the useful result in

Lemma 2.2. *Retain the hypotheses and notation of Theorem 2.2. The set of regular points of F is open in $\mathcal{M}^N$.*

Proof. Let $\xi^0 \in \mathcal{M}^N$ be a regular point of F. When $N_0 = N$ and the domain $\mathcal{M}^N$ is an open subset of $\mathbb{R}^N$, some $n \times n$ submatrix of the Jacobian matrix of F at ξ^0 has nonvanishing determinant; the same submatrix has nonvanishing determinant in a neighborhood of ξ^0, implying the result for that setting. The general case is reduced to the previous case by the use of a coördinate system, in the usual manner. □

Corollary 2.1. *Retain the hypotheses and notation of Theorem 2.2. The restriction of F to its set of regular points is an open mapping.*

Proof. This follows directly from Lemma 2.2 and the proofs of Lemma 2.1 and Theorem 2.2. In fact, in the latter two arguments it was shown that each regular point possesses an open neighborhood (which we may suppose to comprise regular points, according to Lemma 2.2) that is mapped onto an open set in $\mathbb{R}^n$. This implies the open-mapping property claimed. □

As a simple first example of application of Theorem 2.2, we consider the case of an analytic function.

Example. Let $F : \Omega \to \mathbb{C}$ be analytic, where Ω is an open subset of $\mathbb{C}$. To apply Theorem 2.2, we regard Ω as an open subset of $\mathbb{R}^2$ and F as a map into $\mathbb{R}^2$, *i.e.*, now $F_1 = \operatorname{Re} F$ and $F_2 = \operatorname{Im} F$. It is easy to see that the singular points of F are those points in Ω at which $F_{1,1}F_{2,2} - F_{1,2}F_{2,1}$ vanishes; by accounting for the Cauchy-Riemann equations, we see that these are just the points at which $|F_{1,1}|^2 + |F_{2,1}|^2$ vanishes. Consequently, the singular points of F are the zeros of its derivative F' (*qua* analytic function). Moreover, all of the singular points are rank-0 in this case. The points of the included boundary of the range of F, if any, are to be found amongst the singular values, $\{\, F(z) \,|\, F'(z) = 0 \,\}$. However, according to the observation made in Section 1, the included boundary of the range of such a map is void. This is our first example of the common situation in which a map takes some or all of its singular points to the interior of its range. Finally, we also note that, if F is nonconstant, its singular points have no point of accumulation in Ω. Thus, for such mappings we find nothing like the curves of singular points and corresponding curves of singular values that we hypothesize in the later theorems and observe in the examples given in Section 6.

Having established for the Pareto-optimization problem the fundamental importance of identifying the collection of singular points of a given mapping F, we should proceed by next gathering whatever general qualitative and quantitative information is available concerning the set ΣF. Accordingly, in the remainder of this section we supply some orientation on the general structure of the set of singular points, supposing always that $F : \{\mathcal{M}^N \subset \mathbb{R}^{N_0}\} \to \mathbb{R}^n$ (with $N \geq n$) is at least of class C^1. The specific aim here is twofold. First, in Sections 4 and 5 with $N \geq n = 2$, we invariably assume that we are working on a 1-manifold that is *contained in* the set of rank-1 singular points of F, and we want to justify that hypothesis now by indicating that it is, in some sense, the "usual" setting. In addition, we want to review some of the basic relations that one might use to set up a numerical sheme for computing the singular points of a given mapping.

We just observed in Lemma 2.2 that the set of regular points of F is open in $\mathcal{M}^N$, so that the set ΣF of singular points of F is closed in $\mathcal{M}^N$. In general, many singular points of F will be mapped to the interior of the range, *i.e.*, not every singular point of F is mapped to the boundary of the range. However, just from the continuity of F, it is clear that the set $\{\, \xi \in \Sigma F \mid F(\xi) \in \mathcal{R}_F{}^{\circ} \,\}$ of singular points that are mapped to the interior of the range is open in ΣF, so that its complementary set $\{\, \xi \in \Sigma F \mid F(\xi) \in \partial\mathcal{R}_F \,\}$ of singular points mapped to the boundary of the range must be closed in ΣF. Further, we recall Sard's Theorem (*cf.*, *e.g.*, [7]), which says in the present context that the set of singular *values* $F(\Sigma F)$ has Lebesgue measure zero in $\mathbb{R}^n$. This statement always holds, even though the set of singular *points* may have nonzero measure in $\mathcal{M}^N$ (*e.g.*, for a map that is constant on some subset of nonzero measure in $\mathcal{M}^N$).

Beyond these general facts, however, there are evidently few unqualified assertions that can be made. For example, it is not generally true that ΣF is itself a manifold of some dimension, nor will one even find that the singular points of a given rank always form a manifold. On the other hand, there are some statements of this sort that can be made about classes of so-called "generic mappings," which will be dense in the collection of all smooth maps equipped with a certain topology. For example, ARNOL'D, ET AL. [3] present and prove the "corank-product formula," which we shall state for the present specialized setting, with the codomain-manifold being just $\mathbb{R}^n$.

For this, and from this point onward, it is essential to have a notation permitting clear distinction between sets of singular points of various ranks. Accordingly, we shall denote the set of singular points of F of rank ϱ, with $\varrho < n$, by $\Sigma_\varrho F$. The set of regular points of F (for which the differential has its maximal rank n) we then naturally indicate by $\Sigma_n F$. In the Thom-Boardman notation frequently used in the literature on singularities of smooth maps, the symbol $\Sigma^\nu F$ indicates the set of points $\xi \in \mathcal{M}^N$ for which $\mathrm{D}F(\xi)$ has null space of dimension ν; thus, with N carrying its usual significance here, the two notations are related by $\Sigma^\nu F = \Sigma_{N-\nu} F$.

Theorem (the "corank-product formula"). *For a (smooth) "generic" map $F : \mathcal{M}^N \to \mathbb{R}^n$, each set $\Sigma_\varrho F$ ($\varrho \leq n$) is a smooth submanifold of $\mathcal{M}^N$, of codimension equal to the product of the coranks:*

$$N - \dim \Sigma_\varrho F = (N - \varrho)(n - \varrho);$$

in particular, if $N - (N - \varrho)(n - \varrho) < 0$ then $\Sigma_\varrho F$ is empty.

For "generic" maps and $n = 2$, *e.g.*, the product formula asserts that the set $\Sigma_1 F$ of rank-1 singular points forms a 1-submanifold of $\mathcal{M}^N$, while the rank-0 singular points $\Sigma_0 F$ will be absent. On the other hand, it is not difficult to construct apparently innocuous examples for which these results do not hold at all. Thus, while the structure of ΣF must be completely studied for each F from the first, we shall accept the preceding theorem as providing at least some indication of a justification in this initial study for the hypotheses that are imposed in Sections 4 and 5. In any event, the latter hypotheses are local, and never require, say, that $\Sigma_1 F$ be a manifold, which is part of the (global) assertion of the co-rank product formula.

Given $\xi \in \mathcal{M}^N$, there are a number of equivalent ways in which to formulate the condition that the differential $\mathrm{D}F(\xi)$ be rank-deficient. Some of these are useful for the general analysis, and some are useful in the formulation of numerical schemes for actually locating the singular points. In general, it is easy to see that ξ is a singular point of the differentiable mapping $F : \mathcal{M}^N \to \mathbb{R}^n$ iff the set $\{\mathrm{grad}_{\mathcal{M}} F_1(\xi), \ldots, \mathrm{grad}_{\mathcal{M}} F_n(\xi)\}$ of manifold gradients of the components of F is linearly dependent; *cf.*, *e.g.*, Appendix B, for the development of the "manifold gradient." Thus, to locate the singular points one seeks all $\xi \in \mathcal{M}^N$ for which there exists a corresponding n-vector $(\mu_j)_{j=1}^n$ satisfying

$$\sum_{j=1}^{n} \mu_j \mathrm{grad}_{\mathcal{M}} F_j(\xi) = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \mu_j^2 = 1.$$

In particular, the rank-0 singular points will show up as those $\xi \in \mathcal{M}^N$ such that $\mathrm{grad}_{\mathcal{M}} F_j(\xi) = 0$ for $j = 1, \ldots, n$. For example, when $n = 2$ we seek the points ξ in $\mathcal{M}^N$ at which the gradients $\mathrm{grad}_{\mathcal{M}} F_1(\xi)$ and $\mathrm{grad}_{\mathcal{M}} F_2(\xi)$ either are nonzero and parallel or at least one vanishes; if exactly one of the gradients vanishes, then ξ is a rank-1 singular point, while ξ is rank-0 if both vanish.

From this condition involving the linear dependence of the set of gradients, we get a convenient formulation of the singular-point condition in terms of a square matrix: the singular points of F are precisely those points at which the determinant of the $n \times n$ Gram matrix of the collection of gradients of the components of F vanishes, *i.e.*, at which the matrix has nontrivial nullspace. This is so because the rank of the differential $\mathrm{D}F(\xi)$ coincides with the dimension of the subspace spanned by the set of n gradients of the components of F at ξ, which is just the rank of the Gram matrix of the same collection of gradients. The latter assertion follows directly from

Proposition 2.1. *Let $(H, (\cdot\,,\cdot)_H)$ be an inner-product space and $\{\mathrm{x}_m\}_{m=1}^n$ a (finite) subset of H. Then*

$$\dim \mathrm{sp}\, \{\mathrm{x}_m\}_{m=1}^n = \mathrm{rank}\, \{(\mathrm{x}_l, \mathrm{x}_m)_H\}_{n\times n}.$$

Moreover, a linear combination $\sum_{m=1}^n c_m \mathrm{x}_m$ is the zero-element of H iff the n-vector $(c_m)_{m=1}^n$ of coefficients is in the nullspace of the Gram matrix $\{(\mathrm{x}_l, \mathrm{x}_m)_H\}_{n\times n}$.

Proof. The second conclusion here is added for the sake of completeness. The proofs of both statements are straightforward, involving only standard arguments, so we omit them. □

Now we consider the case $F : \mathcal{U} \to \mathbb{R}^n$, with $\mathcal{U}$ an open subset of $\mathbb{R}^N$; this situation is, of course, also important for local examinations which may be necessary in the study of the general manifold-domain case. To locate the singular points in this setting, one can seek the points in $\mathcal{U}$ at which all of the n^{th}-order minors of the $n \times N$ matrix of partial derivatives of components of F vanish. We shall discuss this in further detail for (the bicriterial case) $n = 2$ and under additional smoothness requirements on F, restrictions that we impose in the considerations of Section 4.

Accordingly, let $F : \mathcal{U} \to \mathbb{R}^2$ be of class C^2, with $\mathcal{U}$ open in $\mathbb{R}^N$. For each $m \in \{1, \ldots, N\}$ we introduce the open subset $\mathcal{U}_m \subset \mathcal{U}$ by

$$\mathcal{U}_m := \left\{ \xi \in \mathcal{U} \,\middle|\, (F_{1,m}(\xi))^2 + (F_{2,m}(\xi))^2 > 0 \right\};$$

if one of these sets is empty, the problem has been poorly formulated, so we may suppose that each $\mathcal{U}_m$ is nonvoid. Corresponding to an integer $m \in \{1, \ldots, N\}$, it is convenient to write

$$j_{[m]} := \begin{cases} j, & \text{for} \quad j = 1, \ldots, m-1 \quad (\text{if } 1 < m \le N), \\ j+1, & \text{for} \quad j = m, \ldots, N-1 \quad (\text{if } 1 \le m < N); \end{cases} \tag{2.3}$$

then the ordered $(N-1)$-tuple $\big(j_{[m]}\big)_{j=1}^{N-1}$ is obtained by just removing m from the ordered N-tuple $\big(j\big)_{j=1}^{N}$. With this notation, we define for the same m the mapping $\Psi_m : \mathcal{U}_m \to \mathbb{R}^{N-1}$ by

$$\Psi_m(\xi) := \Big(F_{2,\,m}(\xi)F_{1,\,j_{[m]}}(\xi) - F_{1,\,m}(\xi)F_{2,\,j_{[m]}}(\xi)\Big)_{j=1}^{N-1}, \qquad \text{for each} \quad \xi \in \mathcal{U}_m;$$

$\Psi_m(\xi)$ is essentially the collection of values of the $(N-1)$ 2^{nd}-order minors of the matrix of $\mathrm{D}F(\xi)$ in which the m^{th} column participates. If each such minor vanishes at a point of $\mathcal{U}_m$, then it is easy to see that every 2×2 minor of the matrix of $\mathrm{D}F(\xi)$ vanishes. It follows readily that the zero-set $\big(\Sigma_1 F\big)_m := \{\, \xi \in \mathcal{U}_m \mid \Psi_m(\xi) = 0 \,\}$ of Ψ_m is just the collection of rank-1 singular points of F that lie in $\mathcal{U}_m$:

$$\big(\Sigma_1 F\big)_m = \Sigma_1 F \cap \mathcal{U}_m, \qquad \text{for} \quad m = 1, \ldots, N,$$

and, in fact, $\Sigma_1 F = \bigcup_{m=1}^{N} \big(\Sigma_1 F\big)_m$, *i.e.*, that every rank-1 singular point of F is to be found in at least one of these N zero-sets. Further, it is just as easy to check that each of the subsets

$$\big[\Sigma_1 F\big]_m := \big(\Sigma_1 F\big)_m \cap \Big\{\, \xi \in \mathcal{U}_m \,\Big|\, \operatorname{rank} \mathrm{D}\Psi_m(\xi) = N-1 \Big\}, \qquad \text{for } m = 1, \ldots, N,$$

is either void or a 1-manifold contained in $\Sigma_1 F$. In passing, we note that, when $\big[\Sigma_1 F\big]_m$ does form a 1-manifold, one can find its tangent and normal spaces explicitly by computing appropriate derivatives of Ψ_m.

Thus, we can locate all of the rank-1 singular points of F by determining all of the $\big(\Sigma_1 F\big)_m$, and we can find 1-manifolds contained in $\Sigma_1 F$ by identifying all of the $\big[\Sigma_1 F\big]_m$. To be sure, these sets will not generally be disjoint. For example, if we should find that $\mathcal{U}_m = \mathcal{U}$ for some m, then we can work entirely with the corresponding Ψ_m alone, to find all of $\Sigma_1 F$ and the manifold $\big[\Sigma_1 F\big]_m$, which we would expect to be, in some sense, the "largest" 1-manifold contained in $\Sigma_1 F$.

Here are some first examples illustrating the preceding construction; the conclusion of the corank-product theorem holds for neither.

Example 2.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by taking $F_1(\xi) := \big(\xi_1 - 3\big)^2 - \big(\xi_2 - 4\big)^2$ and $F_2(\xi) := \big(\xi_1 - 3\big)^2 + \big(\xi_2 - 1\big)^2$. The matrix of $\mathrm{D}F(\xi)$ is $2\begin{pmatrix} (\xi_1 - 3) & -(\xi_2 - 4) \\ (\xi_1 - 3) & (\xi_2 - 1) \end{pmatrix}$, whence we find that ΣF consists of the two lines given by $\xi_1 = 3$ and $\xi_2 = 5/2$, and that $\Sigma_0 F$ is void, so that all of the singular points are rank-1. It is easy to check that $\mathcal{U}_1 = \{\, \xi \in \mathbb{R}^2 \mid \xi_1 \ne 3 \,\}$ and $\Psi_1(\xi) = -4(\xi_1 - 3)(2\xi_2 - 5)$ for $\xi_1 \ne 3$. It follows that the set $\big(\Sigma_1 F\big)_1$ and the 1-manifold $\big[\Sigma_1 F\big]_1$ coincide, and comprise the two rays obtained by removing the point $(3, 5/2)$ from the line given by $\xi_2 = 5/2$. Further, one finds $\mathcal{U}_2 = \mathbb{R}^2$ and $\Psi_2(\xi) = 4(\xi_1 - 3)(2\xi_2 - 5)$ for $\xi \in \mathbb{R}^2$. Therefore, the set $\big(\Sigma_1 F\big)_2$ is just the union of the two lines given by $\xi_1 = 3$ and $\xi_2 = 5/2$. Meanwhile, the 1-manifold $\big[\Sigma_1 F\big]_2$ is composed of the four rays obtained by removing the point $(3, 5/2)$ from $\big(\Sigma_1 F\big)_2$, since the map Ψ_2 clearly has but one (rank-0) singular point, at $(3, 5/2)$. Thus, while the set $\Sigma_1 F$ of rank-1 singular points do not form a 1-manifold in this example, each singular point except $(3, 5/2)$ has a relatively open neighborhood in $\Sigma_1 F$ that is a 1-manifold.

Example 2.2. Consider the mapping $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by $F_1(\xi) := \xi_1^2 + \xi_2^2 + \xi_3^2$ and $F_2(\xi) := \xi_3^2$. Here, the matrix of $\mathrm{D}F(\xi)$ is found to be $2\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & \xi_3 \end{pmatrix}$; by inspection, we see that the singular points comprise the 3-axis $\{\, \xi \in \mathbb{R}^3 \,|\, \xi_1 = \xi_2 = 0 \,\}$ and the 1,2-plane $\{\, \xi \in \mathbb{R}^3 \,|\, \xi_3 = 0 \,\}$. Of these singular points, $(0,0,0)$ is rank-0 while all others are rank-1. In particular, $\Sigma_1 F$ here contains a 2-manifold, the punctured 1,2-plane. One can verify that $\mathcal{U}_1 = \{\, \xi \in \mathbb{R}^3 \,|\, \xi_1 \neq 0 \,\}$ and $\Psi_1(\xi) = (0, -\xi_1\xi_3)$ for $\xi_1 \neq 0$. It follows that $(\Sigma_1 F)_1$ is the 1,2-plane with the 2-axis deleted, but $[\Sigma_1 F]_1$ is void, since Ψ_1 is of rank 1 at each point. (Of course, our construction does not pick up the "pathological" 2-manifold of rank-1 singular points here.) The results for $m = 2$ are similar, but for $m = 3$ we get $\mathcal{U}_3 = \{\, \xi \in \mathbb{R}^3 \,|\, \xi_3 \neq 0 \,\}$ and $\Psi_3(\xi) = (\xi_1\xi_3, \xi_2\xi_3)$ for $\xi_3 \neq 0$. From these expressions, it follows that $(\Sigma_1 F)_3$ and $[\Sigma_1 F]_3$ are identical and coincide with the 3-axis with $(0,0,0)$ removed.

In the theorems of Sections 4 and 5, which provide local tests that assist in deciding whether a given singular value lies in the interior or on the boundary of the range, we impose various local-regularity hypotheses on the behavior of F near the singular-point preimage of the singular value under consideration. It seems best to describe those assumptions here, in the discussion of the general structure of the sets of singular points and values.

For the statement of these hypotheses, we first recall that a differentiable map $G : \mathcal{M} \to \mathcal{N}$, taking the m-manifold $\mathcal{M}$ into the n-manifold $\mathcal{N}$ (both without-boundary), is an *immersion* iff the rank of G is m at each point of $\mathcal{M}$ (for which it is necessary that $m \leq n$); G is an *imbedding* iff it is an immersion and also a homeomorphism of $\mathcal{M}$ onto $G(\mathcal{M})$ when the latter is equipped with the topology inherited from $\mathcal{N}$. One can show that the range $G(\mathcal{M})$ of an imbedding G from an m-manifold $\mathcal{M}$ into an n-manifold $\mathcal{N}$ is an m-submanifold of $\mathcal{N}$; *cf.*, *e.g.*, [15].

With $F : \mathcal{M}^N \to \mathbb{R}^2$ a sufficiently smooth mapping carrying the sufficiently smooth N-submanifold $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ into $\mathbb{R}^2$, and $\xi^0 \in \mathcal{M}^N$ a rank-1 singular-point of F, we impose in the later sections the hypotheses (H.1) and (H.2) given here. An additional condition (H.3) is described in Section 4.

H.1. There is an open interval $\mathcal{I}_0 := (t_1, t_2)$ in $\mathbb{R}$ and a C^1-imbedding $\chi : \mathcal{I}_0 \to \mathcal{M}^N$ with $\chi(t) \in \Sigma_1 F$ for $t_1 < t < t_2$ and $\xi^0 = \chi(t_0)$ for some $t_0 \in (t_1, t_2)$.

That is, the hypothesis requires that χ be a homeomorphism of $\mathcal{I}_0$ onto $\chi(\mathcal{I}_0)$ of class C^1 and that $\chi'(t)$ be nonzero for each $t \in \mathcal{I}_0$ (so that χ is of constant rank 1). These conditions guarantee that $\chi(\mathcal{I}_0)$ is a 1-submanifold of $\mathcal{M}^N$ containing the singular point ξ^0 and contained in $\Sigma_1 F$. It suffices to suppose here that there is a C^1-injection χ taking some interval (t_1', t_2') into $\Sigma_1 F$ such that $\chi(t_0) = \xi^0$ and $\chi'(t_0) \neq 0$, with $t_1' < t_0 < t_2'$. The setting of (H.1) can then be realized by possibly shrinking the interval (t_1', t_2') to some (t_1, t_2).

One should note that the conditions of (H.1) do not require that $\chi(\mathcal{I}_0)$ be a relative neighborhood of ξ^0 in $\Sigma_1 F$. To see this, one can, for example, take ξ^0 to be the singular point $(3, 5/2)$ in Example 2.1 above, and let χ be given on $\mathbb{R}$ by either $t \mapsto (3, t)$ (with $t_0 = 5/2$) or $t \mapsto (t, 5/2)$ (with $t_0 = 3$). In fact, $\chi(\mathcal{I}_0)$ could be a subset of a 2-manifold contained in $\Sigma_1 F$ (as in Example 2.2). However, if ξ^0 should have a relative neighborhood in $\Sigma_1 F$ that is a 1-manifold (which will be the "usual" situation), then the range of χ must coincide with that 1-manifold in a neighborhood of ξ^0.

H.2. The composition $F \circ \chi : \mathcal{I}_0 \to \mathbb{R}^2$ is a C^1-imbedding (so that the restriction $F\,|\,\chi(\mathcal{I}_0) : \chi(\mathcal{I}_0) \to \mathbb{R}^2$ is a C^1-imbedding of the manifold $\chi(\mathcal{I}_0)$ into $\mathbb{R}^2$).

More explicitly, the condition requires that $F \circ \chi$ be a homeomorphism of $\mathcal{I}_0$ onto $F \circ \chi(\mathcal{I}_0)$, of class C^1, and $(F \circ \chi)'(t)$ be nonzero for each $t \in \mathcal{I}_0$, so that $F \circ \chi$ is of constant rank 1. Hypothesis (H.2) ensures that $F\big(\chi(\mathcal{I}_0)\big)$ is a 1-submanifold of $\mathbb{R}^2$, containing the singular value $F\big(\xi^0\big)$ and contained in the singular values $F\big(\Sigma_1 F\big)$. It is sufficient to suppose here that $F \circ \chi$ is an injection of class C^1 taking an interval $\big(t_1', t_2'\big)$ into $\mathbb{R}^2$ in such a way that $\chi\big(t_0\big) = \xi^0$ and $\big(F \circ \chi\big)'(t_0) \neq 0$, for some t_0 with $t_1' < t_0 < t_2'$; perhaps by replacing the interval $\big(t_1', t_2'\big)$ with a smaller one $\big(t_1, t_2\big)$, the required setting will obtain.

Clearly, the derivative condition $\big(F\circ\chi\big)'(t_0) \neq 0$ will be fulfilled provided that the tangent vector $\chi'\big(t_0\big)$ to the singular-point curve $\chi\big(\mathcal{I}_0\big)$ at $\xi^0 = \chi\big(t_0\big)$ does not belong to the nullspace $\mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$ of the differential of F at ξ^0. In case the singular point ξ^0 has a relative neighborhood in $\Sigma_1 F$ that is a 1-manifold and the inclusion $\chi'\big(t_0\big) \in \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$ does hold, it is natural and common to find the corresponding singular-value $F\big(\xi^0\big)$ called in the literature a *cusp*, since the singular-value curve $F\big(\chi(\mathcal{I}_0)\big)$ will exhibit such a "spike" at $F\big(\chi(t_0)\big)$; we may use the same term for brevity in referring to this situation, and even when ξ^0 has no 1-manifold neighborhood in $\Sigma_1 F$.

The situation here for hypothesis (H.2) is similar to that noted for (H.1), in that the condition (H.2) does not imply that $F\big(\chi(\mathcal{I}_0)\big)$ is a relative neighborhood of the singular value $F\big(\chi(t_0)\big)$ in the set of singular values. This can be seen by examining the image of the singular point $(3, 5/2)$ of Example 2.1, taking for χ the map $t \mapsto (3, t)$, $t \in \mathbb{R}$. In that case, the image $F\big(\chi(t_0)\big)$ is the point $(-9/4, 9/4)$ and the image $F\big(\chi(\mathcal{I}_0)\big)$ is the parabola in $\mathbb{R}^2$ shown in Figure 11, which is not a relative neighborhood of the image point in the set of singular values (although it is a 1-submanifold in $\mathbb{R}^2$).

3. Procedures for finding the local Pareto minima.

In this section we describe the projected steps in systematic procedures for finding the *local* Pareto minima of an $\mathbb{R}^n$-valued differentiable mapping $F : \mathcal{D}_F \to \mathbb{R}^n$, in each of the cases (A) and (B), identified in Section 1. While the procedures can be demonstrated with simple examples in which they are completely successful, there are formidable numerical difficulties to be overcome in their implementation in a reasonably general setting involving mappings of technical significance. Moreover, a search carried out in case (B) will generally be more difficult and lengthy than one for case (A), if only because the former involves multiple applications of the latter—but usually there will be additional complications. In any event, since there are presently situations that we either cannot recognize or cannot handle, the procedures that we outline are not now definitive. The ultimate interest lies in the *global* Pareto minima; these are to be found in a final comparison once the local Pareto minima have been identified using the procedures described here.

Ignoring these obstacles for the present, our principal aim here is an outline and discussion of the ideas underlying the searches, which are prerequisite to their efficient numerical realization. In particular, we wish to place in their proper context the most important contributions to the procedures that are made here, in Theorem 2.2 and Theorems 4.1 and 5.1.

We already indicated that the procedure in any case is based first on the final assertion of Lemma 1.1, *viz.*, that the image of each local Pareto-minimal point lies on the boundary of the range of F. Now, in case (A) the submanifold-domain $\mathcal{M}^N$ is just right for initiation of the search for $\partial^\star\mathcal{R}_F$ directly from the necessary condition established in Theorem 2.2 (*cf.* the inclusions in (2.2)). However, in case (B) we must first break the sufficiently regular compact domain $\mathcal{K}$ into its interior and the pieces of its (topological) boundary that do form submanifolds, apply case (A) to find the included boundary of the range of *the restriction of F to each piece*, and finally adjust the combined results to arrive at the boundary of the range of F itself (which coincides in this case with the included boundary of the range, since $\mathcal{K}$ is compact).

We proceed to examine each case more fully.

Case A. Now we suppose that $F : \mathcal{M}^N \to \mathbb{R}^n$ is defined on the differentiable N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$, with $N_0 \geq N \geq n \geq 2$. Here are the broad steps of the plan for this case, each of which we shall amplify subsequently:

- **A.I.** Locate the set ΣF of singular points of F.
- **A.II.** Locate and discard those points of ΣF that are mapped to the interior $\mathcal{R}_F{}^\circ$, leaving $F^{-1}\{\partial^\star\mathcal{R}_F\}$.
- **A.III.** Locate $\Pi_{\mathrm{loc}}F$ within $F^{-1}\{\partial^\star\mathcal{R}_F\}$.

It is convenient to summarize the steps as we have for purposes of simplicity of the exposition, but we must append a *caveat*: at each stage we aim to eliminate, by whatever means available and as soon as possible, all singular points that cannot be Pareto minima (or, equivalently, all singular values that cannot be images of Pareto minima). Thus, we may not need to examine all of ΣF when we reach step (A.II), and/or we may not need to search through (or even know) all of $F^{-1}\{\partial^\star\mathcal{R}_F\}$ when we reach step (A.III).

The *partition* of the range given in (2.1) is important in the search procedure, since it implies that we can find $\partial^\star\mathcal{R}_F$ in step (A.II) just by removing from $F(\Sigma F)$ those points that lie in the interior $\mathcal{R}_F{}^\circ$. To explain the utility of this latter course and certain other aspects of the program, as well as to understand the origins of the various difficulties that arise, some preliminary orientation is needed.

Once we have found that some $\xi \in \mathcal{M}^N$ belongs to ΣF, we must determine whether $F(\xi)$ is contained in ${\mathcal{R}_F}^\circ$ or in $\partial\mathcal{R}_F$. Of course, we would much prefer to employ for this a decision procedure that is *local*, rather than *global*, in nature. By "local" we mean that we can decide which inclusion $F(\xi) \in {\mathcal{R}_F}^\circ$ or $F(\xi) \in \partial\mathcal{R}_F$ obtains by examining the behavior of F (and perhaps of its derivatives) in any conveniently small neighborhood of ξ; by "global" we indicate either that the decision requires discovery of some or all of the other preimages of $F(\xi)$ and a study of the mapping behavior near those other points in $\mathcal{M}^N$ or, worse, that we must find and study the action of F on certain open sets whose images do not contain $F(\xi)$ but have it as a limit point. Unfortunately, it usually transpires that we must confront a global examination at some stage(s), so we should concentrate on *minimizing* the amount of domain-wide searching that is required; for this, the partition (2.1) evidently offers the best opportunity, since testing for inclusion in ${\mathcal{R}_F}^\circ$ appears by all means to be easier than testing for inclusion in $\partial\mathcal{R}_F$. Indeed, it seems that there are no *usable* conditions—local or global—sufficient to ensure directly that F maps a given singular point to the "global boundary" $\partial\mathcal{R}_F$. In this connection, Theorems 4.1 and 5.1 are attractive because they offer *local* tests. On the other hand, while the first assertion of each of those theorems provides conditions sufficient to ensure that a point is mapped to what we call the "local included boundary of the range" (the terminology is explained below), values of the latter sort need not be contained in $\partial\mathcal{R}_F$, but may still belong to ${\mathcal{R}_F}^\circ$. However, in the second statement of each theorem we do give local conditions under which it is certain that a singular point is mapped to the interior ${\mathcal{R}_F}^\circ$.

In pursuing a strategy based on the identification of points of ${\mathcal{R}_F}^\circ$, it is important to keep in mind that there are, in a certain definite sense, only two alternatives for the local behavior of a mapping $F : \mathcal{M}^N \to \mathbb{R}^n$: for any $\xi \in \mathcal{M}^N$, either

(*i.*) $F(\xi) \in F(U_\xi)^\circ$ for *every* $\mathcal{M}^N$-neighborhood U_ξ of ξ

or

(*ii.*) $F(\xi) \in \partial F(U_\xi)$ for *some* $\mathcal{M}^N$-neighborhood U_ξ of ξ—in which case also $F(\xi) \in \partial F(V_\xi)$ for every neighborhood V_ξ of ξ that is contained in U_ξ.

In fact, whenever U_ξ is a neighborhood of ξ in $\mathcal{M}^N$ we have the partition $F(U_\xi) = F(U_\xi)^\circ \cup \partial^\star F(U_\xi)$, from which the two alternatives follow. Now, in the former case we certainly have $F(\xi) \in {\mathcal{R}_F}^\circ$, but in the latter case we can draw no conclusion, *i.e.*, even though (*ii*) holds, $F(\xi)$ may still be in either ${\mathcal{R}_F}^\circ$ or $\partial\mathcal{R}_F$, as various examples show. Obviously, alternative (*ii*) must obtain if $F(\xi)$ does belong to $\partial\mathcal{R}_F$, and, indeed, in that case we shall have $F(\xi) \in \partial^\star F(U_\xi)$ for *every* $\mathcal{M}^N$-neighborhood U_ξ of ξ. Leaving aside the possibility of some sort of degeneracy, such as the mapping of a full neighborhood of ξ to a point or to a manifold of dimension less than n, the inclusion in (*ii*) is by itself geometrically suggestive, since it indicates, roughly, that the mapping action may have "folded" the original neighborhood and placed $F(\xi)$ on the "crease"; the terms here are most apt in the case $n = 2$, when the range is in the plane. Now, even though it may appear from a local viewpoint that $F(\xi)$ belongs to $\partial\mathcal{R}_F$ when (*ii*) occurs, it is impossible to decide on the basis of such a local examination alone whether this is indeed the case, for one frequently finds that $F(\xi)$ belongs in fact to ${\mathcal{R}_F}^\circ$. The difficulty here usually arises because a neighborhood of some other preimage of $F(\xi)$ is mapped to a neighborhood of $F(\xi)$. Dealing with precisely this circumstance constitutes one of the main obstacles in the search for Pareto minima with the present method.

In this regard, it is clearly of importance to understand the possible ways in which an image $F(\xi)$ can belong to the interior ${\mathcal{R}_F}^\circ$, *i.e.*, to discern all of the ways in which an open set containing

$F(\xi)$ can be formed in $\mathcal{R}_F$. Again, in a certain sense there are only two possibilities: either there is a preimage ξ' of $F(\xi)$ and a neighborhood $U_{\xi'}$ of ξ' such that $(F(\xi') =)\ F(\xi) \in F\big(U_{\xi'}\big)^\circ$ or there is no such preimage. In the latter case, any neighborhood of $F(\xi)$ that is contained in $\mathcal{R}_F$ must be built up as a union of images that either overlap or just fit together to form a set containing $F(\xi)$ in its interior. This second case, which we have a tendency to regard as "pathological," evidently happens with sufficient frequency that we should be concerned about how to recognize it when it does occur.

We introduce some descriptive terms that are convenient for making reference to the various alternatives:

Definitions. Let $F : \mathcal{M}^N \to \mathbb{R}^n$. Let $x \in \mathcal{R}_F$ and $\xi \in F^{-1}\{x\}$, so that $x = F(\xi)$. We shall say that x is *locally covered by F from ξ* iff $x \in F(U_\xi)^\circ$ whenever U_ξ is an $\mathcal{M}^N$-neighborhood of ξ. We shall say that *the range $\mathcal{R}_F$ is folded at x with respect to ξ* iff there exists an $\mathcal{M}^N$-neighborhood U_ξ of ξ such that $x \in \partial F(U_\xi)$; by $\partial^\star_{\mathrm{loc}}\mathcal{R}_F$ we denote the set of all points at which $\mathcal{R}_F$ is folded (with respect to some preimage). We frequently refer to $\partial^\star_{\mathrm{loc}}\mathcal{R}_F$ as the *local included boundary of the range of F*; it is then natural to distinguish its subset $\partial^\star\mathcal{R}_F$ as the *global included boundary of the range of F*. Alternately, the set $\partial^\star\mathcal{R}_F$ can be called the *genuine included boundary of the range of F* and $\partial^\star_{\mathrm{loc}}\mathcal{R}_F \setminus \partial^\star\mathcal{R}_F = \partial^\star_{\mathrm{loc}}\mathcal{R}_F \bigcap {\mathcal{R}_F}^\circ$ the *spurious included boundary of the range.*

Remarks. (1.) For $x \in \mathcal{R}_F$, precisely one alternative holds *with respect to a given preimage ξ*: either x is locally covered by F from ξ or the range is folded at x with respect to ξ; but *relative to a second preimage*, the other alternative may obtain.

(2.) Clearly, each regular value of F is locally covered by F from any and every (regular-point!) preimage of that regular value; a singular value that is not completely singular will be covered in just the same way, from a regular preimage. However, examples show that a singular value may also be locally covered from a singular-point preimage. In Theorem 4.1.*ii* and Theorem 5.1.*ii* we give conditions sufficient to ensure that $F(\xi)$ is locally covered from the rank-1 singular point ξ in the case $n = 2$ and when F has additional smoothness.

(3.) Of course, if the range of F is folded at $F(\xi)$ with respect to ξ, then ξ is necessarily a singular point of F. Theorem 4.1.*i* and Theorem 5.1.*i* provide sets of conditions under which one is certain that the range of F is folded at $F(\xi)$ with respect to ξ, when the latter is a rank-1 singular point and $n = 2$, again for a smoother F.

(4.) The term "fold" has long been in use in the study of singularities of smooth maps, where it was introduced for the same descriptive purpose. We shall continue to say that the range is "folded at $F(\xi)$ with respect to ξ" and "$F(\xi)$ belongs to the local included boundary" even when some degeneracy occurs, *e.g.*, even when F is constant in a neighborhood of ξ, and the geometric motivation for the term breaks down.

The local mapping direction. As usual, let $F : \mathcal{M}^N \to \mathbb{R}^n$ be of class C^1, with $\mathcal{M}^N$ a differentiable N-submanifold-without-boundary in $\mathbb{R}^{N_0}$. Let ξ be a singular point of F with $F(\xi)$ contained in the local included boundary $\partial^\star_{\mathrm{loc}}\mathcal{R}_F$ of the range. With sufficient local regularity of F, we can introduce in a natural way a vector indicating the "local mapping direction" of F; when $F(\xi)$ is a point of the (genuine) boundary of the range at which the range is regular (*cf*. the definition given in Section 1), this local mapping direction will just coincide with that of the interior normal to $\partial\mathcal{R}_F$. For this, let U_ξ be an $\mathcal{M}^N$-neighborhood of ξ such that $F(\xi) \in \partial F(U_\xi)$ and suppose that $F(U_\xi)$ is regular at $F(\xi)$; in this case, we say that *the local included boundary is regular at $F(\xi)$*, or $F(\xi)$ is *a regular point of the local included boundary*. Thus, there is a neighborhood of $F(\xi)$ in $\partial F(U_\xi)$ that is an

$(n-1)$-submanifold of class C^1 and $F(U_\xi)^\circ$ lies on one side of its boundary in an $\mathbb{R}^n$-neighborhood of $F(\xi)$. Moreover, there is a unique unit-normal $\hat{\nu}$ to $\partial F(U_\xi)$ at $F(\xi)$ such that $F(\xi)+s\hat{\nu}$ lies in $F(U_\xi)^\circ$ for all sufficiently small positive s; we call any positive multiple of such a normal $\hat{\nu}$ a *mapped-side normal at* $F(\xi)$, and refer to the direction of a mapped-side normal as *the local mapping direction at* $F(\xi)$.

In this terminology, in Lemma 1.2 it was shown how the determination of the mapped-side normal at a regular point $F(\xi)$ of the genuine included boundary $\partial^\star\mathcal{R}_F$—which is, in that case, just the interior normal—frequently permits one to decide whether ξ is a Pareto minimum. But it is also sometimes helpful to know how to compute such a normal, more generally, at any given (appropriate) point of $\partial^\star_{\mathrm{loc}}\mathcal{R}_F$, when one is trying to determine whether that point belongs to the interior or to the genuine boundary of the range; we return to this point below. Following the statements of Theorems 4.1 and 5.1, we indicate how one can compute the local mapping direction from the derivatives of F, when $n = 2$ and appropriate conditions are fulfilled.

Now we can lay out in more detail the indicated three steps into which the proposed search procedure naturally falls, keeping in mind the standing hypotheses concerning the mapping F in case (A).

A.I. *Find the set* ΣF *of singular points of* F*, along with the rank of each singular point.*

Numerical procedures for effecting this search can be based upon the points discussed at the end of Section 2 or similar considerations.

A.I.0 *Apply to each singular point all available tests of a local nature.*

This "intermediate step" is inserted here with the aim of minimizing the number of later global searches that may be needed. Thus, given a singular point ξ, we should apply at this stage any local tests that we may have, such as Theorem 4.1 or 5.1, giving conditions under which one can assert whether the range is folded at $F(\xi)$ with respect to ξ or $F(\xi)$ is locally covered from ξ; in the latter case, ξ can be discarded.

For example, one can review Remarks (2) and (3), *supra*, for the information that may be obtained from Theorem 4.1 or 5.1. Those theorems presently apply only under various conditions restricting their applicability, *e.g.*, the bicriterial assumption ($n = 2$) and the hypothesis of strong nondegeneracy.

Theorems 4.1 and 5.1 may be regarded as analogues of the classical "second-derivative test" based on the definiteness properties of the Hessian matrix at a critical point in the case $n = 1$. Indeed, just as the latter test may be inconclusive, Theorem 4.1 or 5.1 may also fail to provide information, even when applicable.

The computation of the local mapping direction is another useful test that should be applied at this stage to as many singular points and values as possible. That is, while it is discussed under step (A.III) at the very end, this test should be employed at the earliest moment to eliminate all singular values known to be on the local boundary of the range (with respect to some preimage) but for which the mapped-side normal is found to have at least one negative component. In this way, we may obviate the conduct of global searches for preimages for large numbers of singular values, which is required in step (A.II). Inspection of Examples 6.4, 6.5, and 6.6 will indicate that this is so. On the other hand, such an examination will also reveal many values in the interior of the range that also lie on the local boundary of the range and have "the correct" local mapping direction, and so cannot be discarded by this tactic.

A.II. *Eliminate from ΣF its intersection with $F^{-1}\{\mathcal{R}_F{}^\circ\}$, leaving $F^{-1}\{\partial^\star \mathcal{R}_F\}$.*

That is, here one seeks to throw out all of the remaining singular points that are mapped to the interior of the range. It is convenient to split this step into three parts.

A.II.1. *Find and discard all remaining $\xi \in \Sigma F$ such that $F(\xi)$ has a regular-point preimage (from which it is then locally covered).*

Here we are to pare the still-eligible singular points down to (perhaps a subset of) the completely singular points $\Sigma^\star F$, by executing for each remaining $\xi \in \Sigma F$ a global search for a regular preimage of $F(\xi)$; as soon as one such preimage is found, we shall know that $F(\xi)$ lies in $\mathcal{R}_F{}^\circ$, so ξ can be eliminated. In particular, those points of the local included boundary $\partial^\star_{\mathrm{loc}}\mathcal{R}_F$ that are also images of regular points in the domain will be eliminated in this computation. A major portion of the global examination is required in this step.

A.II.2. *Find and discard all remaining $\xi \in \Sigma F$ such that $F(\xi)$ is locally covered from the singular point ξ itself.*

The singular points in question here are those for which the testing of step (A.I.0) is either inapplicable or inconclusive, and so must be treated on an *ad hoc* basis. On the other hand, the examinations required now are local, since we can ignore here the possibility that two completely singular points have the same image under F, simply by studying each of these points in turn.

A.II.3. *Find and discard all remaining $\xi \in \Sigma F$ such that $F(\xi)$ belongs to $\mathcal{R}_F{}^\circ$ but is locally covered from no preimage.*

If the singular point ξ has survived to this stage, then the range is folded at $F(\xi)$ with respect to ξ, while $F(\xi)$ is locally covered from none of its preimages, since local covering from both types of preimages—regular and singular—has by now been checked and ruled out. Therefore, as we have already observed, $F(\xi)$ can lie in $\mathcal{R}_F{}^\circ$ only because it is covered by a union of images, none of which contains $F(\xi)$ in its interior. The analysis of this circumstance is difficult but essential, since we are trying to decide which singular points are mapped to the boundary of the range by retaining precisely those that are not mapped to the interior. While here again we may have to rely in general upon *ad hoc* devices such as graphical inspection of the images of many judiciously selected points, we can indicate some approaches that are applicable in various common situations.

First we present a (nonlocal) test pertaining to the case in which the singular value under examination has just one singular-point preimage under F.

Proposition 3.1. *Let $F : \mathcal{M}^N \to \mathbb{R}^n$ be a differentiable mapping of the C^1 N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ into $\mathbb{R}^n$, with $N_0 \geq N \geq n \geq 2$, and assume that F possesses a continuous extension $\widetilde{F}$ to the closure of $\mathcal{M}^N$ in $\mathbb{R}^{N_0}$. Suppose further that*

$$\text{any sequence } \left(\xi^j\right)_{j=1}^\infty \text{ in } \mathcal{M}^N \text{ for which } \left(F\left(\xi^j\right)\right)_{j=1}^\infty \text{ converges in } \mathbb{R}^n \text{ must be bounded.} \qquad (\star)$$

Let $\xi^0 \in \Sigma F$ and suppose that the range of F is folded at $F\left(\xi^0\right)$ with respect to ξ^0.

(i.) If $\widetilde{F}^{-1}\{F(\xi^0)\} = \{\xi^0\}$, i.e., if ξ^0 is the only preimage of $F\left(\xi^0\right)$ under the extension $\widetilde{F}$, then $F\left(\xi^0\right) \in \partial\mathcal{R}_F$.

(ii.) If $\mathcal{M}^N$ is closed in $\mathbb{R}^{N_0}$, then $F\left(\xi^0\right) \in \partial\mathcal{R}_F$ if ξ^0 is the only preimage of $F\left(\xi^0\right)$ under F itself.

Remark. Hypothesis $(\star)$ of Proposition 3.1 obtains if either $\mathcal{M}^N$ is bounded in $\mathbb{R}^{N_0}$ or $\mathcal{M}^N$ is unbounded but we can assert that the image of any unbounded sequence under F is unbounded, as may happen when, *e.g.*, the components of F are (nonconstant) polynomials.

Proof. Obviously, statement (*ii*) will follow immediately once (*i*) is established, so we consider only the proof of the latter. Suppose, contrary to the claim, that $F(\xi^0) \in \mathcal{R}_F{}^\circ$. Then, with U_0 denoting a neighborhood of ξ^0 in $\mathcal{M}^N$ such that $F(\xi^0) \in \partial F(U_0)$, there is a sequence $(\xi^j)_{j=1}^\infty$ such that $(F(\xi^j))_{j=1}^\infty$ lies outside of $F(U_0)$ and converges in $\mathbb{R}^n$ to $F(\xi^0)$. Because of the hypothesis $(\star)$, the sequence $(\xi^j)_{j=1}^\infty$ is bounded, so we may suppose it converges to some ξ^∞ in the closure of $\mathcal{M}^N$, by replacing the original sequence with a convergent subsequence and adjusting the notation if necessary. The limit ξ^∞ belongs to the closure of $\mathcal{M}^N$, with

$$\widetilde{F}(\xi^\infty) = \lim_{j\to\infty} \widetilde{F}(\xi^j) = \lim_{j\to\infty} F(\xi^j) = F(\xi^0),$$

which requires that $\xi^\infty = \xi^0$, since $F(\xi^0)$ has only the preimage ξ^0 under the extension $\widetilde{F}$. But this is impossible, since $(\xi^j)_{j=1}^\infty$ must lie outside of the neighborhood U_0 (since $(F(\xi^j))_{j=1}^\infty$ lies outside of $F(U_0)$), and so cannot converge in $\mathcal{M}^N$ to ξ^0. The contradiction implies that $F(\xi^0) \in \partial\mathcal{R}_F$, completing the proof. $\square$

Now we briefly consider the case in which a completely singular value $F(\xi)$ has two or more singular-point preimages from none of which it is locally covered. In fact, as in Proposition 3.1, when $\mathcal{M}^N$ is not closed in $\mathbb{R}^{N_0}$ but F has continuous extension to the closure of $\mathcal{M}^N$ one should reckon the number of preimages by accounting also for any additional ones that arise under the continuous extension $\widetilde{F}$ of F (and which therefore lie on the topological boundary of $\mathcal{M}^N$). In fact, the existence of a preimage under $\widetilde{F}$ indicates that there is some region at the boundary of $\mathcal{M}^N$ that is mapped by F to a set whose boundary will contain $F(\xi)$, and which therefore may contribute to a neighborhood of $F(\xi)$.

Consider the bicriterial case, $n = 2$: if we discover curves of singular values passing through $F(\xi)$ and nontangent there, then it is difficult to make a general statement. However, in Section 6 we offer two examples, Example 6.3 and Example 6.4, in each of which there is an entire curve Γ of singular values such that each value has two singular-point preimages from neither of which it is locally covered. Now, in Example 6.3 at each value on Γ the local mapping directions associated with the two preimages are opposite, and one discovers the two half-neighborhoods of each value combining to form a full neighborhood, whence Γ is in the interior of the range of F; this can be seen in Figure 7, by studying the blue and the cyan mapped-side normals. On the other hand, Example 6.4 exhibits the opposite behavior: at each value on Γ the mapped-side normals coincide, so full neighborhoods are not formed from the half-neighborhoods. Since there are no other preimages of any of these singular values, they must in fact lie on the global boundary of the range; the graphical evidence appearing in Figure 12 illustrates this phenomenon.

Even these simple examples display the complexity of the mapping interactions that are possible, and so must be anticipated, in the analysis of this step (A.II.3).

A.III. *Find the local Pareto minima* $\Pi_{\mathrm{loc}}F$ *by examining the mapping behavior in a neighborhood of each point of* $F^{-1}\{\partial^\star\mathcal{R}_F\}$.

We assume now that we have completely determined the preimage $F^{-1}\{\partial^\star\mathcal{R}_F\}$. We must examine each singular point ξ such that $F(\xi) \in \partial\mathcal{R}_F$ and retain only those that prove to be local Pareto minima. If $\mathcal{R}_F$ is regular at $F(\xi)$, then we should calculate the mapped-side normal at $F(\xi)$, *i.e.*, the interior normal to the boundary of the range at $F(\xi)$ (if this has not already been found in step (A.I.0)), and apply Lemma 1.2; the exceptional cases for which this computation is inconclusive (*i.e.*, when the components of the interior normal are nonnegative and at least one is zero) must be separately considered. We already mentioned that the determination of the mapped-side normal at appropriate points of $\partial^\star_{\mathrm{loc}}\mathcal{R}_F$ is described in Sections 4 and 5 for the bicriterial case, in conjunction with the statements of Theorems 4.1 and 5.1. On the other hand, if $\mathcal{R}_F$ fails to be regular at $F(\xi)$, we have presently no systematic procedure to offer; some *ad hoc* investigation proceeding from the fundamental condition given in Lemma 1.1 is required to decide whether ξ is a local Pareto minimum. For example, included here are those cases in which the range degenerates locally to form a manifold of dimension less than n (such as a curve in the bicriterial setting); the interior of the range must be locally void in such situations, so that there will be no regular points on such a manifold.

Case B. Now we suppose that $F : \mathcal{K} \to \mathbb{R}^n$ is defined on the compact and regularly closed subset $\mathcal{K} \subset \mathbb{R}^{N_0}$, with $N_0 \geq n \geq 2$. We assume also that $\mathcal{K}$ has "sufficiently regular" boundary. As always, our initial aim is the identification of the included boundary of the range, which here is the entire boundary $\partial F(\mathcal{K})$. Already, it was indicated that the strategy for analysis in this case (B) begins with the separate location of the included boundaries of the images $F(\mathcal{K}^\circ)$ of the interior and $F(\partial\mathcal{K})$ of the boundary. The first of these is a case-(A) problem; in the simplest situation, when $\partial\mathcal{K}$ is an (N_0-1)-submanifold of class C^1 in $\mathbb{R}^{N_0}$, the second task will also fall under case (A). In any event, we must know how to combine the results of the separate included-boundary determinations to arrive ultimately at the desired boundary $\partial F(\mathcal{K})$ of the entire range. To shed light on this question, we offer

Proposition 3.2. *Let $F : \mathcal{K} \to \mathbb{R}^n$ be continuous, in which the domain $\mathcal{K} \subset \mathbb{R}^N$ is compact and regularly closed.*

- (*i.*) *The boundary $\partial F(\mathcal{K})$ satisfies*

$$\partial F(\mathcal{K}) \subset \Big\{\partial F(\mathcal{K}^\circ) \cap F(\mathcal{K}^\circ)\Big\} \cup \Big\{\partial F(\partial\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\}\Big\}; \tag{3.1}$$

  the inclusion in (3.1) *may be proper.*

- (*ii.*) *We have $F(\mathcal{K}) = \overline{F(\mathcal{K}^\circ)}$ and $\partial F(\mathcal{K}) \subset \partial F(\mathcal{K}^\circ)$.*
- (*iii.*) *The equality $F(\mathcal{K})^\circ = F(\mathcal{K}^\circ)^\circ$ holds iff $\partial F(\mathcal{K}) = \partial F(\mathcal{K}^\circ)$ holds; when they obtain, the inclusion in* (3.1) *can be replaced by equality.*

Remarks. Maintain the setting of the Proposition. Of course, here we have $\partial F(\mathcal{K}) = \partial^\star F(\mathcal{K})$ and $\partial F(\partial\mathcal{K}) = \partial^\star F(\partial\mathcal{K})$, since $F(\mathcal{K})$ and $F(\partial\mathcal{K})$ are (compact and so) closed. The two sets appearing within large braces on the right in (3.1) are obviously disjoint. The inclusion (3.1) says that we *can* find all of $\partial F(\mathcal{K})$ by a two-step procedure, looking separately in $F(\mathcal{K}^\circ)$ for $\partial^\star F(\mathcal{K}^\circ) := \partial F(\mathcal{K}^\circ) \cap F(\mathcal{K}^\circ)$ (involving an application of case (A)) and in $F(\partial\mathcal{K})$ for $\partial F(\partial\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\}$ (involving further application(s) of case (A)). Of course, for the latter we must also have some means for deciding whether a point lies in $F(\mathcal{K}^\circ)$ or in its complement. However, when the inclusion is proper, we will pick up in this way points that do not lie in $\partial F(\mathcal{K})$; some other examination is necessary to identify and discard such points.

Proof. The results are almost purely topological, *i.e.*, are essentially independent of any special properties of the present setting involving continuous maps between euclidean spaces. We prove first (*ii*), then the first half of (*iii*), then establish all but the final assertion of (*i*) and complete the proof of (*iii*). A counterexample then serves to finish the proof of (*i*).

(ii). From the equality $\mathcal{K} = \overline{\mathcal{K}^\circ}$ and the continuity of F, we see that $F(\mathcal{K}) = F\big(\overline{\mathcal{K}^\circ}\big) \subset \overline{F(\mathcal{K}^\circ)}$. On the other hand, we certainly have $F(\mathcal{K}^\circ) \subset F(\mathcal{K})$, so, since the compactness of $F(\mathcal{K})$ implies that it is closed, the reverse inclusion $\overline{F(\mathcal{K}^\circ)} \subset F(\mathcal{K})$ also follows. This establishes the equality $F(\mathcal{K}) = \overline{F(\mathcal{K}^\circ)}$. From the latter result and the inclusion $F(\mathcal{K}^\circ)^\circ \subset F(\mathcal{K})^\circ$, we get

$$\partial F(\mathcal{K}^\circ) = \overline{F(\mathcal{K}^\circ)} \setminus F(\mathcal{K}^\circ)^\circ = F(\mathcal{K}) \setminus F(\mathcal{K}^\circ)^\circ \supset F(\mathcal{K}) \setminus F(\mathcal{K})^\circ = \overline{F(\mathcal{K})} \setminus F(\mathcal{K})^\circ = \partial F(\mathcal{K}),$$

completing the proof of (ii).

(iii). By assuming that $F(\mathcal{K})^\circ = F(\mathcal{K}^\circ)^\circ$, we find, having (ii) and again using the fact that $F(\mathcal{K})$ is closed, that

$$\partial F(\mathcal{K}) = F(\mathcal{K}) \setminus F(\mathcal{K})^\circ = \overline{F(\mathcal{K}^\circ)} \setminus F(\mathcal{K}^\circ)^\circ = \partial F(\mathcal{K}^\circ).$$

Conversely, if we suppose that $\partial F(\mathcal{K}) = \partial F(\mathcal{K}^\circ)$ and note that $\overline{A} = A^\circ \cup \partial A$ whenever A is a subset of a topological space, we get

$$F(\mathcal{K})^\circ = F(\mathcal{K}) \setminus \partial F(\mathcal{K}) = \overline{F(\mathcal{K}^\circ)} \setminus \partial F(\mathcal{K}) = \Big\{F(\mathcal{K}^\circ)^\circ \cup \partial F(\mathcal{K}^\circ)\Big\} \setminus \partial F(\mathcal{K}^\circ) = F(\mathcal{K}^\circ)^\circ,$$

the latter equality holding because $F(\mathcal{K}^\circ)^\circ$ and $\partial F(\mathcal{K}^\circ)$ are disjoint. Thus, the two equalities are indeed equivalent. We complete the proof of (iii) during the proof of (i).

(i). We begin with the partition

$$\partial F(\mathcal{K}) = \Big\{\partial F(\mathcal{K}) \cap F(\mathcal{K}^\circ)\Big\} \cup \Big\{\partial F(\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\}\Big\},$$

whence it is clear that (3.1) will follow once we have proven that

$$\partial F(\mathcal{K}) \cap F(\mathcal{K}^\circ) \subset \partial F(\mathcal{K}^\circ) \cap F(\mathcal{K}^\circ) \tag{3.2}$$

and

$$\partial F(\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\} \subset \partial F(\partial\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\}, \tag{3.3}$$

while the proof of (iii) will be complete if we show that equality holds in both (3.2) and (3.3) when the equivalent equalities in the statement of (iii) obtain. Clearly, (3.2) holds in view of the inclusion $\partial F(\mathcal{K}) \subset \partial F(\mathcal{K}^\circ)$, established in (ii), while the inclusion in (3.2) can certainly be strengthened to equality when we know that $\partial F(\mathcal{K}) = \partial F(\mathcal{K}^\circ)$. Turning to the verification of (3.3), let us begin by showing that

$$\partial F\big(\mathcal{K}\big) \cap F\big(\partial\mathcal{K}\big) \subset \partial F\big(\partial\mathcal{K}\big). \tag{3.4}$$

Since $\partial\mathcal{K}$ is compact, $F(\partial\mathcal{K})$ is also compact, and therefore closed, so we have the partition $F(\partial\mathcal{K}) = F(\partial\mathcal{K})^\circ \cup \partial F(\partial\mathcal{K})$. Now let $x \in \partial F\big(\mathcal{K}\big) \cap F\big(\partial\mathcal{K}\big)$, and suppose that $x \in F(\partial\mathcal{K})^\circ$: since $F(\partial\mathcal{K})^\circ \subset F(\mathcal{K})^\circ$, we would then have $x \in F\big(\mathcal{K}\big)^\circ \cap \partial F\big(\mathcal{K}\big)$, which is impossible. We conclude then that $x \in \partial F\big(\partial\mathcal{K}\big)$, establishing (3.4). (Actually, one can show further that $\partial F\big(\mathcal{K}\big) \cap F\big(\partial\mathcal{K}\big) = \partial F\big(\mathcal{K}\big) \cap \partial F\big(\partial\mathcal{K}\big)$.) Now, returning to (3.3), let $x \in \partial F(\mathcal{K}) \cap \big\{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\big\}$: then $x \in F(\mathcal{K}) \setminus F(\mathcal{K}^\circ)$, which implies that $x \in F\big(\partial\mathcal{K}\big)$, since we have the partition $\mathcal{K} = \mathcal{K}^\circ \cup \partial\mathcal{K}$. Therefore, we have $x \in \partial F\big(\mathcal{K}\big) \cap F\big(\partial\mathcal{K}\big)$, whence (3.4) gives $x \in \partial F\big(\partial\mathcal{K}\big)$; this effectively completes the proof of (3.3). Finally, suppose that we also know that $F(\mathcal{K})^\circ = F(\mathcal{K}^\circ)^\circ$; to see that the inclusion in (3.3) can then be replaced by equality, let $x \in \partial F(\partial\mathcal{K}) \cap \big\{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\big\}$. Then $x \in F(\mathcal{K})$ but $x \notin F(\mathcal{K}^\circ)$; it follows that $x \notin F(\mathcal{K}^\circ)^\circ$, so with our assumed relation we also get $x \notin F(\mathcal{K})^\circ$. Therefore, $x \in F(\mathcal{K}) \setminus F(\mathcal{K})^\circ = \partial F(\mathcal{K})$. With this, we can assert that equality must hold in (3.3). Now (iii) has been proven.

We conclude the proof of (i), and of the Proposition, by citing Example 3.1, *infra*, in which equality between the sets $F(\mathcal{K})^\circ$ and $F(\mathcal{K}^\circ)^\circ$ and between the sets $\partial F(\mathcal{K})$ and $\partial F(\mathcal{K}^\circ)$ fails to hold and the inclusion in (3.1) is strict. The example indicates that difficulties arise when dealing with a mapping whose range does not lie locally on one side of its boundary. $\square$

Example 3.1. Let $N = n = 2$. Take $\mathcal{K}$ to be the closure of the upper half of the unit disk centered at the origin; let F be the map with components given by $F_1(x_1, x_2) := x_1^2 - x_2^2$, $F_2(x_1, x_2) := 2x_1x_2$, *i.e.*, F is just $z \mapsto z^2$ with $z = x_1 + \mathrm{i}x_2$. Then $F(\mathcal{K})$ is the closed unit disk, while $F(\mathcal{K}^\circ)$ is the open disk with the closed slit from $(0,0)$ to $(1,0)$ removed (and so is open); $\partial F(\partial\mathcal{K})$ is the union of the unit circle and the slit, so $\partial F(\partial\mathcal{K}) = \partial F(\mathcal{K}^\circ)$. Clearly, $F(\mathcal{K})^\circ \supsetneqq F(\mathcal{K}^\circ)^\circ$ and $\partial F(\mathcal{K}) \subsetneqq \partial F(\mathcal{K}^\circ)$, while $\partial F(\mathcal{K}^\circ) \cap F(\mathcal{K}^\circ)$ is void and $\partial F(\partial\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\} = \partial F(\partial\mathcal{K})$, so equality does not hold in (3.1).

We shall consider case (B) in more detail only in the nicest situation, already identified, in which the boundary $\partial\mathcal{K}$ is an $(N_0 - 1)$-submanifold of $\mathbb{R}^{N_0}$ of class C^1. We also suppose that, by some means, we have shown that equality holds in (3.1). Under these simplifying circumstances, the strategy indicated for case (B) and the conclusions of Proposition 3.2 are implemented through these steps:

B.I. *Find the included boundary* $\partial^\star F(\mathcal{K}^\circ)$.

Here, one applies the procedure of case (A) to the restriction $F \,|\, \mathcal{K}^\circ$ of F to the interior of $\mathcal{K}$.

B.II. *Find the (included) boundary* $\partial F(\partial\mathcal{K})$.

Now the procedure of case (A) is to be used for analyzing the restriction $F \,|\, \partial\mathcal{K}$ of F to the boundary of $\mathcal{K}$.

B.III. *Find and discard all the points of* $\partial F(\partial\mathcal{K})$ *that also belong to* $F(\mathcal{K}^\circ)$.

Evidently, here we are forced to conduct global searches for preimages of the points of $\partial F(\partial\mathcal{K})$ that belong to $\mathcal{K}^\circ$. Succeeding in this, we produce the set $\partial F(\partial\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\}$, whence, according to the assumed equality in (3.1), we shall have identified the entire boundary $\partial F(\mathcal{K})$ as the union of two disjoint sets. Consequently, we also have the desired preimage $F^{-1}\{\partial F(\mathcal{K})\}$.

B.IV. *Find* $\Pi_{\mathrm{loc}}F$ *within* $F^{-1}\{\partial F(\mathcal{K})\}$.

Here, just as in case (A), we can use Lemma 1.2 to test a point ξ of the preimage $F^{-1}\{\partial F(\mathcal{K})\}$ such that the range $F(\mathcal{K})$ is regular at $F(\xi)$, by computing and inspecting the local mapping direction at $F(\xi)$. For regular points of $\partial F(\mathcal{K}^\circ) \cap F(\mathcal{K}^\circ)$ the local mapping direction is available from the computations of step (B.I); for regular points of $\partial F(\partial\mathcal{K}) \cap \{\mathbb{R}^n \setminus F(\mathcal{K}^\circ)\}$ the local mapping direction is known from the analysis of step (B.II). Once again, however, we have no systematic attack to recommend for cases in which the range is not regular at $F(\xi)$; a special study must be undertaken for such points.

We complete our discussion of case (B) by remarking on the situation in which F has no singular points in the interior of its domain; this seems to occur sufficiently frequently in applications that it is worthwhile clarifying the implications. (If F has no singular points in case (A), the included boundary of the range of F must be empty, which implies simply that F has no Pareto minima.) Suppose then that the restriction of $F : \mathcal{K} \to \mathbb{R}^n$ to the interior $\mathcal{K}^\circ$ has no singular points. Then no point of $\mathcal{K}^\circ$ is mapped to the boundary of the range of F, so $\partial F(\mathcal{K})$ lies entirely in the image $F(\partial\mathcal{K})$ of the boundary of the domain, and in fact, according to Proposition 3.2, lies in the boundary $\partial F(\partial\mathcal{K})$ of that image. The entire boundary of the range can then be identified by finding and discarding those points of $\partial F(\partial\mathcal{K})$ that are contained in $F(\mathcal{K}^\circ)$.

This completes our outline of the proposed search procedures for both cases (A) and (B).

4. Discrimination of singular values: the open-set domain.

As discussed in Section 3, once we have identified the singular points of F we need further tests for distinguishing between those (completely) singular points with images lying on the boundary of the range and those with images contained in the interior of the range, which will enable us to discard the latter. For a next step in this analysis, we develop conditions sufficient to identify singular values at which the range is *folded* with respect to a given singular-point preimage (which therefore have a chance of belonging to the boundary of the range) and singular values that are *locally covered from a singular preimage* (which therefore lie in the interior of the range). From this point on, we restrict attention to the bicriterial case $n = 2$. In the present section, in particular, for the main result in Theorem 4.1, we take up the case of a mapping $F : \mathcal{U} \to \mathbb{R}^2$ defined on an open set $\mathcal{U} \subset \mathbb{R}^N$, with $N \geq 2$, and consider in the next Section 5 the more general case of an F defined on a submanifold in a euclidean space.

The statement in Theorem 4.1 can be regarded as a sort of analogue of the Hessian-matrix criterion for identifying whether a critical value of a *real*-valued function $f : \mathcal{U} \to \mathbb{R}$ is a local minimum, local maximum, or saddle point. In particular, we require more smoothness of F in the formulation of the test.

As always, we require at least that F be differentiable at each point of $\mathcal{U}$. Then for each $\xi \in \mathcal{U}$ the differential of F at ξ is the linear operator $\mathrm{D}F(\xi) : \mathbb{R}^N \to \mathbb{R}^2$ given by

$$\mathrm{D}F(\xi)h := \begin{pmatrix} \mathrm{d}F_1(\xi)h = \operatorname{grad} F_1(\xi) \cdot h = \displaystyle\sum_{j=1}^{N} F_{1},_{j}(\xi)h_j \\ \\ \mathrm{d}F_2(\xi)h = \operatorname{grad} F_2(\xi) \cdot h = \displaystyle\sum_{j=1}^{N} F_{2},_{j}(\xi)h_j \end{pmatrix}, \qquad \text{for} \quad h \in \mathbb{R}^N. \tag{4.1}$$

For a regular point ξ, this operator has (maximal) rank 2, so that the set $\{\operatorname{grad} F_1(\xi), \operatorname{grad} F_2(\xi)\}$ of gradients is linearly independent; $\mathrm{D}F(\xi)$ then carries $\mathbb{R}^N$ onto $\mathbb{R}^2$, with $(N-2)$-dimensional null space $\mathcal{N}\big(\mathrm{D}F(\xi)\big)$ formed by the orthogonal complement of $\operatorname{sp}\{\operatorname{grad} F_1(\xi), \operatorname{grad} F_2(\xi)\}$ in $\mathbb{R}^N$. For a singular point ξ of rank 1, which is the case of greatest interest for us, the differential $\mathrm{D}F(\xi)$ has rank 1, *i.e.*, has range of dimension 1, and so $\mathcal{N}\big(\mathrm{D}F(\xi)\big)$ is of dimension $N-1$. This rank-1 case occurs precisely when the set $\{\operatorname{grad} F_1(\xi), \operatorname{grad} F_2(\xi)\}$ of gradients is linearly dependent but not both gradients vanish; the span of the gradients is then of dimension 1 and the $(N-1)$-dimensional orthogonal complement of this span in $\mathbb{R}^N$ coincides with the null space $\mathcal{N}\big(\mathrm{D}F(\xi)\big)$. Further, it is easy to see that the range of $\mathrm{D}F(\xi)$ at the rank-1 singular point $\xi \in \mathcal{U}$ is the span of the vector $F_{1},_{j}(\xi)\varepsilon^{(1)} + F_{2},_{j}(\xi)\varepsilon^{(2)}$ in $\mathbb{R}^2$, with j denoting any integer in $\{1, \dots, N\}$ such that $\big[F_{1},_{j}(\xi)\big]^2 + \big[F_{2},_{j}(\xi)\big]^2 > 0$ (there exists at least one such j). Finally, $\xi \in \mathcal{U}$ is a singular point of rank 0 iff $\mathrm{D}F(\xi)$ is the trivial operator, *i.e.*, iff $\operatorname{grad} F_1(\xi) = \operatorname{grad} F_2(\xi) = 0$; the null space is then all of $\mathbb{R}^N$.

In all of the reasoning of this section, we impose at least hypotheses (H.1) and (H.2) of Section 2. Thus, $\xi^0 \in \mathcal{U}$ always denotes a rank-1 singular point of F contained in a 1-manifold of rank-1 singular points that is the range of a C^1-imbedding $\chi : (t_1, t_2) \to \mathcal{U}$, so we have $\chi(t) \in \Sigma_1 F$ for $t_1 < t < t_2$ and $\xi^0 = \chi(t_0)$ for some $t_0 \in (t_1, t_2)$. Moreover, the composition $t \mapsto F\big(\chi(t)\big) \in \mathbb{R}^2$, $t_1 < t < t_2$, is supposed to be a C^1-embedding of the interval (t_1, t_2) into $\mathbb{R}^2$, whose range is then a 1-manifold comprising singular values of F. The derivative of the latter imbedding is $t \mapsto \mathrm{D}F\big(\chi(t)\big)\chi'(t) = \{\operatorname{grad} F_1\big(\chi(t)\big) \cdot \chi'(t)\}\varepsilon^{(1)} + \{\operatorname{grad} F_2\big(\chi(t)\big) \cdot \chi'(t)\}\varepsilon^{(2)}$, which must be nonzero for $t_1 < t < t_2$, *i.e.*,

$\chi'(t) \notin \mathcal{N}\big(\mathrm{D}F(\chi(t))\big)$ for those t. Recalling the common terminology noted in Section 2, we can say that no point of the curve of singular values, including $F(\xi^0)$, is a cusp.

It is important to point out how one can compute tangent (and normal) vectors to the curve $t \mapsto F\big(\chi(t)\big)$ of singular values without explicit calculation of $\chi'(t)$—indeed, even without knowledge of χ. The simple recipe for this is given in Proposition 4.1.*ii*, and follows readily from a fundamental fact describing how any smooth curve passing through a rank-1 singular point in a certain manner must behave under the mapping F.

Proposition 4.1. *Let $N \geq 2$. Let $F : \mathcal{U} \to \mathbb{R}^2$ be of class C^1 on the open set $\mathcal{U} \subset \mathbb{R}^N$. Suppose that ξ^0 is a rank-1 singular point of F for which hypotheses (H.1) and (H.2) hold.*

- (i.) *Let $t \mapsto \zeta(t) \in \mathcal{U}$, for $t_1 < t < t_2$, be a class-C^1, smooth curve with range in $\mathcal{U}$ such that $\zeta(t_0) = \chi(t_0)$ $(= \xi^0)$ and $\zeta'(t_0) \notin \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$. Then the curves $t \mapsto F\big(\chi(t)\big)$ and $t \mapsto F\big(\zeta(t)\big)$ have the same tangent vectors at the point $F(\xi^0)$.*
- (ii.) *Let j denote any integer in $\{1, \ldots, N\}$ such that $\big[F_{1,j}(\xi^0)\big]^2 + \big[F_{2,j}(\xi^0)\big]^2 > 0$ (there exists at least one such j). Then the vector $F_{1,j}(\xi^0)\varepsilon^{(1)} + F_{2,j}(\xi^0)\varepsilon^{(2)}$ is a tangent vector to the singular-value curve $t \mapsto F\big(\chi(t)\big)$ at the point $F(\xi^0)$.*

Proof. (i). Since $\mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$ has dimension $N-1$ and does not contain $\chi'(t_0)$ (which is nonzero), we can write $\zeta'(t_0) = \alpha_0\chi'(t_0) + \nu_0$ for some $\alpha_0 \in \mathbb{R}$ and $\nu_0 \in \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$; α_0 must be nonzero, since $\zeta'(t_0) \notin \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$. With these observations, we get

$$\big(F \circ \zeta\big)'(t_0) = \mathrm{D}F\big(\zeta(t_0)\big)\zeta'(t_0) = \mathrm{D}F(\xi^0)\big\{\alpha_0\chi'(t_0) + \nu_0\big\} = \alpha_0\mathrm{D}F\big(\chi(t_0)\big)\chi'(t_0) = \alpha_0\big(F \circ \chi\big)'(t_0),$$

which completes the proof of (i).

(ii). Construct a curve ζ in $\mathbb{R}^N$ through ξ^0 and parallel to the unit basis-vector $\mathrm{e}^{(j)}$ by setting $\zeta(t) := \xi^0 + (t - t_0)\mathrm{e}^{(j)}$, for $t_1 < t < t_2$. Then $\zeta(t_0) = \xi^0$ and $\zeta'(t_0) = \mathrm{e}^{(j)}$, so that

$$\mathrm{D}F(\xi^0)\zeta'(t_0) = \big\{\mathrm{grad}\, F_1(\xi^0) \cdot \mathrm{e}^{(j)}\big\}\varepsilon^{(1)} + \big\{\mathrm{grad}\, F_2(\xi^0) \cdot \mathrm{e}^{(j)}\big\}\varepsilon^{(2)} = F_{1,j}(\xi^0)\varepsilon^{(1)} + F_{2,j}(\xi^0)\varepsilon^{(2)} \neq 0,$$

and we can apply part (i) of the Proposition. We conclude that a tangent vector to the singular-value curve at $F(\xi^0)$ is given by the derivative of $t \mapsto F\big(\zeta(t)\big)$ evaluated at $t = t_0$; a computation almost the same as the one just effected then produces the result claimed. ∎

Remarks. (1.) Retain the setting of Proposition 4.1. Then $\mathbf{n}_0 := F_{2,j}(\xi^0)\varepsilon^{(1)} - F_{1,j}(\xi^0)\varepsilon^{(1)}$ is a normal vector to the singular-value curve $t \mapsto F\big(\chi(t)\big)$ $(t_1 < t < t_2)$ at the point $F(\xi^0)$. If $F(\xi^0)$ is a regular point of the local boundary of the range, then one of $\pm\mathbf{n}_0$ will give a mapped-side normal; Theorem 4.1 permits the formulation of a test for determining which normal gives the local mapping direction, so we complete the argument begun here following the statement of that theorem. Recalling Lemma 1.2, and now supposing that $F(\xi^0)$ lies on the genuine boundary of the range, already it is apparent that if the slope of the line of action of $\mathbf{n}_0$ is *negative*, *i.e.*, if the components $F_{1,j}(\xi^0)$ and $F_{2,j}(\xi^0)$ are nonzero and of the same sign, then ξ^0 cannot be a Pareto minimum. Otherwise, if the slope of the line of action of $\mathbf{n}_0$ is *positive*, *i.e.*, if the components $F_{1,j}(\xi^0)$ and $F_{2,j}(\xi^0)$ are nonzero and of opposite sign, then we must examine further, to decide which of $\pm\mathbf{n}_0$ is the mapped-side normal and inspect the signs of its components.

(2.) Maintaining the previous setting, if $F_{1,j}(\xi^0) = 0$ while $F_{2,j}(\xi^0) \neq 0$, then the singular-value curve at $F(\xi^0)$ has a vertical tangent; the tangent will be horizontal if the second component vanishes and the first is nonvanishing. These simple observations are useful in finding points on the boundary of the range that may delimit images of Pareto minima.

Now we can develop the local discrimination tests of Theorem 4.1, under the restrictions noted. That is, while maintaining the hypotheses (H.1) and (H.2), we shall provide conditions sufficient to allow us to decide whether the given singular value $F(\xi^0)$ is a fold point of the range or a point locally covered from ξ^0. The reasoning involves simple applications of Taylor's formula to study the manner in which F maps points of the translated $(N-1)$-dimensional nullspace of $\mathrm{D}F(\xi^0)$. Consequently, in preparation for these arguments, we record some of the implications of Taylor's formula in such a case.

First, consider a real function $f : \mathfrak{U} \to \mathbb{R}$ of class C^3. Let $\xi \in \mathfrak{U}$ and $h \in \mathbb{R}^N$, and suppose that the closed line segment from ξ to $\xi + h$ lies in $\mathfrak{U}$. Then there is a $\vartheta(\xi, h) \in [0,1]$ such that

$$f(\xi+h) = f(\xi) + \mathrm{d}f(\xi)h + \frac{1}{2!}\mathrm{d}^2 f(\xi; h) + \frac{1}{3!}\mathrm{d}^3 f(\xi + \vartheta(\xi,h)h; h),$$

in which

$$\left.\begin{aligned} &\mathrm{d}^2 f(\zeta; h) := \sum_{j,k=1}^{N} f,_{jk}(\zeta) h_j h_k \\ &\text{and} \\ &\mathrm{d}^3 f(\zeta; h) := \sum_{j,k,l=1}^{N} f,_{jkl}(\zeta) h_j h_k h_l \end{aligned}\right\}, \qquad \text{for} \quad \zeta \in \mathfrak{U} \quad \text{and} \quad h \in \mathbb{R}^N.$$

It is frequently more convenient to use the *integral form* of the remainder term, given by

$$\frac{1}{3!}\mathrm{d}^3 f(\xi + \vartheta(\xi,h)h; h) = \frac{1}{2}\sum_{j,k,l=1}^{N}\left\{\int_0^1 (1-s)^2 f,_{jkl}(\xi + sh)\, ds\right\} h_j h_k h_l.$$

Then for $F : \mathfrak{U} \to \mathbb{R}^2$ of class C^3, and under the same hypotheses on ξ and h, by applying the preceding to each component of F we find

$$F(\xi+h) = F(\xi) + \mathrm{D}F(\xi)h + \frac{1}{2}\mathrm{D}^2 F(\xi; h) + R_3 F(\xi; h),$$

with

$$\mathrm{D}^2 F(\xi; h) := \begin{pmatrix} \displaystyle\sum_{j,k=1}^{N} F_1,_{jk}(\xi) h_j h_k \\ \displaystyle\sum_{j,k=1}^{N} F_2,_{jk}(\xi) h_j h_k \end{pmatrix}$$

and

$$R_3 F(\xi; h) := \frac{1}{2}\begin{pmatrix} \displaystyle\sum_{j,k,l=1}^{N}\left\{\int_0^1 (1-s)^2 F_1,_{jkl}(\xi + sh)\, ds\right\} h_j h_k h_l \\ \displaystyle\sum_{j,k,l=1}^{N}\left\{\int_0^1 (1-s)^2 F_2,_{jkl}(\xi + sh)\, ds\right\} h_j h_k h_l \end{pmatrix}.$$

For fixed ξ, the map $h \mapsto \mathrm{D}^2 F(\xi; h) \in \mathbb{R}^2$, for h in any given subspace of $\mathbb{R}^N$, is a (symmetric, homogeneous) *biquadratic map*; *cf*., Appendix A. It is shown in Proposition A.1 that the range of such a map is a convex cone with vertex at the origin (including the possible degenerate cases) in the plane; this is of central importance in the formulation of the sufficient conditions given here.

Now, with $\chi : (t_1, t_2) \to \mathbb{R}^N$ denoting, as always, the C^1-imbedding of hypotheses (H.1) and (H.2), with range in $\Sigma_1 F$, for each $t \in (t_1, t_2)$ let $\{\nu_t^{(j)}\}_{j=1}^{N-1}$ denote a basis for $\mathcal{N}(\mathrm{D}F(\chi(t)))$. Then, when $t \in (t_1, t_2)$ and the argument $\chi(t)$ is displaced by an element $\sum_{j=1}^{N-1} \xi_j \nu_t^{(j)}$ of $\mathcal{N}(\mathrm{D}F(\chi(t)))$, we get just

$$F\Big(\chi(t) + \sum_{j=1}^{N-1} \xi_j \nu_t^{(j)}\Big) = F(\chi(t)) + \frac{1}{2}\mathrm{D}^2 F\Big(\chi(t); \sum_{j=1}^{N-1} \xi_j \nu_t^{(j)}\Big) + R_3 F\Big(\chi(t); \sum_{j=1}^{N-1} \xi_j \nu_t^{(j)}\Big), \tag{4.2}$$

in which we have, explicitly,

$$\mathrm{D}^2 F\Big(\chi(t); \sum_{j=1}^{N-1} \xi_j \nu_t^{(j)}\Big) = \begin{pmatrix} \sum_{p,q=1}^{N} F_{1,pq}(\chi(t)) \sum_{j=1}^{N-1} \xi_j \nu_{tp}^{(j)} \sum_{k=1}^{N-1} \xi_k \nu_{tq}^{(k)} \\ \sum_{p,q=1}^{N} F_{2,pq}(\chi(t)) \sum_{j=1}^{N-1} \xi_j \nu_{tp}^{(j)} \sum_{k=1}^{N-1} \xi_k \nu_{tq}^{(k)} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j,k=1}^{N-1} \Big\{ \sum_{p,q=1}^{N} F_{1,pq}(\chi(t)) \nu_{tp}^{(j)} \nu_{tq}^{(k)} \Big\} \xi_j \xi_k \\ \sum_{j,k=1}^{N-1} \Big\{ \sum_{p,q=1}^{N} F_{2,pq}(\chi(t)) \nu_{tp}^{(j)} \nu_{tq}^{(k)} \Big\} \xi_j \xi_k \end{pmatrix}, \tag{4.3}$$

with $\nu_{tp}^{(j)}$ denoting the component of $\nu_t^{(j)}$ with respect to the standard unit-basis vector $\mathrm{e}^{(p)}$ for $\mathbb{R}^N$.

As we noted, the singular-point discrimination test of Theorem 4.1 has been formulated by using the previous expressions to study the manner in which F maps the translated null space $\chi(t) + \mathcal{N}(\mathrm{D}F(\chi(t)))$ of its differential at $\chi(t)$. In particular, (4.2) and (4.3) are crucial throughout the development, bringing out the natural origin of the rôle of biquadratic maps. These forms indicate that, in a certain sense, the behavior of F is essentially determined near singular points not by its first differential, but by its second—unless the latter degenerates, which is a case that we leave for a later investigation.

The test consists simply in examining the definiteness properties, *i.e.*, the nature of the eigenvalues, of a certain quadratic form on the nullspace $\mathcal{N}(\mathrm{D}F(\xi^0))$ that is associated with the selected rank-1 singular point ξ^0 and constructed from the derivatives of F at ξ^0. There is a simple geometric interpretation of both the quadratic form and its definiteness properties, which we explain after stating the result. Actually, we prefer to use a form $\mathcal{Q}_{\xi^0} : \mathbb{R}^{N-1} \to \mathbb{R}$, defined instead on $\mathbb{R}^{N-1}$, by virtue of the choice of a basis for the $(N-1)$-dimensional subspace $\mathcal{N}(\mathrm{D}F(\xi^0))$ of $\mathbb{R}^N$. Accordingly, let $\{\nu_0^{(j)}\}_{j=1}^{N-1}$ denote a basis for $\mathcal{N}(\mathrm{D}F(\xi^0))$ and choose any integer m_0 from $\{1, \ldots, N\}$ such that $[F_{1,m_0}(\xi^0)]^2 + [F_{2,m_0}(\xi^0)]^2 > 0$. Set

$$\begin{aligned} \mathcal{Q}_{\xi^0}(\xi) &:= F_{2,m_0}(\xi^0)\,\mathrm{d}^2 F_1\Big(\xi^0; \sum_{j=1}^{N-1} \xi_j \nu_0^{(j)}\Big) - F_{1,m_0}(\xi^0)\,\mathrm{d}^2 F_2\Big(\xi^0; \sum_{j=1}^{N-1} \xi_j \nu_0^{(j)}\Big) \\ &= \sum_{j,k=1}^{N-1} \Big\{ \sum_{p,q=1}^{N} \Big[F_{2,m_0}(\xi^0) F_{1,pq}(\xi^0) - F_{1,m_0}(\xi^0) F_{2,pq}(\xi^0)\Big] \nu_{0p}^{(j)} \nu_{0q}^{(k)} \Big\} \xi_j \xi_k, \\ &\qquad \text{for each } \xi \equiv (\xi_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}. \end{aligned} \tag{4.4}$$

Clearly, the definiteness properties of $\mathcal{Q}_{\xi^0}$ are independent of the particular choice of basis for the null space, so that the essential aspects of the quadratic form depend upon only F and ξ^0.

We also find it convenient—but it is not clear to what extent it is necessary—to restrict attention to singular points ξ^0 that are "not too pathological." Specifically, we shall consider here only those singular points that we call "strongly nondegenerate."

Definition. Let $F : \mathcal{M}^N \to \mathbb{R}^n$ be of class C^2 on the sufficiently smooth N-submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ and mapping into $\mathbb{R}^n$. A singular point $\xi \in \mathcal{M}^N$ of F is *strongly nondegenerate* iff $\mathrm{D}^2F(\xi;\nu)$ is nonzero whenever ν is a nonzero element of $\mathcal{N}(\mathrm{D}F(\xi))$.

It may be helpful here to recall that the set of points at which a quadratic form vanishes is a cone, called the *zero-cone* of the form; this cone contains the *zero-subspace* of the form, which is just the nullspace of the matrix of the quadratic form. Then we can say that a singular point ξ of F is strongly nondegenerate precisely when the intersection of the zero-cones of the quadratic forms $\mathrm{d}^2F_k(\xi;\cdot)$ on $\mathbb{R}^N$ for $k = 1, \ldots, n$ and $\mathcal{N}(\mathrm{D}F(\xi))$ is just $\{0\}$. In that case, we see that the magnitude of $\mathrm{D}^2F(\xi;\cdot)$ has a positive lower bound c_0 on the unit sphere in $\mathcal{N}(\mathrm{D}F(\xi))$, so that

$$\left|\mathrm{D}^2F(\xi;\nu)\right|_n \geq c_0|\nu|_N^2 \qquad \text{for} \quad \nu \in \mathcal{N}(\mathrm{D}F(\xi)).$$

With this terminology, we can state the additional hypothesis on which we rely in the proofs of both Theorem 4.1 and Theorem 5.1; it is not presently clear how this restriction can be relaxed:

H.3. The rank-1 singular point ξ^0 of F is strongly nondegenerate.

Observe that the hypothesis (H.3) does not imply that ξ^0 has a relative neighborhood in $\Sigma_1 F$ that is a 1-manifold; a counterexample is afforded by the point $(3, 5/2)$ in Example 2.1.

Theorem 4.1. *Let $F : \mathcal{U} \to \mathbb{R}^2$ be of class C^3 on the open set $\mathcal{U} \subset \mathbb{R}^N$, with $N \geq 2$. Suppose that $\xi^0 \in \mathcal{U}$ is a rank-1 singular point of F for which the conditions (H.1)–(H.3) hold. Let the corresponding quadratic form $\mathcal{Q}_{\xi^0}$ on $\mathbb{R}^{N-1}$ be defined as in* (4.4).

- (i.) *If the quadratic form $\mathcal{Q}_{\xi^0}$ is definite, then the range is folded at the singular value $F(\xi^0)$ with respect to ξ^0.*
- (ii.) *If $F : \mathcal{U} \to \mathbb{R}^2$ is of class C^5 and the quadratic form $\mathcal{Q}_{\xi^0}$ is indefinite, then the singular value $F(\xi^0)$ is locally covered by F from ξ^0.*
- (iii.) *If the quadratic form $\mathcal{Q}_{\xi^0}$ is semidefinite, no conclusion can be drawn about the singular value $F(\xi^0)$, i.e., it may be either on the local boundary of the range of F or locally covered by F from ξ^0.*

Remarks. Some remarks preliminary to the proof will aid in maintaining orientation.

(1.) While the hypothesis of strong nondegeneracy of the singular point ξ^0 is apparently used explicitly only in the proof of assertion (ii), this condition is necessary for the definiteness of the form $\mathcal{Q}_{\xi^0}$ that is required in assertion (i).

(2.) The proof of the theorem given here is rather tedious in some places and rather delicate in others. In following the argument, it is helpful to keep in mind the underlying geometry associated with the mapping F and its differentials, and the geometric significance of the conditions (i)–(iii). The needed facts are isolated and proven in Proposition A.2, which describes how one can discern the position of a line through the origin relative to the convex-cone range of a biquadratic map, according to the definiteness properties of a certain associated quadratic form. To use those results in the interpretation of the present situation, we first note that, by Proposition 4.1.ii, the vector $F_{1,m_0}(\xi^0)\varepsilon^{(1)} + F_{2,m_0}(\xi^0)\varepsilon^{(2)}$ is a tangent to the singular-value curve $t \mapsto F(\chi(t))$ at the point $F(\xi^0) = F(\chi(t_0))$. Moreover, the map given on $\mathbb{R}^{N-1}$ by $\xi \mapsto \mathrm{D}^2F\left(\xi^0; \sum_{j=1}^{N-1} \xi_j \nu_0^{(j)}\right)$ is a biquadratic map, so its range comprises a (perhaps degenerate) convex cone with vertex at the origin in $\mathbb{R}^2$ (Proposition A.1). Consequently, by Proposition A.2, the conditions of (i) definiteness, (ii) indefiniteness, or (iii) semidefiniteness of $\mathcal{Q}_{\xi^0}$ imply, respectively, that the line through the origin and parallel to the tangent vector $F_{1,m_0}(\xi^0)\varepsilon^{(1)} + F_{2,m_0}(\xi^0)\varepsilon^{(2)}$ (i) meets this cone only at the origin, (ii) contains a ray belonging to the interior of the cone, or (iii) contains a ray lying in the boundary of the cone; here, we are using the hypothesis that ξ^0 is a strongly nondegenerate singular point of F. Thus, for the present purposes, we may say that the important aspects of the mapping behavior of F near a singular point are determined by *the action of its second quadratic differential on the null space of its first differential at the singular point*, in particular, by *the position of the convex-cone image of the nullspace relative to the singular values*. Roughly speaking, if (i) the cone does not "cover the singular value," then neither does the map itself, and a fold occurs, so that the singular value is at least on the local boundary of the range, and must be studied further. On the other hand, if (ii) the cone "covers the singular value," then the full map does likewise, whence the singular value lies in the interior of the range. The borderline case, in which (iii) the cone boundary is tangent to the curve of singular values, must be examined separately.

(3.) In the case $N = 2$, the null space $\mathcal{N}(\mathrm{D}F(\xi^0))$ has dimension 1, so the quadratic form $\mathcal{Q}_{\xi^0}$ is defined on $\mathbb{R}$, and therefore cannot be indefinite, *i.e.*, must be either definite or trivial (semidefinite). Put another way, when $N = 2$ the cone-image of the nullspace $\mathcal{N}(\mathrm{D}F(\xi^0))$ under the second quadratic differential is just a (one-dimensional) ray or line. In any event, only (i) and (iii) can occur for $N = 2$.

(4.) According to the theorem, the test of a point ξ^0 can be conducted by finding and inspecting the signs of the eigenvalues of the $(N-1) \times (N-1)$ matrix with typical entry appearing in the braces on the right in (4.4). For this, a basis $\{\nu_0^{(j)}\}_{j=1}^{N-1}$ for the null space $\mathcal{N}(\mathrm{D}F(\xi^0))$ must actually be constructed. In fact, in the proof of the theorem we need to know also that there is for each t in an interval (t_1, t_2) about t_0 a basis $\{\nu_t^{(j)}\}_{j=1}^{N-1}$ for the null space $\mathcal{N}(\mathrm{D}F(\chi(t)))$ such that the resultant maps $t \mapsto \nu_t^{(j)}$ are continuous or even of class C^1, so we shall at this stage offer one method for producing such bases. We can proceed most easily here by noting that the subspace $\mathcal{N}(\mathrm{D}F(\chi(t)))$ is just the orthogonal complement of the span of $\operatorname{grad} F_l(\chi(t))$ in $\mathbb{R}^N$, with $l \in \{1,2\}$ chosen so that the indicated gradient is nonzero. Recall the definition of the integer $j_{[m]}$, corresponding to $m \in \{1,\dots,N\}$ and $j \in \{1,\dots,N-1\}$, which is given in (2.3). With the integer m_0 as in the statement of the theorem, we can suppose that l_0 has been chosen and the interval (t_1,t_2) has been adjusted so that $F_{l_0,m_0}(\chi(t)) \neq 0$ for every $t \in (t_1,t_2)$. Then it is easy to check that a basis for the $(N-1)$-dimensional null space of $\mathrm{D}F(\chi(t))$ is given for each $t \in (t_1,t_2)$ by $\{\nu_t^{(j)}\}_{j=1}^{N-1}$, with

$$\nu_t^{(j)} := F_{l_0,j_{[m_0]}}(\chi(t))\mathrm{e}^{(m_0)} - F_{l_0,m_0}(\chi(t))\mathrm{e}^{(j_{[m_0]})}, \qquad \text{for} \quad j = 1,\dots,N-1,$$

and, moreover, that the dependence $t \mapsto \nu_t^{(j)}$ is of class C^1 for each j. In the proof of Theorem 4.1, we shall suppose that the interval (t_1,t_2) is sufficiently short and that this extension has been effected. For brevity, we shall continue to write $\nu_0^{(j)}$ in place of $\nu_{t_0}^{(j)}$, which should cause no confusion.

Finding a mapped-side normal. Maintain the setting and notation of Theorem 4.1. We already established, in Lemma 1.2 and Section 3, the importance of the computation of a mapped-side normal at a regular point of the local included boundary of the range, and especially at a point of the global included boundary itself. Further, our observation in Remark 1 following Proposition 4.1 enables us to compute a normal $\mathbf{n}_0$ to the curve of singular values described by $F \circ \chi$ at $F(\xi^0)$. Now, the statement of Theorem 4.1 permits us to complete the determination of a mapped-side normal by choosing the correct one of $\pm\mathbf{n}_0$; we point this out prior to the proof, to avoid obscuring the simplicity of the calculation. Thus, suppose we have determined that the quadratic form $\mathcal{Q}_{\xi^0}$ is definite, as in statement (i) of the theorem, so that the range is folded at $F\big(\xi^0\big)$ with respect to $\xi^0 = \chi(t_0)$, and assume that the local boundary of the range of F is regular at $F\big(\xi^0\big)$. For any $\xi \in \mathbb{R}^{N-1}$, the vector $\mathrm{D}^2F\Big(\xi^0; \sum_{j=1}^{N-1} \xi_j \nu_0^{(j)}\Big)$ will point from $F\big(\xi^0\big)$ into the image of a neighborhood of ξ^0 under F. That is, for sufficiently small $s > 0$, the point $F\big(\xi^0\big) + s\mathrm{D}^2F\Big(\xi^0; \sum_{j=1}^{N-1} \xi_j \nu_0^{(j)}\Big)$ will lie in the image of such a neighborhood. Therefore, if the inner product of $\mathrm{D}^2F\Big(\xi^0; \sum_{j=1}^{N-1} \xi_j \nu_0^{(j)}\Big)$ and the normal $F_{2^{,}m_0}\big(\xi^0\big)\varepsilon^{(1)} - F_{1^{,}m_0}\big(\xi^0\big)\varepsilon^{(2)}$ to the singular-value curve at the point $F\big(\xi^0\big)$ is positive, then the direction of this normal is the local mapping direction, *i.e.*, the normal is a mapped-side normal; otherwise (since we can suppose the inner product to be nonzero), the negative of that normal vector will be a mapped-side normal. Upon reviewing the definition of $\mathcal{Q}_{\xi^0}$, we see that the conclusion can be phrased alternately in terms of the specific definiteness property of the quadratic form $\mathcal{Q}_{\xi^0}$. Thus, when the local boundary of the range of F is regular at $F\big(\xi^0\big)$,

> *if $\mathcal{Q}_{\xi^0}$ is positive-definite [negative-definite], then the normal $F_{2^{,}m_0}\big(\xi^0\big)\varepsilon^{(1)} - F_{1^{,}m_0}\big(\xi^0\big)\varepsilon^{(2)}$ [negative of this normal] is a mapped-side normal to the local boundary of the range at $F\big(\xi^0\big)$.*

This second formulation is convenient, since we must determine the definiteness properties of the quadratic form $\mathcal{Q}_{\xi^0}$ to apply Theorem 4.1 in any event, so we can identify the mapped-side normal with essentially no extra work when it turns out that $\mathcal{Q}_{\xi^0}$ is definite and the conclusion of (i) is assured. Finally, when we know that $F\big(\xi^0\big)$ is actually in $\partial\mathcal{R}_F$, we can use our determination of the interior normal in conjunction with Lemma 1.2 to decide whether ξ^0 is a local Pareto minimum (unless some component of the normal vanishes while all components are nonnegative). The conclusions drawn here are corroborated in various of the examples discussed in Section 6.

Proof of Theorem 4.1. Since it is helpful to keep in mind that ξ^0 is the image $\chi(t_0)$, we shall as a rule use the latter notation to indicate the singular point under study.

$(i.)$ Suppose that the quadratic form $\mathcal{Q}_{\xi^0}$ is definite: we shall construct an $\mathbb{R}^N$-neighborhood U_δ of $\chi(t_0)$ such that $F\big(\chi(t_0)\big) \in \partial F\big(U_\delta\big)$, thereby showing directly that $F\big(\chi(t_0)\big)$ is a fold point of the range. The neighborhood U_δ is to be built up by "leafing together" neighborhoods of zero in $\mathcal{N}\big(\mathrm{D}F\big(\chi(t)\big)\big)$ translated by $\chi(t)$, as t runs through (t_1, t_2). That is, let us define the map $\Psi : (t_1, t_2) \times \mathbb{R}^{N-1} \to \mathbb{R}^N$ by setting

$$\Psi(t, \xi) := \chi(t) + \sum_{j=1}^{N-1} \xi_j \nu_t^{(j)}, \qquad \text{for} \quad t_1 < t < t_2 \quad \text{and} \quad \xi \in \mathbb{R}^{N-1}.$$

As explained above, we can suppose that the maps $t \mapsto \nu_t^{(j)}$ are of class C^1 on (t_1, t_2), so that Ψ is of class C^1. A short computation shows that the matrix of the differential $\mathrm{D}\Psi(t_0, 0)$ is given by

$$\Big[\chi'(t_0) \quad \nu_{t_0}^{(1)} \quad \cdots \quad \nu_{t_0}^{(N-1)}\Big];$$

here, the column-vectors of the matrix are displayed. By hypothesis, $F \circ \chi$ is an imbedding, so that $\chi'(t_0)$ does not belong to the nullspace $\mathcal{N}\big(\mathrm{D}F(\chi(t_0))\big)$ $\big(F(\chi(t_0))$ is not a cusp$\big)$. It follows that the columns of the matrix of $\mathrm{D}\Psi(t_0,0)$ are linearly independent, and therefore that $\mathrm{D}\Psi(t_0,0) : \mathbb{R}^N \to \mathbb{R}^N$ is injective. Immediately, the inverse-function theorem shows that Ψ maps an open neighborhood of $(t_0,0) \in \mathbb{R}^N$ contained in $(t_1,t_2) \times \mathbb{R}^{N-1}$ diffeomorphically onto a full open $\mathbb{R}^N$-neighborhood of $\chi(t_0)$. Then, for any sufficiently small $\delta > 0$, Ψ maps the open neighborhood $(t_0-\delta, t_0+\delta) \times \mathrm{B}^{N-1}_\delta(0)$ of $(t_0,0)$ onto a "leaved" open neighborhood $U_\delta := \Psi\big((t_0-\delta, t_0+\delta) \times \mathrm{B}^{N-1}_\delta(0)\big)$ of $\chi(t_0)$ in $\mathbb{R}^N$.

We claim that $F\big(\chi(t_0)\big) \in \partial F\big(U_\delta\big)$ for some $\delta > 0$. To see that this is so, consider the line $\mathfrak{l}_\nu \subset \mathbb{R}^2$ normal to the singular-value curve at $F\big(\chi(t_0)\big)$. According to Proposition 4.1.*ii* and the notation established here, $\mathfrak{l}_\nu$ can be parametrized by

$$\sigma \mapsto F\big(\chi(t_0)\big) + \sigma \begin{pmatrix} F_{2,m_0}\big(\chi(t_0)\big) \\ -F_{1,m_0}\big(\chi(t_0)\big) \end{pmatrix} \qquad \text{for } \sigma \in \mathbb{R}.$$

Let $\mathfrak{l}_\nu^-$ and $\mathfrak{l}_\nu^+$ denote the parts of $\mathfrak{l}_\nu$ corresponding to $\sigma < 0$ and $\sigma > 0$, respectively. To verify our claim, it clearly suffices to produce a positive δ such that the image $F\big(U_\delta\big)$ does not meet (at least) one of $\mathfrak{l}_\nu^-$ and $\mathfrak{l}_\nu^+$.

Then let $\delta > 0$. Suppose that $\sigma \neq 0$ and there are $t^\sigma \in \mathbb{R}$ with $t_0 - \delta < t^\sigma < t_0 + \delta$ and $\xi^\sigma \in \mathbb{R}^{N-1}$ with $|\xi^\sigma|_{N-1} < \delta$ such that F maps the corresponding point $\Psi\big(t^\sigma, \xi^\sigma\big) \in U_\delta$ to the point on the normal line corresponding to σ, *i.e.*, such that

$$F\Big(\chi(t^\sigma) + \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big) = F\big(\chi(t_0)\big) + \sigma \begin{pmatrix} F_{2,m_0}\big(\chi(t_0)\big) \\ -F_{1,m_0}\big(\chi(t_0)\big) \end{pmatrix}; \tag{4.5}$$

we suppose through the remainder of the proof of (i) that σ, t^σ, and ξ^σ satisfy these conditions and are related by (4.5). We shall show that, provided δ is sufficiently small, either all such σ are positive or all such σ are negative, thereby completing the proof of (i). To this end, we first use (4.2) to rewrite the lefthand side of (4.5); a rearrangement then yields

$$\sigma \begin{pmatrix} F_{2,m_0}\big(\chi(t_0)\big) \\ -F_{1,m_0}\big(\chi(t_0)\big) \end{pmatrix} \\ = \Big\{F\big(\chi(t^\sigma)\big) - F\big(\chi(t_0)\big)\Big\} + \frac{1}{2}\mathrm{D}^2F\Big(\chi(t^\sigma); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big) + R_3F\Big(\chi(t^\sigma); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big). \tag{4.6}$$

Upon taking the $\mathbb{R}^2$-inner products in (4.6) successively with the vectors $\tau_{t_0} := F_{1,m_0}\big(\chi(t_0)\big)\varepsilon^{(1)} + F_{2,m_0}\big(\chi(t_0)\big)\varepsilon^{(2)}$ and $\mathrm{n}_{t_0} := F_{2,m_0}\big(\chi(t_0)\big)\varepsilon^{(1)} - F_{1,m_0}\big(\chi(t_0)\big)\varepsilon^{(2)}$, a (nonzero) tangent and normal to the curve of singular values at $F\big(\chi(t_0)\big)$, respectively, we obtain

$$0 = \Big\{F\big(\chi(t^\sigma)\big) - F\big(\chi(t_0)\big)\Big\} \cdot \tau_{t_0} + \Big\{\frac{1}{2}\mathrm{D}^2F\Big(\chi(t^\sigma); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big) + R_3F\Big(\chi(t^\sigma); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big)\Big\} \cdot \tau_{t_0}, \tag{4.6.1}$$

and

$$\sigma\big|\mathrm{n}_{t_0}\big|^2 = \Big\{F\big(\chi(t^\sigma)\big) - F\big(\chi(t_0)\big)\Big\} \cdot \mathrm{n}_{t_0} + \frac{1}{2}\mathfrak{Q}_{\xi^0}(\xi^\sigma) \\ + \frac{1}{2}\Big\{\mathrm{D}^2F\Big(\chi(t^\sigma); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big) - \mathrm{D}^2F\Big(\chi(t_0); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t_0}^{(j)}\Big)\Big\} \cdot \mathrm{n}_{t_0} + R_3F\Big(\chi(t^\sigma); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t^\sigma}^{(j)}\Big) \cdot \mathrm{n}_{t_0}; \tag{4.6.2}$$

in (4.6.2), we used the equality $\mathcal{Q}_{\xi^0}(\xi^\sigma) = \mathrm{D}^2 F\Big(\chi(t_0); \sum_{j=1}^{N-1} \xi_j^\sigma \nu_{t_0}^{(j)}\Big) \cdot \mathrm{n}_{t_0}$. We shall establish from (4.6.1) an estimate of the form

$$\Big|F(\chi(t^\sigma)) - F(\chi(t_0))\Big|_2 \le M_2 |\xi^\sigma|_{N-1}^2, \tag{4.6.3}$$

holding for some positive constant M_2, independent of σ, provided that δ is sufficiently small (recall that $|t^\sigma - t_0| < \delta$, $|\xi^\sigma|_{N-1} < \delta$, and (4.5) holds), and then use this to show that the first term on the right in (4.6.2) has a bound of the form

$$\Big|\Big\{F(\chi(t^\sigma)) - F(\chi(t_0))\Big\} \cdot \mathrm{n}_{t_0}\Big| \le M_1 |\xi^\sigma|_{N-1}^2 f(t^\sigma), \tag{4.6.4}$$

again holding for some positive constant M_1 and for any sufficiently small positive δ, in which f is a nonnegative function such that $\lim_{t\to t_0} f(t) = 0$.

Let us first suppose that (4.6.3) and (4.6.4) are true for such δ and check that then the claimed result follows. Because $\mathcal{Q}_{\xi^0}$ is definite in the present case, there is a positive constant c_0 for which

$$|\mathcal{Q}_{\xi^0}(\xi)| \ge c_0 |\xi|_{N-1}^2, \qquad \text{for all} \quad \xi \in \mathbb{R}^{N-1}.$$

Also, it is easy to see that the third and fourth terms on the right in (4.6.2) are bounded in magnitude by $M_3 |t^\sigma - t_0| |\xi^\sigma|_{N-1}^2$ and $M_4 |\xi^\sigma|_{N-1}^3$, respectively, with positive-constant multiples M_3 and M_4, again independent of σ, and again for all sufficiently small positive δ. In view of (4.6.4) and the limit property cited for the nonnegative function f figuring in that estimate, it is now clear that whenever $\delta > 0$ is sufficiently small and (4.5) holds with t^σ and ξ^σ satisfying $|t^\sigma - t_0| < \delta$ and $|\xi^\sigma|_{N-1} < \delta$, the absolute value of the sum of the first, third, and fourth terms on the righthand side of (4.6.2) is bounded by

$$\begin{aligned} &M_1 |\xi^\sigma|_{N-1}^2 f(t^\sigma) + M_3 |t^\sigma - t_0| |\xi^\sigma|_{N-1}^2 + M_4 |\xi^\sigma|_{N-1}^3 \\ &\qquad = \Big\{M_1 f(t^\sigma) + M_3 |t^\sigma - t_0| + M_4 |\xi^\sigma|_{N-1}\Big\} |\xi^\sigma|_{N-1}^2 \le \frac{c_0}{4} |\xi^\sigma|_{N-1}^2 < \frac{1}{2} |\mathcal{Q}_{\xi^0}(\xi^\sigma)|, \end{aligned}$$

implying, in turn, that the entire righthand side of (4.6.2) has, under the same conditions, the same sign as $\mathcal{Q}_{\xi^0}(\xi^\sigma)$; the same is then true of σ. As explained, this completes the proof of statement (i), *modulo* the verification of (4.6.3) and (4.6.4).

For the latter proofs, we denote by $\hat{\tau}_{t_0}$ and $\hat{\mathrm{n}}_{t_0}$ the unit vectors corresponding to τ_{t_0} and n_{t_0}, respectively. We derive the estimate (4.6.3) from the obvious equality

$$\Big|F(\chi(t^\sigma)) - F(\chi(t_0))\Big|_2^2 = \Big\{\Big(F(\chi(t^\sigma)) - F(\chi(t_0))\Big) \cdot \hat{\tau}_{t_0}\Big\}^2 + \Big\{\Big(F(\chi(t^\sigma)) - F(\chi(t_0))\Big) \cdot \hat{\mathrm{n}}_{t_0}\Big\}^2. \tag{4.6.5}$$

In fact, from the derivative relation

$$\lim_{t\to t_0} \frac{F(\chi(t)) - F(\chi(t_0))}{t - t_0} = \alpha_{t_0} \tau_{t_0},$$

holding for some nonzero α_{t_0}, we find that

$$\lim_{t\to t_0} \left| \frac{F(\chi(t)) - F(\chi(t_0))}{t - t_0} \cdot \hat{\tau}_{t_0} \right| = |\alpha_{t_0}| |\tau_{t_0}|_2 > 0 \qquad \text{and} \qquad \lim_{t\to t_0} \frac{F(\chi(t)) - F(\chi(t_0))}{t - t_0} \cdot \hat{\mathrm{n}}_{t_0} = 0.$$

Clearly, there is then a $\delta_1 > 0$ for which

$$\left| \frac{F(\chi(t)) - F(\chi(t_0))}{t - t_0} \cdot \hat{\mathrm{n}}_{t_0} \right| < \frac{1}{2} |\alpha_{t_0}| |\tau_{t_0}|_2 < \left| \frac{F(\chi(t)) - F(\chi(t_0))}{t - t_0} \cdot \hat{\tau}_{t_0} \right| \qquad \text{for} \quad 0 < |t - t_0| < \delta_1,$$

so that $\left| \{ F(\chi(t)) - F(\chi(t_0)) \} \cdot \hat{\mathrm{n}}_{t_0} \right| < \left| \{ F(\chi(t)) - F(\chi(t_0)) \} \cdot \hat{\tau}_{t_0} \right|$ for such t, which gives, with (4.6.5),

$$\left| F(\chi(t^\sigma)) - F(\chi(t_0)) \right|_2^2 < 2 \Big\{ \Big(F(\chi(t^\sigma)) - F(\chi(t_0)) \Big) \cdot \hat{\tau}_{t_0} \Big\}^2 \qquad \text{for} \quad 0 < |t^\sigma - t_0| < \delta_1.$$

The latter inequality, when coupled with an estimate of the form

$$\left| \Big(F(\chi(t^\sigma)) - F(\chi(t_0)) \Big) \cdot \tau_{t_0} \right| \le M' |\xi^\sigma|_{N-1}^2 \qquad \text{for} \quad 0 < |t^\sigma - t_0| < \delta_2$$

which clearly follows from the relation (4.6.1), now yields the estimate claimed in (4.6.3), provided that δ is a sufficiently small positive number.

With (4.6.3) established, (4.6.4) follows readily. Indeed, since $F(\chi(\,\cdot\,))$ is injective on an interval containing t_0 we can write, again for any sufficiently small $\delta > 0$,

$$\begin{aligned} \left| \Big(F(\chi(t^\sigma)) - F(\chi(t_0)) \Big) \cdot \mathrm{n}_{t_0} \right| &= \left| F(\chi(t^\sigma)) - F(\chi(t_0)) \right|_2 \left| \frac{F(\chi(t^\sigma)) - F(\chi(t_0))}{\left| F(\chi(t^\sigma)) - F(\chi(t_0)) \right|_2} \cdot \mathrm{n}_{t_0} \right| \\ &= \left| F(\chi(t^\sigma)) - F(\chi(t_0)) \right|_2 \left| \left\{ \frac{F(\chi(t^\sigma)) - F(\chi(t_0))}{\left| F(\chi(t^\sigma)) - F(\chi(t_0)) \right|_2} - \iota \hat{\tau}_{t_0} \right\} \cdot \mathrm{n}_{t_0} \right| \\ &\le M_1 |\xi^\sigma|_{N-1}^2 \left| \frac{F(\chi(t^\sigma)) - F(\chi(t_0))}{\left| F(\chi(t^\sigma)) - F(\chi(t_0)) \right|_2} - \iota \hat{\tau}_{t_0} \right|_2, \end{aligned}$$

in which ι is adjusted to be either $+1$ or -1 so that

$$\lim_{t \to t_0} \frac{F(\chi(t)) - F(\chi(t_0))}{\left| F(\chi(t)) - F(\chi(t_0)) \right|_2} = \iota \hat{\tau}_{t_0}$$

obtains. This is just the form claimed in (4.6.4).

Proof of (*ii*.) Now we suppose that the quadratic form $\mathcal{Q}_{\xi^0}$ is indefinite and show that $F(\chi(t_0)) \in F(U)^\circ$ for every $\mathbb{R}^N$-neighborhood U of $\chi(t_0)$. For this, it suffices to show that for each sufficiently small $\delta > 0$ there can be found $r_\delta^\star > 0$ such that the ball $\mathrm{B}^2_{r_\delta^\star}(F(\chi(t_0))) \subset \mathbb{R}^2$, of radius $r_\delta^\star$ and centered at $F(\chi(t_0))$, is contained in the image $F(U_\delta)$, in which the neighborhoods $U_\delta \subset \mathbb{R}^N$ are just those introduced in part (*i*) of the proof. By the indefiniteness of the quadratic form $\mathcal{Q}_{\xi^0}$, we can find and fix ξ^- and $\xi^+ \in \mathbb{R}^{N-1}$ for which $\mathcal{Q}_{\xi^0}(\xi^-) < 0$ and $\mathcal{Q}_{\xi^0}(\xi^+) > 0$. With δ denoting throughout any positive number sufficiently small that $U_\delta \subset \mathcal{U}$, we shall in fact prove (*ii*) by showing that

$$\left.\begin{array}{l} \text{there exist } t_\delta \in (t_0 - \delta, t_0 + \delta) \text{ and } r_\delta^\star > 0 \text{ such that whenever } \Delta \in \mathrm{B}^2_{r_\delta^\star}(0) \text{ there} \\ \text{can be found } \zeta^{\delta,\Delta} \in \mathbb{R}^2 \text{ satisfying } \chi(t_\delta) + \sum_{j=1}^{N-1} \Big\{ (\zeta^{\delta,\Delta})_1 \xi_j^- + (\zeta^{\delta,\Delta})_2 \xi_j^+ \Big\} \nu_{t_\delta}^{(j)} \in U_\delta \\ \text{and} \\ \qquad F(\chi(t_0)) + \Delta = F\Big(\chi(t_\delta) + \displaystyle\sum_{j=1}^{N-1} \Big\{ (\zeta^{\delta,\Delta})_1 \xi_j^- + (\zeta^{\delta,\Delta})_2 \xi_j^+ \Big\} \nu_{t_\delta}^{(j)} \Big), \end{array}\right\} \qquad (4.7)$$

i.e., by showing that the covering can be accomplished at a constant value of t chosen sufficiently close to t_0. We conduct the proof of this claim in two steps: first, roughly speaking, we secure the result when F is replaced by the first approximating terms from Taylor's formula for F restricted to the translated null spaces $\chi(t)+\mathcal{N}\big(\mathrm{D}F\big(\chi(t)\big)\big)$; subsequently, we use the first step in conjunction with a perturbation result closely connected to the Newton-Kantorovich method to verify the statement for the full mapping F. (It is in the use of the latter that we require the additional smoothness of F.)

Step (*ii*.1). For each $t \in (t_1, t_2)$, we introduce the corresponding linear operator $N_t : \mathbb{R}^2 \to \mathcal{N}\big(\mathrm{D}F\big(\chi(t)\big)\big)$ by setting

$$N_t\zeta := \sum_{j=1}^{N-1}\Big\{\zeta_1\xi_j^- + \zeta_2\xi_j^+\Big\}\nu_t^{(j)} = \zeta_1 N_t^1 + \zeta_2 N_t^2, \qquad \text{for} \quad \zeta \in \mathbb{R}^2,$$

in which we have written

$$N_t^1 := \sum_{j=1}^{N-1}\xi_j^-\nu_t^{(j)} \qquad \text{and} \qquad N_t^2 := \sum_{j=1}^{N-1}\xi_j^+\nu_t^{(j)}, \tag{4.8}$$

and then define the biquadratic map $B(\,\cdot\,;t) : \mathbb{R}^2 \to \mathbb{R}^2$, depending upon the parameter t, as essentially the restriction of the second quadratic differential of F at $\chi(t)$ to the subspace of $\mathcal{N}\big(\mathrm{D}F\big(\chi(t)\big)\big)$ spanned by N_t^1 and N_t^2:

$$B(\zeta;t) := \frac{1}{2}\mathrm{D}^2F\Big(\chi(t); N_t\zeta\Big) = \begin{pmatrix} B_1(\zeta;t) := \big(\mathcal{A}_t^{(1)}\zeta\big)\cdot\zeta \\ \\ B_2(\zeta;t) := \big(\mathcal{A}_t^{(2)}\zeta\big)\cdot\zeta \end{pmatrix}, \qquad \text{for} \quad \zeta \in \mathbb{R}^2, \quad t \in \big(t_1, t_2\big); \tag{4.9}$$

here, the 2×2 matrices $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ have the entries

$$\big(\mathcal{A}_t^{(l)}\big)_{jk} = \frac{1}{2}\sum_{p,q=1}^{N} F_{l,\,pq}\big(\chi(t)\big)N_{tp}^j N_{tq}^k, \qquad \text{for} \quad j,\, k,\, l = 1 \text{ and } 2. \tag{4.10}$$

The strong nondegeneracy of the singular point $\chi(t_0)$ clearly implies the strong nondegeneracy of the biquadratic map $B(\,\cdot\,;t_0)$, *i.e.*, according to the definition given in Appendix A, the positivity of $\big|B(\,\cdot\,;t_0)\big|_2$ on the unit circle $\mathrm{S}_1^2(0)$. It follows that $B(\,\cdot\,;t)$ is strongly nondegenerate for all t sufficiently near to t_0, *i.e.*, that there are positive numbers c_0 and η_0 such that

$$\big|B(\zeta;t)\big|_2 \geq c_0|\zeta|_2^2 \qquad \text{whenever} \quad \zeta \in \mathbb{R}^2 \quad \text{and} \quad |t - t_0| < \eta_0. \tag{4.11}$$

In fact, the map $(t,\zeta) \mapsto \big|B(\zeta;t)\big|_2 = \big|\frac{1}{2}\mathrm{D}^2F\big(\chi(t);N_t\zeta\big)\big|_2$ is continuous on $(t_1,t_2)\times\mathbb{R}^2$ and positive when $t = t_0$ and $\zeta \neq 0$. In particular, the map is bounded below by a positive c_1 on the compact set $\{t_0\}\times\mathrm{S}_1^2(0)$, whence it follows by a simple continuity and compactness argument that there is an $\eta_0 > 0$ for which the map is bounded below by $c_0 := c_1/2$ on $\big(t_0 - \eta_0, t_0 + \eta_0\big)\times\mathrm{S}_1^2(0)$. From this, (4.11) follows immediately.

Each $B(\,\cdot\,;t)$ carries $\mathbb{R}^2$ onto a convex cone with vertex at the origin in $\mathbb{R}^2$ (*cf*. Proposition A.1); we denote this cone by $\mathcal{C}_t := \mathcal{R}_{B(\,\cdot\,;t)} := B(\mathbb{R}^2;t)$. By Proposition A.2.*ii*, the strong nondegeneracy of the map $B(\,\cdot\,;t_0)$, the indefiniteness of $\mathcal{Q}_{\xi^0}$, and the choice of ξ^- and ξ^+ together imply that the cone $\mathcal{C}_{t_0}$ contains at least one ray in the tangent line to the singular-value curve at $F\big(\chi(t_0)\big)$. Indeed, to verify this we need only observe that the quadratic form

$$\zeta \mapsto F_{2,\,m_0}\big(\chi(t_0)\big)B_1(\zeta;t_0) - F_{1,\,m_0}\big(\chi(t_0)\big)B_2(\zeta;t_0) = \frac{1}{2}\mathcal{Q}_{\xi^0}\big(\zeta_1\xi^- + \zeta_2\xi^+\big), \qquad \zeta \in \mathbb{R}^2,$$

is indefinite, since its value is negative at $(1,0)$ and positive at $(0,1)$.

In this Step $(ii.1)$, we show that there is a positive δ_{t_0} and (at least) one of the intervals $\big(t_0 - \delta_{t_0}, t_0\big)$ and $\big(t_0, t_0+\delta_{t_0}\big)$, call it $\mathfrak{I}_{t_0}$, such that for any t contained in $\mathfrak{I}_{t_0}$, the point $F\big(\chi(t_0)\big) - F\big(\chi(t)\big)$ lies in the interior $\mathcal{C}_t^\circ$ of the range of the map $B(\,\cdot\,;t)$, *i.e.*,

$$\left.\begin{aligned}&\text{for each } t \in \mathfrak{I}_{t_0} \text{ there exists } r_t > 0 \text{ such that for every } \Delta \in \mathrm{B}^2_{r_t}(0) \text{ there can be found}\\ &\zeta_0^{t,\Delta} \in \mathbb{R}^2 \text{ satisfying}\\ &\qquad\qquad\qquad B(\zeta_0^{t,\Delta};t) = F\big(\chi(t_0)\big) + \Delta - F\big(\chi(t)\big).\end{aligned}\right\} \tag{4.12}$$

To establish (4.12), we introduce a quadratic form $\mathrm{k}_{\xi^0}(\,\cdot\,;t)$ on $\mathbb{R}^2$, depending upon the parameter $t \in (t_1, t_2)$, by

$$\begin{aligned}\mathrm{k}_{\xi^0}(\zeta;t) &:= \Big\{F_2\big(\chi(t_0)\big) - F_2\big(\chi(t)\big)\Big\}\mathrm{d}^2F_1\Big(\chi(t);N_t\zeta\Big) - \Big\{F_1\big(\chi(t_0)\big) - F_1\big(\chi(t)\big)\Big\}\mathrm{d}^2F_2\Big(\chi(t);N_t\zeta\Big)\\ &= 2\bigg\{\Big\{F_2\big(\chi(t_0)\big) - F_2\big(\chi(t)\big)\Big\}B_1(\zeta;t) - \Big\{F_1\big(\chi(t_0)\big) - F_1\big(\chi(t)\big)\Big\}B_2(\zeta;t)\bigg\},\\ &\qquad\qquad\qquad\qquad \text{for} \quad \zeta \in \mathbb{R}^2 \quad \text{and} \quad t_1 < t < t_2.\end{aligned}$$

Upon inspection of this definition, we see with the help of Proposition A.2.*ii* that if t is sufficiently near t_0 and there exist ζ' and $\zeta'' \in \mathbb{R}^2$ such that the product $\mathrm{k}_{\xi^0}(\zeta';t)\mathrm{k}_{\xi^0}(\zeta'';t)$ is negative, so that the quadratic form is indefinite for that t, then we can assert that at least one of $\pm\Big\{F\big(\chi(t_0)\big) - F\big(\chi(t)\big)\Big\}$ lies in the interior of the convex-cone range of $B(\,\cdot\,;t)$. Let us show that this condition obtains, with $\zeta' = (1,0)$ and $\zeta'' = (0,1)$, for each t in some neighborhood of t_0.

For this, we first observe that the chain rule clearly implies, for each $\zeta \in \mathbb{R}^2$,

$$\begin{aligned}\lim_{t\to t_0}\frac{1}{t-t_0}\mathrm{k}_{\xi^0}(\zeta;t) = &-\Big\{\operatorname{grad} F_2\big(\chi(t_0)\big)\cdot\chi'(t_0)\Big\}\mathrm{d}^2F_1\Big(\chi(t_0);N_{t_0}\zeta\Big)\\ &+\Big\{\operatorname{grad} F_1\big(\chi(t_0)\big)\cdot\chi'(t_0)\Big\}\mathrm{d}^2F_2\Big(\chi(t_0);N_{t_0}\zeta\Big).\end{aligned}$$

Now we exploit the hypothesis that $\xi^0 = \chi(t_0)$ is a rank-1 singular point of F. In fact, since the matrix $\Big\{F_{l\,,\,m}\big(\chi(t_0)\big)\Big\}_{2\times N}$ has rank 1, each of its 2×2 minors must vanish; in particular, the $N-1$ minors involving the $(m_0)^{\text{th}}$ column vanish, so we have

$$F_{1\,,\,m}\big(\chi(t_0)\big)F_{2\,,\,m_0}\big(\chi(t_0)\big) - F_{2\,,\,m}\big(\chi(t_0)\big)F_{1\,,\,m_0}\big(\chi(t_0)\big) = 0 \qquad \text{for} \quad m = 1,\dots,N.$$

Therefore, by setting

$$\alpha_m(t_0) := \begin{cases}\dfrac{F_{1\,,\,m}\big(\chi(t_0)\big)}{F_{1\,,\,m_0}\big(\chi(t_0)\big)} & \text{if} \quad F_{1\,,\,m_0}\big(\chi(t_0)\big) \neq 0,\\[3ex] \dfrac{F_{2\,,\,m}\big(\chi(t_0)\big)}{F_{2\,,\,m_0}\big(\chi(t_0)\big)} & \text{if} \quad F_{1\,,\,m_0}\big(\chi(t_0)\big) = 0,\end{cases}$$

we can write

$$\left.\begin{aligned}F_{1\,,\,m}\big(\chi(t_0)\big) &= \alpha_m(t_0)F_{1\,,\,m_0}\big(\chi(t_0)\big)\\ F_{2\,,\,m}\big(\chi(t_0)\big) &= \alpha_m(t_0)F_{2\,,\,m_0}\big(\chi(t_0)\big)\end{aligned}\right\}, \qquad \text{for} \quad m = 1,\dots,N.$$

By using these relations and recalling the definition of $\mathcal{Q}_{\xi^0}$, we find

$$\lim_{t\to t_0} \frac{1}{t-t_0}\mathrm{k}_{\xi^0}(\zeta;t) = -\left\{\sum_{m=1}^{N} \alpha_m(t_0)\chi_m'(t_0)\right\}\mathcal{Q}_{\xi^0}\big(\zeta_1\xi^- + \zeta_2\xi^+\big), \qquad \text{for all} \quad \zeta \equiv (\zeta_1,\zeta_2) \in \mathbb{R}^2;$$

it is important to observe, and easy to check, that the factor in braces here is nonzero because the composition $F\circ\chi$ has nonzero derivative at t_0. Consequently, we can conclude that there is a positive δ_0 for which $\frac{1}{t-t_0}\mathrm{k}_{\xi^0}((1,0);t)$ and $\frac{1}{t-t_0}\mathrm{k}_{\xi^0}((0,1);t)$ are of constant but opposite sign for $0 < |t-t_0| < \delta_0$. Then $\mathrm{k}_{\xi^0}((1,0);t)$ and $\mathrm{k}_{\xi^0}((0,1);t)$ themselves are of constant but opposite sign for $t_0-\delta_0 < t < t_0$ and also for $t_0 < t < t_0+\delta_0$. Therefore, for each t with $0 < |t-t_0| < \delta_0$ at least one of the points $\pm\{F(\chi(t_0)) - F(\chi(t))\}$ lies in the interior $\mathcal{C}_t^\circ$; if both points lie in the interior then the cone $\mathcal{C}_t$ must be all of $\mathbb{R}^2$. In turn, there are just two alternatives: either

(*a*.) there is a positive δ_{t_0} and at least one of the corresponding one-sided intervals $(t_0-\delta_{t_0}, t_0)$ and $(t_0, t_0+\delta_{t_0})$ such that $F(\chi(t_0)) - F(\chi(t))$ lies in the interior $\mathcal{C}_t^\circ$ for every t in the interval while $-\{F(\chi(t_0)) - F(\chi(t))\}$ lies in the interior for no t in the interval

or

(*b*.) there is no such interval as in (*a*), *i.e.*, there are sequences $(t_n^-)_{n=1}^\infty$ and $(t_n^+)_{n=1}^\infty$ with $t_n^- \nearrow t_0$ and $t_n^+ \searrow t_0$, such that $-\{F(\chi(t_0)) - F(\chi(t_n^\pm))\} \in \mathcal{C}^\circ_{t_n^\pm}$ for $n \geq 1$.

In case (*a*), (4.12) clearly holds and we are done, so we proceed to the examination of case (*b*). In fact, we shall show that (*b*) implies the existence of an open interval about t_0 such that $\mathcal{C}_t = \mathbb{R}^2$ for every t in the interval; of course, with this we shall be able to conclude that (4.12) holds also in this second case.

Continuing under case (*b*), we also have, owing to the hypothesis that $F(\chi(\cdot))$ is injective, the inclusions $\hat{\tau}_n^\pm := \big|F(\chi(t_n^\pm)) - F(\chi(t_0))\big|_2^{-1}\{F(\chi(t_n^\pm)) - F(\chi(t_0))\} \in \mathcal{C}^\circ_{t_n^\pm}$ for all $n \geq 1$. Letting $n\to\infty$ in each of these, it is easy to see that we get

$$\lim_{n\to\infty}\hat{\tau}_n^\pm = \lim_{n\to\infty}\frac{F(\chi(t_n^\pm)) - F(\chi(t_0))}{\big|F(\chi(t_n^\pm)) - F(\chi(t_0))\big|_2} = \pm\frac{\mathrm{D}F(\chi(t_0))\chi'(t_0)}{\big|\mathrm{D}F(\chi(t_0))\chi'(t_0)\big|_2} =: \pm\hat{\tau}_0;$$

clearly, the limiting vectors $\pm\hat{\tau}_0$ here are just the two unit-tangent vectors to the singular-value curve at $F(\chi(t_0))$, *i.e.*, the unit vectors generated from $F_{1,m_0}(\chi(t_0))\varepsilon^{(1)} + F_{2,m_0}(\chi(t_0))\varepsilon^{(2)}$ and its negative. We claim that both of these lie in the cone $\mathcal{C}_{t_0}$. Once we have proven this, it will follow immediately that $\mathcal{C}_{t_0} = \mathbb{R}^2$, since the cone is convex and we have already decided that at least one of the limits lies in the *interior* of $\mathcal{C}_{t_0}$. Consider first the "+" case: for each $n \geq 1$, let $\zeta_n^+ \in \mathbb{R}^2$ with $\hat{\tau}_n^+ = B(\zeta_n^+; t_n^+)$. It is easy to see that the resultant sequence $(\zeta_n^+)_{n=1}^\infty$ is bounded in $\mathbb{R}^2$. Indeed, by virtue of inequality (4.11) and the convergence of $(t_n^+)_{n=1}^\infty$ to t_0, we have

$$c_0\big|\zeta_n^+\big|_2^2 \leq \big|B(\zeta_n^+; t_n^+)\big|_2 = \big|\hat{\tau}_n^+\big|_2 = 1 \qquad \text{for all sufficiently large } n.$$

Then there exists a subsequence $(\zeta_{n_k}^+)_{k=1}^\infty$ converging to some $\zeta_0^+ \in \mathbb{R}^2$. From the equalities $\hat{\tau}_{n_k}^+ = B(\zeta_{n_k}^+; t_{n_k}^+)$ we obtain $\hat{\tau}_0 = B(\zeta_0^+; t_0)$ by letting $k\to\infty$, and so conclude that $\hat{\tau}_0 \in \mathcal{C}_{t_0}$. The argument in the "−" case is analogous, and leads to the inclusion $-\hat{\tau}_0 \in \mathcal{C}_{t_0}$. As noted, we can now assert that $\mathcal{C}_{t_0} = \mathbb{R}^2$. The reasoning is completed by observing that the entries of the matrices defining the biquadratic maps $B(\cdot\,;t): \mathbb{R}^2 \to \mathbb{R}^2$ for $t_1 < t < t_2$ are continuous functions of the parameter t, since they are obtained from the second partial derivatives of the components F_1 and F_2 of F and the continuous basis-vector maps $t \mapsto \nu_t^{(j)}$. In Proposition A.4 it is shown under these conditions that the set $\{\, t \in (t_1,t_2) \,|\, \mathcal{C}_t := B(\mathbb{R}^2;t) = \mathbb{R}^2 \,\}$ is open. Therefore, in this case there exists an entire open interval containing t_0 such that we can find $\zeta_0^{t,\Delta} \in \mathbb{R}^2$ satisfying (4.12) for each t in the interval and *every* $\Delta \in \mathbb{R}^2$.

This completes the first step of the proof of (*ii*). Before passing to the second step, we prepare an estimate for the norms $\left|\zeta_0^{t,\Delta}\right|_2$ of the elements figuring in (4.12). First, since $F\big(\chi(\,\cdot\,)\big)$ is injective on (t_1, t_2), we may, and shall, suppose that the length of the interval $\mathfrak{I}_{t_0}$ and the radii of the discs in (4.12) are adjusted so that

$$0 < r_t < \left|F\big(\chi(t)\big) - F\big(\chi(t_0)\big)\right|_2^2 \le \left|F\big(\chi(t)\big) - F\big(\chi(t_0)\big)\right|_2 \qquad \text{for} \quad t \in \mathfrak{I}_{t_0}. \tag{4.13.1}$$

It follows directly from (4.12) that $B\big(\zeta_0^{t,\Delta}; t\big)$ does not vanish for $t \in \mathfrak{I}_{t_0}$ and $\left|\Delta\right|_2 < r_t$, and the same then must be true of $\zeta_0^{t,\Delta}$. Of course, (4.13.1) also implies that $r_t\left|F\big(\chi(t)\big) - F\big(\chi(t_0)\big)\right|_2^{-1} \to 0$ as $t \to t_0$ in $\mathfrak{I}_{t_0}$. Moreover, from (4.12) and the nondegeneracy inequality (4.11) we find that

$$\begin{aligned} c_0\left|\zeta_0^{t,\Delta}\right|_2^2 &\le \left|B(\zeta_0^{t,\Delta}; t)\right|_2 = \left|F\big(\chi(t_0)\big) - F\big(\chi(t)\big) + \Delta\right|_2 \\ &\le \left|F\big(\chi(t_0)\big) - F\big(\chi(t)\big)\right|_2 + r_t \le 2\left|F\big(\chi(t_0)\big) - F\big(\chi(t)\big)\right|_2 \\ &\qquad \text{for} \quad t \in \mathfrak{I}_{t_0} \text{ with } \left|t - t_0\right| < \eta_0 \quad \text{and} \quad \Delta \in \mathbb{R}^2 \text{ with } \left|\Delta\right|_2 < r_t. \end{aligned} \tag{4.13.2}$$

Step (*ii*.2). Now we establish the desired full result in (4.7). Observe at the outset that the equality required in (4.7) can be recast in the form

$$\frac{1}{2}\mathrm{D}^2 F\big(\chi(t_\delta); N_{t_\delta}\zeta^{\delta,\Delta}\big) + R_3 F\big(\chi(t_\delta); N_{t_\delta}\zeta^{\delta,\Delta}\big) = F\big(\chi(t_0)\big) + \Delta - F\big(\chi(t_\delta)\big),$$

or

$$B(\zeta^{\delta,\Delta}; t_\delta) + R_{[3]}(\zeta^{\delta,\Delta}; t_\delta) = F\big(\chi(t_0)\big) + \Delta - F\big(\chi(t_\delta)\big), \tag{4.14}$$

in which we have introduced the notation $R_{[3]}(\zeta; t) := R_3 F\big(\chi(t); N_t\zeta\big)$, so that

$$R_{[3]}(\zeta; t) = \frac{1}{2}\begin{pmatrix} \displaystyle\sum_{j,k,l=1}^{N} \left\{\int_0^1 (1-s)^2 F_{1,\,jkl}\big(\chi(t) + sN_t\zeta\big)\,ds\right\}\big(N_t\zeta\big)_j\big(N_t\zeta\big)_k\big(N_t\zeta\big)_l \\ \displaystyle\sum_{j,k,l=1}^{N} \left\{\int_0^1 (1-s)^2 F_{2,\,jkl}\big(\chi(t) + sN_t\zeta\big)\,ds\right\}\big(N_t\zeta\big)_j\big(N_t\zeta\big)_k\big(N_t\zeta\big)_l \end{pmatrix}$$
$$\text{for} \quad t_1 < t < t_2 \quad \text{and} \quad \zeta \in \mathbb{R}^2 \quad \text{with} \quad \left|\zeta\right|_2 \text{ sufficiently small.} \tag{4.15}$$

It is important to observe that the products of the components of $N_t\zeta$ appearing in (4.15) have the form

$$\big(N_t\zeta\big)_j\big(N_t\zeta\big)_k\big(N_t\zeta\big)_l = \alpha_{jkl}^{11}(t)\zeta_1^3 + \alpha_{jkl}^{12}(t)\zeta_1^2\zeta_2 + \alpha_{jkl}^{21}(t)\zeta_1\zeta_2^2 + \alpha_{jkl}^{22}(t)\zeta_2^3, \qquad \text{for} \quad \zeta \in \mathbb{R}^2, \quad t \in (t_1, t_2), \tag{4.16}$$

in which the $\alpha_{jkl}^{pq}(\,\cdot\,)$ are continuous on (t_1, t_2). The smoothness properties of $R_{[3]}$ in both its arguments follow from those of χ, the original map F, and the basis-vector maps $t \mapsto \nu_t^{(j)}$.

To prove the existence of the numbers required in (4.7), we shall apply the following statement of Kantorovich and Akilov [11], describing sufficient conditions under which the solvability of a "perturbed" problem follows from the solvability of a "base" problem.

Lemma 4.1. *Let X and Y be Banach spaces with maps π, $R : \{\mathrm{B}_{\varrho^\star}(\mathrm{x}_0) \subset X\} \to Y$ of class C^1 on the ball $\mathrm{B}_{\varrho^\star}(\mathrm{x}_0)$, where $\varrho^\star > 0$ and $\mathrm{x}_0 \in X$ satisfies*

$$\pi(\mathrm{x}_0) = 0.$$

Suppose also that the second derivatives π'' and R'' are continuous on some closed ball $\Omega_\varrho := \overline{\mathrm{B}_\varrho(\mathrm{x}_0)}$ (with $0 < \varrho \le \varrho^\star$), while the derivative $\pi'(\mathrm{x}_0) : X \to Y$ is invertible. Set

$$\gamma_0 := \left\| [\pi'(\mathrm{x}_0)]^{-1} R(\mathrm{x}_0) \right\|_X,$$

$$\gamma_1 := \left\| [\pi'(\mathrm{x}_0)]^{-1} R'(\mathrm{x}_0) \right\|_{\mathcal{B}(X)},$$

$$\gamma_2 := \sup_{\mathrm{x} \in \Omega_\varrho} \left\| [\pi'(\mathrm{x}_0)]^{-1} \left(\pi'' + R''\right)(\mathrm{x}) \right\|_{\mathcal{B}(X^2, X)}.$$

If the inequalities

$$\gamma_1 < 1, \qquad h := \frac{\gamma_0 \gamma_2}{(1-\gamma_1)^2} \le \frac{1}{2}, \qquad \text{and} \qquad \frac{1 - \sqrt{1-2h}}{h} \, \frac{\gamma_0}{1-\gamma_1} \le \varrho \tag{4.17}$$

hold, then there exists $\mathrm{x}_1 \in \Omega_\varrho$ such that $\pi(\mathrm{x}_1) + R(\mathrm{x}_1) = 0$.

Proof. This is an immediate consequence of Theorem XVIII.2.1 of Kantorovich and Akilov [11]; *cf.*, the explicit development in Section XVIII.2.6 of [11]. □

Remarks. (1) The finiteness of γ_2 is not an issue in our application of this result, since the Banach spaces will be finite-dimensional, so that the closed ball Ω_ϱ will be compact. Then the inequalities (4.17) required in the statement of Lemma 4.1 will clearly be satisfied if the norms $\left\|R(\mathrm{x}_0)\right\|_X$ and $\left\|R'(\mathrm{x}_0)\right\|_{\mathcal{B}(X,Y)}$ of the perturbation and of its derivative at x_0 are sufficiently small. Indeed, satisfaction of the three inequalities quantifies what is meant here by a "small perturbation."

(2) Here, the second derivatives $\pi''(\mathrm{x})$ and $R''(\mathrm{x})$ are regarded in the natural manner as elements of the Banach space $\mathcal{B}(X^2, Y)$ of bounded bilinear Y-valued operators on $X \times X$; *cf.* [11]. Then, for example, $[\pi'(\mathrm{x}_0)]^{-1}(\pi'' + R'')(\mathrm{x})$ is an element of $\mathcal{B}(X^2, X)$; the norm of such a composition is bounded above by the product $\left\|[\pi'(\mathrm{x}_0)]^{-1}\right\|_{\mathcal{B}(Y,X)} \left\|(\pi'' + R'')(\mathrm{x})\right\|_{\mathcal{B}(X^2,Y)}$ of the norms, just as in more familiar settings.

In applying Lemma 4.1 to establish (4.7), the forms of (4.12) and (4.14) suggest that we take $X = Y = \mathbb{R}^2$, $\pi(\,\cdot\,) = B(\,\cdot\,; t_\delta) - \{F(\chi(t_0)) + \Delta - F(\chi(t_\delta))\}$, $R(\,\cdot\,) = R_{[3]}(\,\cdot\,; t_\delta)$, and $\mathrm{x}_0 = \zeta_0^{t_\delta, \Delta}$ (as in (4.12)), all for an appropriate value of $t_\delta \in (t_1, t_2)$ with $|t_\delta - t_0| < \delta$, chosen to ensure that the hypotheses of the Lemma are fulfilled, and for any $\Delta \in \mathbb{R}^2$ with $|\Delta|_2 < r_\delta^\star$ for $r_\delta^\star := r_{t_\delta}$.

With $\mathfrak{I}_{t_0}$ as in (4.11), for $\eta > 0$ we shall denote by $\mathfrak{I}_{t_0}(\eta)$ the interval $\{\, t \in \mathfrak{I}_{t_0} \mid |t - t_0| < \eta \,\}$. Since $\chi(t_0)$ lies in the open set $\mathcal{U}$, a simple continuity argument shows that there is a positive ϱ_0 such that

$$\chi(t) + \sum_{j=1}^{N-1} \left\{ \zeta_1 \xi_j^- + \zeta_2 \xi_j^+ \right\} \nu_t^{(j)} \in \mathcal{U} \qquad \text{whenever} \quad |t - t_0| < 2\varrho_0 \quad \text{and} \quad |\zeta|_2 < 2\varrho_0.$$

Then, for each t with $\left|t-t_0\right| < 2\varrho_0$ the map $\zeta \to F\big(\chi(t)+\zeta_1 N_t^1+\zeta_2 N_t^2\big)$ is defined in the disc $\mathrm{B}^2_{2\varrho_0}(0) \subset \mathbb{R}^2$, since we may also suppose that the interval $(t_0-2\varrho_0, t_0+2\varrho_0)$ is contained in (t_1, t_2). Setting

$$\varrho_\delta := \min\left\{\frac{2\varrho_0}{3}, \frac{\delta}{2\sqrt{2}\max\{|\xi^-|_{N-1}, |\xi^+|_{N-1}\}}\right\},$$

we can find $\eta_1 > 0$ with $\eta_1 < \min\{2\varrho_0, \eta_0\}$ for which

$$\frac{2}{c_0}\left|F\big(\chi(t)\big)-F\big(\chi(t_0)\big)\right|_2 < \varrho_\delta^2 \qquad \text{whenever} \quad t \in \mathfrak{I}_{t_0}(\eta_1).$$

According to (4.13.2), we have then also $\left|\zeta_0^{t,\Delta}\right|_2 < \varrho_\delta$ for $t \in \mathfrak{I}_{t_0}(\eta_1)$ and $\left|\Delta\right|_2 < r_t$. It follows that the map $\zeta \to F\big(\chi(t)+\zeta_1 N_t^1+\zeta_2 N_t^2\big)$ is defined in a neighborhood of the closed disc $\overline{\mathrm{B}^2_{\varrho_\delta}(\zeta_0^{t,\Delta})}$, in fact, in the open disc $\mathrm{B}^2_{2\varrho_\delta}(\zeta_0^{t,\Delta})$, whenever $t \in \mathfrak{I}_{t_0}(\eta_1)$ and $\Delta \in \mathbb{R}^2$ with $\left|\Delta\right|_2 < r_t$, since then $|\zeta|_2 < 2\varrho_0$ if $\left|\zeta-\zeta_0^{t,\Delta}\right|_2 < 2\varrho_\delta$.

The explicit form of the map B is given in (4.9), (4.10), and (4.8), while that of the remainder-map $R_{[3]}$ is displayed in (4.15) with (4.16). The latter two equalities show that there is a positive M_0 such that

$$\left|R_{[3]}(\zeta_0;t)\right|_2 \le M_0\left|\zeta_0\right|_2^3 \qquad \text{for} \quad \left|\zeta_0\right|_2 \le \varrho_0 \quad \text{and} \quad \left|t-t_0\right| \le \varrho_0. \tag{4.18.1}$$

Whenever they are defined, the derivatives $B'(\zeta_0;t)$ and $R'_{[3]}(\zeta_0;t)$ (reckoned with respect to the first argument) are just the operators that we have been denoting by $\mathrm{D}B(\zeta_0;t)$ and $\mathrm{D}R_{[3]}(\zeta_0;t)$, respectively. The action of the derivative $B'\big(\zeta_0;t\big)$ is given by

$$B'\big(\zeta_0;t\big)\zeta = 2\begin{pmatrix} \big(\mathcal{A}_t^{(1)}\zeta_0\big)\cdot\zeta \\ \\ \big(\mathcal{A}_t^{(2)}\zeta_0\big)\cdot\zeta \end{pmatrix}, \qquad \text{for} \quad \zeta \in \mathbb{R}^2, \quad t \in \big(t_1, t_2\big), \tag{4.19}$$

with the matrices $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ appearing in (4.10). We shall use these explicit expressions to prove that there are positive numbers c_3 and η_3 such that

$$\left.\begin{array}{l} \text{the operator } B'\big(\zeta_0^{t,\Delta};t\big) \text{ is injective and the norm of its inverse is bounded by} \\ \left\|[B'(\zeta_0^{t,\Delta};t)]^{-1}\right\| \le \dfrac{c_3}{\left|\zeta_0^{t,\Delta}\right|_2} \qquad \text{for} \quad t \in \mathfrak{I}_{t_0}(\eta_3) \quad \text{and} \quad \Delta \in \mathbb{R}^2 \text{ with } \left|\Delta\right|_2 < r_t \end{array}\right\} \tag{4.20}$$

(recall that none of the $\zeta_0^{t,\Delta}$ vanish); the proof of this essential bound we defer until after it is clear that it does indeed enable us to establish (4.7) (*cf*. (4.14)). For the norm of the operator $R'_{[3]}(\zeta_0;t)$ it suffices to use the simple upper bound given in terms of the partial derivatives of the component-functions of $R_{[3]}$ by

$$\left\|R'_{[3]}(\zeta_0;t)\right\|_{\mathcal{B}(\mathbb{R}^2)} \le \left\{\sum_{l=1}^{2}\sum_{m=1}^{2}\left|R_{[3]l,\,m}(\zeta_0;t)\right|^2\right\}^{\frac{1}{2}}, \qquad \text{for} \quad \left|\zeta_0\right|_2 < 2\varrho_0, \quad \left|t-t_0\right| < 2\varrho_0.$$

In fact, from this and an inspection of (4.15) and (4.16), by recalling that F is now supposed to be of class C^5 we see that there is a positive M_1 such that

$$\left\|R'_{[3]}(\zeta_0;t)\right\|_{\mathcal{B}(\mathbb{R}^2)} \le M_1\left|\zeta_0\right|_2^2 \qquad \text{for} \quad \left|\zeta_0\right|_2 \le \varrho_0 \quad \text{and} \quad \left|t-t_0\right| \le \varrho_0. \tag{4.18.2}$$

The second derivatives $B''(\zeta;t)$ and $R''_{[3]}(\zeta;t)$ are bilinear operators taking $\mathbb{R}^2 \times \mathbb{R}^2$ into $\mathbb{R}^2$. More precisely, when $|\zeta|_2 < 2\varrho_0$ and $|t-t_0| < 2\varrho_0$ we have

$$R''_{[3]}(\zeta;t)\,[\zeta',\zeta''] = \begin{pmatrix} \displaystyle\sum_{p,q=1}^{N} R_{[3]1,\,pq}(\zeta;t)\zeta'_p\zeta''_q \\ \displaystyle\sum_{p,q=1}^{N} R_{[3]2,\,pq}(\zeta;t)\zeta'_p\zeta''_q \end{pmatrix} \qquad \text{for} \quad \zeta',\,\zeta'' \in \mathbb{R}^2,$$

with a similar expression for the action of $B''(\zeta;t)$; again, we rely here on the additional smoothness hypothesized for F. From these forms, upper bounds on the bilinear-operator norms in terms of the second-order partial derivatives of the component-functions are found as

$$\|B''(\zeta;t)\|_{\mathcal{B}((\mathbb{R}^2)^2,\mathbb{R}^2)} \le 2\left\{\sum_{l=1}^{2}\sum_{p,q=1}^{2} \left|\big(\mathcal{A}_t^{(l)}\big)_{pq}\right|^2\right\}^{\frac{1}{2}},$$

and

$$\|R''_{[3]}(\zeta;t)\|_{\mathcal{B}((\mathbb{R}^2)^2,\mathbb{R}^2)} \le \left\{\sum_{l=1}^{2}\sum_{p,q=1}^{2} \left|R_{[3]l,\,pq}(\zeta;t)\right|^2\right\}^{\frac{1}{2}},$$

(at least) for $|\zeta|_2 < 2\varrho_0$ and $|t-t_0| < 2\varrho_0$. It follows that there is an $M_2 > 0$ such that

$$\|B''(\zeta;t)\|_{\mathcal{B}((\mathbb{R}^2)^2,\mathbb{R}^2)} < M_2 \quad \text{and} \quad \|R''_{[3]}(\zeta;t)\|_{\mathcal{B}((\mathbb{R}^2)^2,\mathbb{R}^2)} < M_2$$
$$\text{if} \quad |\zeta|_2 \le \frac{4\varrho_0}{3} \quad \text{and} \quad |t-t_0| \le \varrho_0 \tag{4.18.3}$$

(a factor $|\zeta|_2$ can be appended to the second bound, but is not needed here).

Use of (4.20) along with (4.18.1), (4.18.2), and (4.18.3), respectively, shows that there is a positive $\eta_2 \le \min\{\varrho_0,\eta_1,\eta_3\}$ such that for any $t \in \mathfrak{I}_{t_0}(\eta_2)$ and $\Delta \in \mathbb{R}^2$ with $|\Delta|_2 < r_t$ we have the inequalities

$$\gamma_0 = \left|[B'(\zeta_0^{t,\Delta};t)]^{-1}R_{[3]}(\zeta_0^{t,\Delta};t)\right|_2 \le \frac{c_3}{|\zeta_0^{t,\Delta}|_2}M_0|\zeta_0^{t,\Delta}|_2^3 = \widetilde{M}_0|\zeta_0^{t,\Delta}|_2^2,$$

$$\gamma_1 = \left\|[B'(\zeta_0^{t,\Delta};t)]^{-1}R'_{[3]}(\zeta_0^{t,\Delta};t)\right\|_{\mathcal{B}(\mathbb{R}^2)} \le \frac{c_3}{|\zeta_0^{t,\Delta}|_2}M_1|\zeta_0^{t,\Delta}|_2^2 = \widetilde{M}_1|\zeta_0^{t,\Delta}|_2,$$

and

$$\gamma_2 = \sup_{|\zeta-\zeta_0^{t,\Delta}|_2\le\varrho_\delta} \left\|[B'(\zeta_0^{t,\Delta};t)]^{-1}\big(B''+R''_{[3]}\big)(\zeta;t)\right\|_{\mathcal{B}((\mathbb{R}^2)^2,\mathbb{R}^2)} \le \frac{c_3}{|\zeta_0^{t,\Delta}|_2}2M_2 = \frac{\widetilde{M}_2}{|\zeta_0^{t,\Delta}|_2}.$$

With the preparatory estimates thus completed, we see next that we can find $\eta_4 > 0$ with $\eta_4 \le \eta_0$ for which

$$\frac{2}{c_0}\left|F\big(\chi(t)\big) - F\big(\chi(t_0)\big)\right|_2 < \left\{\min\left\{\frac{1}{2\widetilde{M}_1},\frac{1}{16\widetilde{M}_0\widetilde{M}_2},\frac{\sqrt{\varrho_\delta}}{2\sqrt{\widetilde{M}_0}},\varrho_\delta\right\}\right\}^2 \qquad \text{whenever} \quad t \in \mathfrak{I}_{t_0}(\eta_4).$$

Therefore, recalling (4.13.2), we get

$$\left|\zeta_0^{t,\Delta}\right|_2 < \min\left\{\frac{1}{2\widetilde{M}_1}, \frac{1}{16\widetilde{M}_0\widetilde{M}_2}, \frac{\sqrt{\varrho_\delta}}{2\sqrt{\widetilde{M}_0}}, \varrho_\delta\right\} \qquad \text{whenever} \quad t \in \mathfrak{I}_{t_0}(\eta_4) \quad \text{and} \quad \left|\Delta\right|_2 < r_t.$$

Now we choose any $t_\delta \in \mathfrak{I}_{t_0}$ with $\left|t_\delta - t_0\right| < \min\{\delta, \eta_2, \eta_4\}$, and set $r_\delta^\star := r_{t_\delta}$. Then for any $\Delta \in \mathbb{R}^2$ with $\left|\Delta\right|_2 < r_\delta^\star$, we find

$$\gamma_1 \leq \widetilde{M}_1\left|\zeta_0^{t_\delta,\Delta}\right|_2 < \frac{1}{2} \qquad \text{and} \qquad \gamma_0 \leq \widetilde{M}_0\left|\zeta_0^{t_\delta,\Delta}\right|_2^2 < \frac{\varrho_\delta}{4},$$

whence

$$h := \frac{\gamma_0\gamma_2}{(1-\gamma_1)^2} \leq 4\frac{\widetilde{M}_2}{\left|\zeta_0^{t_\delta,\Delta}\right|_2}\gamma_0 \leq 4\widetilde{M}_2\widetilde{M}_0\left|\zeta_0^{t_\delta,\Delta}\right|_2 < \frac{1}{4} \quad \text{and} \quad \frac{1-\sqrt{1-2h}}{h}\,\frac{\gamma_0}{1-\gamma_1} < 2(2\gamma_0) < \varrho_\delta,$$

in which we used the fact that the real function $s \mapsto s^{-1}\{1-\sqrt{1-2s}\}$ is increasing on $\left(-\infty, \frac{1}{2}\right]$. By comparing the inequalities required in (4.17), we see finally that we can apply Lemma 4.1 to conclude that there exists $\zeta^{\delta,\Delta}$ with $\left|\zeta^{\delta,\Delta} - \zeta_0^{t_\delta,\Delta}\right|_2 \leq \varrho_\delta$ and satisfying (4.14). Moreover, we find that

$$\left|\zeta^{\delta,\Delta}\right|_2 \leq \left|\zeta_0^{t_\delta,\Delta}\right|_2 + \varrho_\delta < 2\varrho_\delta < \frac{\delta}{\sqrt{2}\max\{|\xi^-|_{N-1}, |\xi^+|_{N-1}\}},$$

from which it follows readily that $\left|\left(\zeta^{\delta,\Delta}\right)_1\xi^- + \left(\zeta^{\delta,\Delta}\right)_2\xi^+\right|_{N-1} < \delta$ whenever $\left|\Delta\right|_2 < r_\delta^\star$; for those same $\Delta \in \mathbb{R}^2$, upon recalling that $\left|t_\delta - t_0\right| < \delta$ we see that

$$\chi(t_\delta) + \sum_{j=1}^{N-1}\left\{\left(\zeta^{\delta,\Delta}\right)_1\xi_j^- + \left(\zeta^{\delta,\Delta}\right)_2\xi_j^+\right\}\nu_{t_\delta}^{(j)} \in U_\delta,$$

as required in (4.7).

Now the proof of statement (ii) is completed by returning to verify (4.20). The argument begins with an explicit computation and norm-estimation of $[B'(\zeta_0;t)]^{-1}\zeta$ for $\zeta \in \mathbb{R}^2$, when $\zeta_0 \in \mathbb{R}^2$ is not a singular point of $B(\,\cdot\,;t)$. From the expression for the differential $B'(\zeta_0;t)\zeta$, given in (4.19) for ζ_0, $\zeta \in \mathbb{R}^2$, it is easy to see that $B'(\zeta_0;t) : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible for some t, *i.e.*, that ζ_0 is a regular point of the biquadratic map $B(\,\cdot\,;t)$, iff the quadratic form $\beta(\,\cdot\,;t)$ given on $\mathbb{R}^2$ by

$$\beta(\zeta;t) := \left(\mathcal{A}_t^{(1)}\zeta\right)_1\left(\mathcal{A}_t^{(2)}\zeta\right)_2 - \left(\mathcal{A}_t^{(1)}\zeta\right)_2\left(\mathcal{A}_t^{(2)}\zeta\right)_1$$

is nonzero at ζ_0; *cf.*, also, Proposition A.3. In that case, a simple computation followed by an obvious estimation produces $[B'(\zeta_0;t)]^{-1}\zeta$ and the inequality

$$\left|[B'(\zeta_0;t)]^{-1}\zeta\right|_2^2 \leq \frac{\left|\mathcal{A}_t^{(1)}\zeta_0\right|_2^2 + \left|\mathcal{A}_t^{(2)}\zeta_0\right|_2^2}{4\left(\beta(\zeta_0;t)\right)^2}\left|\zeta\right|_2^2 \qquad \text{for} \quad \zeta \in \mathbb{R}^2 \quad \text{and} \quad t \in (t_1, t_2).$$

With this, it is clear that (4.20) will follow once we have shown that there can be found $c_3' > 0$ and $\eta_3 > 0$ such that

$$\left|\beta(\zeta_0^{t,\Delta};t)\right| \geq c_3'\left|\zeta_0^{t,\Delta}\right|_2^2 \qquad \text{for} \quad t \in \mathfrak{I}_{t_0}(\eta_3) \quad \text{and} \quad \Delta \in \mathbb{R}^2 \text{ with } \left|\Delta\right|_2 < r_t. \tag{4.21}$$

For this, suppose the contrary: then there are sequences $(t_n)_{n=1}^{\infty}$ from $\mathcal{I}_{t_0}$ with $t_n \to t_0$ as $n \to \infty$ and $(\Delta_n)_{n=1}^{\infty}$ from $\mathbb{R}^2$ with $\Delta_n < r_{t_n}$ for each $n \geq 1$, such that

$$\left|\beta(\zeta_0^{t_n,\Delta_n};t_n)\right| < \frac{1}{n}\left|\zeta_0^{t_n,\Delta_n}\right|_2^2 \qquad \text{for each} \quad n \geq 1.$$

We have remarked that none of the $\zeta_0^{t,\Delta}$ vanish; the normalized elements $\hat{\zeta}_0^{t_n,\Delta_n} := \left|\zeta_0^{t_n,\Delta_n}\right|_2^{-1}\zeta_0^{t_n,\Delta_n}$ then satisfy $\left|\beta(\hat{\zeta}_0^{t_n,\Delta_n};t_n)\right| < \frac{1}{n}$ for each n and contain a subsequence converging to some $\hat{\zeta}_0^0$ with $\left|\hat{\zeta}_0^0\right|_2 = 1$ for which we find $\beta(\hat{\zeta}_0^0;t_0) = 0$. The strong nondegeneracy implies that $B(\hat{\zeta}_0^0;t_0) \neq 0$, as one can see from (4.11). According to Proposition A.3.*ii*, $\hat{\zeta}_0^0$ is therefore a singular point of the biquadratic map $B(\,\cdot\,;t_0)$ and the image $B(\hat{\zeta}_0^0;t_0)$ lies on the boundary $\partial\mathcal{C}_{t_0}$ of the range $\mathcal{C}_{t_0} := \mathcal{R}_{B(\,\cdot\,;t_0)}$ of $B(\,\cdot\,;t_0)$.

On the other hand, by recalling that none of the $B(\zeta_0^{t,\Delta};t)$ vanish, and by denoting convergent subsequences again with the same symbols as the original, we also have

$$\begin{aligned}\frac{B(\hat{\zeta}_0^0;t_0)}{\left|B(\hat{\zeta}_0^0;t_0)\right|_2} &= \lim_{n\to\infty}\frac{B(\hat{\zeta}_0^{t_n,\Delta_n};t_n)}{\left|B(\hat{\zeta}_0^{t_n,\Delta_n};t_n)\right|_2} = \lim_{n\to\infty}\frac{B(\zeta_0^{t_n,\Delta_n};t_n)}{\left|B(\zeta_0^{t_n,\Delta_n};t_n)\right|_2}\\ &= \lim_{n\to\infty}\frac{F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)+\Delta_n}{\left|F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)+\Delta_n\right|_2} = \lim_{n\to\infty}\frac{F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)}{\left|F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)\right|_2}; \end{aligned} \tag{4.22}$$

here, the final equality on the right holds because

$$\frac{\left|\Delta_n\right|}{\left|F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)\right|_2} < \frac{r_{t_n}}{\left|F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)\right|_2} \to 0 \qquad \text{as} \quad n\to\infty,$$

by (4.13.1), and

$$\left|\frac{F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)+\Delta_n}{\left|F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)+\Delta_n\right|_2} - \frac{F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)}{\left|F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)\right|_2}\right|_2^2 = 2-2\frac{\widehat{\Phi}_{t_n}\cdot\left\{\widehat{\Phi}_{t_n}+\dfrac{\Delta_n}{\left|\Phi_{t_n}\right|_2}\right\}}{\left|\widehat{\Phi}_{t_n}+\dfrac{\Delta_n}{\left|\Phi_{t_n}\right|_2}\right|_2},$$

in which we used the abbreviation $\Phi_{t_n} := F\big(\chi(t_0)\big)-F\big(\chi(t_n)\big)$ and denoted by $\widehat{\Phi}_{t_n}$ the corresponding vector of unit norm. The limit in (4.22) is clearly a (unit-) tangent vector to the singular-value curve at $F\big(\chi(t_0)\big)$. But this is impossible, since it implies that at least one ray on the tangent line to the singular-value curve at $F\big(\chi(t_0)\big)$ lies in the boundary of the convex-cone range $\mathcal{C}_{t_0}$ of $B(\,\cdot\,;t_0)$, contradicting our previous observation that the tangent line has at least one ray lying in the *interior* of that convex cone (based on the original indefiniteness hypothesis, the choice of ξ^- and ξ^+, and the strong nondegeneracy of the singular point $\chi(t_0)$). We conclude that (4.21) is correct. As noted, this completes the proof of statement (*ii*) of the theorem.

Proof of (*iii*): Here we cite two simple examples to show that either possibility can obtain in the case of a semidefinite quadratic form $\mathcal{Q}_{\xi^0}$, *i.e.*, the singular value $F(\xi^0)$ in such a case belongs to the local boundary in some examples but is locally covered by F from ξ^0 in others.

Example 4.1.1. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by setting

$$\left.\begin{aligned} F_1(\xi) &:= \xi_1^2+\xi_2^2+\xi_3^2\\ F_2(\xi) &:= 2\xi_1^2+\xi_2^2+\xi_3^2 \end{aligned}\right\}, \qquad \text{for} \quad \xi \equiv (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3,$$

i.e., F is the biquadratic map on $\mathbb{R}^3$ generated by the symmetric matrices

$$\mathcal{A}^{(1)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{A}^{(2)} := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

cf., Appendix A. We shall show that the point $\xi^0 := (0,0,1)$ (for example) is a nondegenerate, rank-1 singular point of F for which the corresponding singular value $F(\xi^0) = (1,1)$ lies on the boundary $\partial\mathcal{R}_F$, while the quadratic form $\mathcal{Q}_{\xi^0}$ is semidefinite.

It is easy to identify the range of F here. In fact, we know that $\mathcal{R}_F$ is a convex cone in $\mathbb{R}^2$ with vertex at the origin (Appendix A), and it is obvious now that the cone lies in the first quadrant of $\mathbb{R}^2$. Moreover, it is simple to check that the slope of the ray in $\mathbb{R}^2$ from the origin through $F(\xi)$ satisfies $1 \leq F_2(\xi)/F_1(\xi) \leq 2$ for any nonzero $\xi \in \mathbb{R}^3$ and that the ratio does take on the limit-values 1 and 2; in particular, the slope of the ray through $F(\xi^0)$ is just 1. From these observations it follows that the range of F is the closed cone in the first quadrant bounded by the rays from the origin with slopes 1 and 2, hence that $F(\xi^0)$ belongs to the boundary $\partial\mathcal{R}_F$.

From the matrix $2\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ 2\xi_1 & \xi_2 & \xi_3 \end{pmatrix}$ of partial derivatives, one finds the singular points as the union of the 1-axis $\{\, (\xi_1,0,0) \mid \xi_1 \in \mathbb{R} \,\}$ and the 2,3-plane $\{\, (0,\xi_2,\xi_3) \mid \xi_2,\xi_3 \in \mathbb{R} \,\}$; all of these are rank-1 except $(0,0,0)$, which is rank-0. The Hessian matrices of F_1 and F_2 are the constant matrices $2\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, respectively, which are nonsingular, so all of the singular points are strongly nondegenerate except $(0,0,0)$. Since F maps the 1-axis onto the ray from the origin with slope 2 and the 2,3-plane onto the ray from the origin with slope 1, we see that the set of singular values here coincides with the boundary of the range.

Now consider, *e.g.*, the rank-1 singular point $\xi^0 := (0,0,1)$, which F takes to the singular value $(1,1)$ on the boundary of the range. We find that $\mathcal{N}(\mathrm{D}F(\xi^0)) = \mathrm{sp}\,\{\mathrm{e}^{(1)}, \mathrm{e}^{(2)}\}$. As the curve χ of singular points we may take, say, a parametrization of any line segment through ξ^0, lying in the 2,3-plane, and not parallel to the basis-vector $\mathrm{e}^{(2)}$ (to avoid a cusp on the corresponding image-curve of singular values). A short computation shows that the quadratic form $\mathcal{Q}_{\xi^0}$ on $\mathbb{R}^2$ is given by $\mathcal{Q}_{\xi^0}(\zeta) = -8\zeta_1^2$, for $\zeta \equiv (\zeta_1,\zeta_2) \in \mathbb{R}^2$, which is negative-semidefinite.

Example 4.1.2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\left.\begin{aligned} F_1(\xi) &:= \left(\xi_1^2 - 1\right)^3 \\ F_2(\xi) &:= \xi_1\left(\xi_2 - \xi_1\right)^2 \end{aligned}\right\}, \qquad \text{for} \quad \xi \equiv (\xi_1,\xi_2) \in \mathbb{R}^2.$$

This example is discussed in Section 6, where it appears as Example 6.3. There, the singular points and singular values are found and displayed graphically. In particular, it is shown that the two lines $\{\, \xi \in \mathbb{R}^2 \mid \xi_1 = \pm 1 \,\}$ comprise singular points, all of which are rank-1 and strongly nondegenerate except for the two $(\pm 1, \pm 1)$. Moreover, at each such rank-1 singular point ξ^0, the corresponding quadratic form $\mathcal{Q}_{\xi^0}$ reduces to the zero-form. The corresponding sets of singular values are the positive and negative rays on the 2-axis, *i.e.*, the mapping action of F folds the line given by $\xi_1 = -1$ [by $\xi_1 = +1$] at the point $(-1,-1)$ [at the point $(1,1)$] and takes both halves onto the ray $\{\, (x_1,x_2) \mid x_1 = 0,\ x_2 \leq 0\ [x_2 \geq 0] \,\}$. The graphical evidence indicates that each point on the latter rays of singular values (except perhaps the origin) is locally covered from each of its two singular-point preimages (at which the quadratic forms are trivial). We shall verify this for one such singular pair, say, the singular point $\xi^0 := (1,2)$ and the corresponding singular value $F(\xi^0) = (0,1)$, which will complete this example and the proof of Theorem 4.1.

Let the map $G : \mathcal{D}_G \to \mathbb{R}^2$ be defined on the open square $\mathcal{D}_G := (-1,1) \times (0,2)$ in $\mathbb{R}^2$ according to

$$\left.\begin{aligned} G_1(x) &:= \sqrt{x_1^{1/3} + 1} \\ G_2(x) &:= \sqrt{\frac{x_2}{\sqrt{x_1^{1/3} + 1}}} + \sqrt{x_1^{1/3} + 1} \end{aligned}\right\}, \qquad \text{for} \quad -1 < x_1 < 1, \quad 0 < x_2 < 2.$$

G is clearly continuous. Moreover, one can show that G maps $\mathcal{D}_G$ into the domain

$$\mathcal{D}(F) := \left\{ \xi \equiv (\xi_1, \xi_2) \,\middle|\, 0 < \xi_1 < \sqrt{2}, \quad \xi_1 < \xi_2 < \xi_1 + \sqrt{\frac{2}{\xi_1}} \right\}$$

with $F(G(x)) = x$ for every $x \in \mathcal{D}_G$, while F takes $\mathcal{D}(F)$ into $\mathcal{D}_G$ with with $G(F(\xi)) = \xi$ for every $\xi \in \mathcal{D}(F)$, so the restriction $F|\mathcal{D}(F)$ is injective with inverse G. That is, F takes the domain $\mathcal{D}(F)$ containing $\xi^0 = (1,2)$ homeomorphically onto $\mathcal{D}_G$ containing $F(\xi^0) = (0,1)$, whence we conclude that every neighborhood of ξ^0 is mapped to a neighborhood of $F(\xi^0)$, *i.e.*, $F(\xi^0)$ is locally covered by F from its singular preimage ξ^0. $\square$

5. Discrimination of singular values: the general submanifold-domain.

Now we take up the more general case in which the domain $\mathcal{D}_F$ is a differentiable N-dimensional submanifold-without-boundary $\mathcal{M}^N$ in some $\mathbb{R}^{N_0}$, with the aim of extending the tests contained in Theorem 4.1 (where the domain is an open set in $\mathbb{R}^N$). In fact, since the identification of fold points and locally covered points is a completely local examination, we could proceed in practice by covering $\mathcal{M}$ with coördinate patches, using coördinate functions to transfer the analysis down into open subsets of $\mathbb{R}^{N-1}$, and applying Theorem 4.1. But we prefer to formulate a test involving only quantities intrinsic to $\mathcal{M}$, *i.e.*, having a form that is independent of any particular coördinate system. We present such a test in Theorem 5.1; the result was obtained by first following the coördinate-system procedure just described and subsequently translating the outcome to a coördinate-system independent form.

We begin with the generalization of Proposition 4.1 to the present circumstances.

Proposition 5.1. *Let $N_0 \geq N \geq 2$. Let $F : \mathcal{M}^N \to \mathbb{R}^2$ be a mapping of class C^1 on the N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ of class C^1. Suppose that ξ^0 is a rank-1 singular point of F for which hypotheses $(H.1)$ and $(H.2)$ hold.*

- *$(i.)$ Let $t \mapsto \zeta(t) \in \mathcal{M}^N$, $t_1 < t < t_2$, be a smooth curve of class C^1 with range in $\mathcal{M}^N$ and such that $\zeta(t_0) = \chi(t_0)$ $(= \xi^0)$ and $\zeta'(t_0) \notin \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$. Then the curves $t \mapsto F\big(\chi(t)\big)$ and $t \mapsto F\big(\zeta(t)\big)$ have the same tangent vectors at the singular value $F(\xi^0)$.*
- *$(ii.)$ Let τ_0 denote any element of $\mathsf{T}_{\xi^0}\mathcal{M}$ that is not in the nullspace $\mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$, i.e., which satisfies $\big|\mathrm{grad}_{\mathcal{M}}F_1(\xi^0)\cdot\tau_0\big|^2 + \big|\mathrm{grad}_{\mathcal{M}}F_2(\xi^0)\cdot\tau_0\big|^2 > 0$ (such a τ_0 exists). Then a tangent to the singular-value curve $t \mapsto F\big(\chi(t)\big)$ at $F(\xi^0)$ is given by $\{\mathrm{grad}_{\mathcal{M}}F_1(\xi^0)\cdot\tau_0\}\varepsilon^{(1)} + \{\mathrm{grad}_{\mathcal{M}}F_2(\xi^0)\cdot\tau_0\}\varepsilon^{(2)}$.*

Proof. The proof consists in checking that the same sorts of computations used in the proof of Proposition 4.1 can be extended to the present more general setting.

(i). The null space $\mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$ has dimension $N-1$, lies in the N-dimensional tangent space $\mathsf{T}_{\xi^0}\mathcal{M}$, and does not contain $\chi'(t_0)$ (which is nonzero). Therefore, we can represent the (nonzero) element $\zeta'(t_0)$ of $\mathsf{T}_{\xi^0}\mathcal{M}$ by $\zeta'(t_0) = \alpha_0\chi'(t_0) + \nu_0$ for some $\alpha_0 \in \mathbb{R}$ and $\nu_0 \in \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$; α_0 must be nonzero, since $\zeta'(t_0) \notin \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$. Consequently, we compute

$$\big(F\circ\zeta\big)'(t_0) = \mathrm{D}F\big(\zeta(t_0)\big)\zeta'(t_0) = \mathrm{D}F(\xi^0)\{\alpha_0\chi'(t_0)+\nu_0\} = \alpha_0\mathrm{D}F\big(\chi(t_0)\big)\chi'(t_0) = \alpha_0\big(F\circ\chi\big)'(t_0),$$

which completes the proof of (i).

(ii). We construct a smooth curve ζ with range in $\mathcal{M}^N$, passing through $\xi^0 = \chi(t_0)$, and having tangent vector τ_0 at $\chi(t_0)$: Let (U,h) be a coördinate system for $\mathcal{M}$ with $\xi^0 \in U$. Since $\mathrm{D}h^{-1}\big(h(\xi^0)\big)$ is an isomorphism of $\mathbb{R}^N$ onto $\mathsf{T}_{\xi^0}\mathcal{M}$, there is a (unique) $\tilde{\tau}_0 \in \mathbb{R}^N$ with $\mathrm{D}h^{-1}\big(h(\xi^0)\big)\tilde{\tau}_0 = \tau_0$. By setting $\zeta(t) := h^{-1}\Big(h(\xi^0) + (t-t_0)\tilde{\tau}_0\Big)$ for all t with $|t-t_0|$ sufficiently small, it is clear that the range of ζ lies in $\mathcal{M}$, while $\zeta(t_0) = \xi^0$ and $\zeta'(t_0) = \mathrm{D}h^{-1}\big(h(\xi^0)\big)\tilde{\tau}_0 = \tau_0$, so that ζ has the required properties. Since $\zeta'(t_0) \notin \mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$, we can apply part (i) of the Proposition to conclude that a tangent vector to the singular-value curve at $F(\xi^0)$ in $\mathbb{R}^2$ is given by the derivative of $t \mapsto F\big(\zeta(t)\big)$ evaluated at $t = t_0$, which we compute as

$$\mathrm{D}F\big(\zeta(t_0)\big)\zeta'(t_0) = \mathrm{D}F(\xi^0)\tau_0 = \{\mathrm{grad}_{\mathcal{M}}F_1(\xi^0)\cdot\tau_0\}\varepsilon^{(1)} + \{\mathrm{grad}_{\mathcal{M}}F_2(\xi^0)\cdot\tau_0\}\varepsilon^{(2)},$$

producing the result claimed. □

Remarks. (1.) Of course, an element τ_0 of $\mathsf{T}_{\xi^0}\mathcal{M}$ with the property required in statement (*ii*) of Proposition 5.1 exists because ξ^0 is a rank-1 singular point of F, so that the null space of the differential at ξ^0 is of dimension $N-1$ in the tangent space of dimension N. For example, at least one of $\mathrm{grad}_{\mathcal{M}}F_1(\xi^0)$ and $\mathrm{grad}_{\mathcal{M}}F_2(\xi^0)$ is nonzero and that nonzero vector will serve as τ_0. When $N = N_0$, τ_0 can be taken as an appropriate one of the standard unit-basis vectors, as in Section 4.

(2.) With obvious modifications, the remarks made following Proposition 4.1 carry over to the present situation. That is, Proposition 5.1 provides the first step in computing the local mapping direction at a sufficiently regular point of the local boundary of the range, since it implies a simple recipe for computing a normal to the curve of singular values passing through such a value $F(\xi^0)$; Theorem 5.1 will provide the second step, *i.e.*, the means for selecting which of the normals is the mapped-side normal. Similarly, the conditions for a horizontal or vertical tangent to a singular-value curve are easy to deduce from Proposition 5.1.

Our extension of Theorem 4.1 to the present setting retains the same form: to test whether a given singular value is a fold point of the range or is locally covered, we identify an intrinsic quadratic form whose definiteness properties provide the tell-tale, except in a borderline case.

For the statement of this theorem, we must prepare some definitions and notations. First, consider a real-valued function $f : \mathcal{M}^N \to \mathbb{R}$ of class C^2. Let $\xi \in \mathcal{M}$. The definition of the (first) differential $\mathrm{d}f(\xi)$, a linear form on the tangent space $\mathsf{T}_\xi\mathcal{M}$, is reviewed in Appendix B. Now we introduce the *second quadratic differential of f at ξ* as a quadratic form $\mathrm{d}^2f(\xi;\,\cdot\,)$ on $\mathsf{T}_\xi\mathcal{M}$. For this, we choose any coördinate system (U,h) for $\mathcal{M}$ with the coördinate patch U containing ξ. We write $h_\zeta \equiv h(\zeta)$ for each $\zeta \in U$, for brevity. The N $\mathbb{R}^{N_0}$-vectors $\{h^{-1}_{\,,1}(h_\zeta), h^{-1}_{\,,2}(h_\zeta), \ldots, h^{-1}_{\,,N}(h_\zeta)\}$ form a basis for $\mathsf{T}_\zeta\mathcal{M}$. The metric, or first fundamental, tensor of the manifold $\mathcal{M}$ for (U,h) has (covariant) components given on $h(U)$ by

$$g_{jk}(z) := h^{-1}_{\,,j}(z)\cdot h^{-1}_{\,,k}(z), \qquad \text{for} \quad z \in h(U), \qquad \text{for} \quad j,\,k = 1,\ldots,N;$$

the inverse of the matrix $\{g_{jk}(z)\}_{N\times N}$ is denoted by $\{g^{jk}(z)\}_{N\times N}$. The Christoffel symbols of the second kind Γ^j_{kl} are defined on $h(U)$ by

$$\Gamma^j_{kl}(z) := \sum_{m=1}^{N} \frac{g^{mj}(z)}{2}\Big\{g_{km\,,\,l}(z) - g_{kl\,,\,m}(z) + g_{ml\,,\,k}(z)\Big\}, \quad \text{for} \quad z \in h(U), \quad \text{for} \quad j,\,k,\,l = 1,\ldots,N;$$

obviously, these vanish when $N = N_0$. Now we can define the quadratic form $\tau \mapsto \mathrm{d}^2f(\xi;\tau)$ on $\mathsf{T}_\xi\mathcal{M}$ by using covariant differentiation according to

$$\mathrm{d}^2f(\xi;\tau) := \sum_{p,q=1}^{N}\Big\{\big(f\circ h^{-1}\big)_{,\,pq}(h_\xi) - \sum_{m=1}^{N}\Gamma^m_{pq}(h_\xi)\big(f\circ h^{-1}\big)_{,\,m}(h_\xi)\Big\}\tau^p\tau^q, \qquad \text{for each} \quad \tau \in \mathsf{T}_\xi\mathcal{M},$$

in which the τ^p are the *contravariant components* of τ, such that $\tau = \sum_{p=1}^N \tau^p h^{-1}_{\,,p}(h_\xi)$; explicitly, these components are given by $\tau^p = \sum_{q=1}^N g^{pq}(h_\xi)\{\tau\cdot h^{-1}_{\,,q}(h_\xi)\}$. Finally, when the mapping $F : \mathcal{M}^N \to \mathbb{R}^n$ is of class C^2, the *second quadratic differential of F at $\xi \in \mathcal{M}$* is defined to be the n-quadratic form $\mathsf{D}^2F(\xi;\,\cdot\,) : \mathsf{T}_\xi\mathcal{M} \to \mathbb{R}^n$ on the tangent space $\mathsf{T}_\xi\mathcal{M}$ with components $\mathrm{d}^2F_l(\xi;\,\cdot\,)$, $l = 1,\ldots,n$. It is clear that these definitions do generalize those given for the case in which $N = N_0$.

Remarks. (1.) One can show that the second quadratic differential defined in this way is an object *intrinsic* to the manifold $\mathcal{M}^N$ and the function f or mapping F, *i.e.*, it is independent of the particular coördinate system for $\mathcal{M}^N$ that is used in its definition.

(2.) It is useful to have an alternate means for computing the second quadratic differential. At least when $N = N_0 - 1$, if $\tilde{f}$ is a C^2-extension of $f : \mathcal{M}^N \to \mathbb{R}$ to a full $\mathbb{R}^{N_0}$-neighborhood of $\mathcal{M}^N$, then one also finds, for $\xi \in \mathcal{M}$,

$$\mathrm{d}^2 f(\xi;\tau) = \sum_{p,q=1}^{N_0} \Big\{ \tilde{f}_{,pq}(\xi) + \kappa_{\mathrm{n}}(\xi;\hat{\tau}) \cdot \operatorname{grad} \tilde{f}(\xi) \delta_{pq} \Big\} \tau_p \tau_q, \qquad \text{for each} \quad \tau \in \mathsf{T}_\xi \mathcal{M},$$

in which the τ_p are the components of τ with respect to the standard unit-basis vectors of $\mathbb{R}^{N_0}$ and $\kappa_{\mathrm{n}}(\xi;\hat{\tau})$ is the normal-curvature vector of $\mathcal{M}$ at ξ for the direction of τ, *i.e.*, the (normal) curvature vector of the geodesic passing through ξ and having tangent vector τ there (and δ_{pq} is the Kronecker symbol). This result is independent of the particular extension $\tilde{f}$ that is chosen for f; in particular, when $\tilde{f}$ is such that its gradient at $\xi \in \mathcal{M}$ lies in the tangent space $\mathsf{T}_\xi\mathcal{M}$ (in which case it coincides there with the manifold gradient $\operatorname{grad}_{\mathcal{M}} f(\xi)$), the second term within the braces vanishes and the result takes on an especially simple form involving only the second partial derivatives $\tilde{f}_{,pq}(\xi)$. A verification of these assertions is somewhat lengthy, although straightforward; since we presently have no real need for the results, we omit their proofs.

Let us introduce the quadratic form playing the central rôle in Theorem 5.1, the counterpart and generalization of that introduced in Section 4. With $\xi^0 \in \mathcal{M}^N$ denoting a rank-1 singular point of the C^2-map $F : \mathcal{M}^N \to \mathbb{R}^2$, let $\{\nu_0^{(j)}\}_{j=1}^{N-1}$ be a basis for the $(N-1)$-dimensional nullspace $\mathcal{N}\big(\mathrm{D}F(\xi^0)\big)$ in the tangent space $\mathsf{T}_{\xi^0}\mathcal{M}$. The contravariant components of $\nu_0^{(j)}$ are denoted by $\nu_0^{(j)p}$, so that $\nu_0^{(j)} = \sum_{p=1}^{N-1} \nu_0^{(j)p} h^{-1}_{;p}(h_{\xi^0})$, in which we again use the abbreviation $h_{\xi^0} \equiv h(\xi^0)$. We choose any element τ_0 of $\mathsf{T}_{\xi^0}\mathcal{M}$ such that $\left|\operatorname{grad}_{\mathcal{M}} F_1(\xi^0) \cdot \tau_0\right|^2 + \left|\operatorname{grad}_{\mathcal{M}} F_2(\xi^0) \cdot \tau_0\right|^2 > 0$; again, such τ_0 exist because ξ^0 is a rank-1 singular point of F. With this notation, the quadratic form $\mathcal{Q}_{\xi^0}$ on $\mathbb{R}^{N-1}$ is defined by setting

$$\begin{aligned}
\mathcal{Q}_{\xi^0}(\zeta) :={}& \Big\{\operatorname{grad}_{\mathcal{M}} F_2(\xi^0) \cdot \tau_0\Big\} \mathrm{d}^2 F_1\Big(\xi^0; \sum_{j=1}^{N-1} \zeta_j \nu_0^{(j)}\Big) - \Big\{\operatorname{grad}_{\mathcal{M}} F_1(\xi^0) \cdot \tau_0\Big\} \mathrm{d}^2 F_2\Big(\xi^0; \sum_{j=1}^{N-1} \zeta_j \nu_0^{(j)}\Big) \\
={}& \sum_{j,k=1}^{N-1} \Bigg\{ \sum_{p,q=1}^{N} \Bigg\{ \Big[\operatorname{grad}_{\mathcal{M}} F_2(\xi^0) \cdot \tau_0\Big] \Big[(F_1 \circ h^{-1})_{,pq}(h_{\xi^0}) - \sum_{m=1}^{N} \Gamma^m_{pq}(h_{\xi^0}) (F_1 \circ h^{-1})_{,m}(h_{\xi^0}) \Big] \\
&- \Big[\operatorname{grad}_{\mathcal{M}} F_1(\xi^0) \cdot \tau_0\Big] \Big[(F_2 \circ h^{-1})_{,pq}(h_{\xi^0}) - \sum_{m=1}^{N} \Gamma^m_{pq}(h_{\xi^0}) (F_2 \circ h^{-1})_{,m}(h_{\xi^0}) \Big] \Bigg\} \nu_0^{(j)p} \nu_0^{(k)q} \Bigg\} \zeta_j \zeta_k, \\
&\qquad\qquad\qquad\qquad \text{for each } \zeta \equiv (\zeta_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}. \qquad (5.1)
\end{aligned}$$

It is clear that the definition reduces to that of Section 4 when $N = N_0$.

Just as in the preceding section, we make a statement only about the "strongly nondegenerate singular points" of our mappings.

Definition. Let $F : \mathcal{M}^N \to \mathbb{R}^n$ be of class C^2. A singular point ξ of F is *strongly nondegenerate* iff $\mathrm{D}^2 F(\xi;\nu)$ is nonzero whenever ν is a nonzero element of $\mathcal{N}\big(\mathrm{D}F(\xi)\big) \subset \mathsf{T}_\xi\mathcal{M}$.

Theorem 5.1. *Let $N_0 \geq N \geq 2$. Let $F : \mathcal{M}^N \to \mathbb{R}^2$ be a mapping of class C^3 on the N-dimensional submanifold-without-boundary $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ of class C^3. Suppose that $\xi^0 \in \mathcal{M}^N$ is a rank-1 singular point of F for which the hypotheses (H.1)–(H.3) hold. Let the quadratic form $\mathcal{Q}_{\xi^0}$ on $\mathbb{R}^{N-1}$ be* *defined as in* (5.1), *with the tangent vector* τ_0, *the nullspace-basis* $\{\nu_0^{(j)}\}_{j=1}^{N-1}$, *and other notation as established in that definition.*

(*i*.) *If the quadratic form* $\mathcal{Q}_{\xi^0}$ *is definite, then the range is folded at the singular value* $F(\xi^0)$ *with respect to* ξ^0.

(*ii*.) *Suppose that* F *and* $\mathcal{M}$ *are of class* C^5. *If the quadratic form* $\mathcal{Q}_{\xi^0}$ *is indefinite, then the singular value* $F(\xi^0)$ *is locally covered by* F *from* ξ^0.

(*iii*.) *If the quadratic form* $\mathcal{Q}_{\xi^0}$ *is semidefinite, then no conclusion can be drawn about the singular value* $F(\xi^0)$, *i.e., it may be either on the local boundary of the range of* F *or locally covered by* F *from* ξ^0.

Remarks. (1.) The conditions (*i*)–(*iii*) have the same geometric significance as in the setting of Theorem 4.1. That is, the respective conditions indicate the situation of the tangent to the curve of singular values at $F(\xi^0)$ in $\mathbb{R}^2$ relative to the translated convex-cone range of the restriction of the second quadratic differential of F to the nullspace of its first differential, obtaining according to whether the translated cone (*i*) meets the tangent line only at its vertex $F(\xi^0)$, (*ii*) contains in its interior (at least) a ray in the tangent line, or (*iii*) contains in its boundary (at least) a ray in the tangent line.

(2.) Again, when $N = 2$ the quadratic form $\mathcal{Q}_{\xi^0}$ is defined just on $\mathbb{R}$, and so can be only definite or trivial-semidefinite.

(3.) Following the statement of Theorem 4.1, we remarked on the identification of the local mapping direction when conclusion (*i*) obtains, so that $F(\xi^0)$ is on the local boundary of the range. With the necessary (and obvious) modifications, those remarks can be extended to the present case.

Proof of Theorem 5.1. The statements in the case $N = N_0$ are proven in Theorem 4.1, so we suppose now that $N_0 > N$. In fact, we reduce the present proof entirely to that previous case by first using a coördinate system to transfer the setting to an open set in $\mathbb{R}^N$, in which we can apply the appropriate form of Theorem 4.1. It transpires that the quadratic form figuring in that application of Theorem 4.1 is just a multiple of the form defined in the statement of the present theorem, and so the two forms have the same definiteness properties. Thus, the proof will be effectively completed by reduction to the previous case.

With this plan in mind, let (U, h) be a coördinate system for $\mathcal{M}$ with ξ^0 contained in the patch U, and consider the mapping $F \circ h^{-1} : h(U) \to \mathbb{R}^2$, of class C^3 on the open set $h(U) \subset \mathbb{R}^N$, which has the same range as the restriction of F to U. It is clear that $t \mapsto h(\chi(t))$ is a C^1-smooth curve of rank-1 singular points of $F \circ h^{-1}$. Moreover, since h carries U homeomorphically onto $h(U)$, it is also easy to see that $F(\xi^0) = F(\chi(t_0))$ is locally covered by F from $\xi^0 = \chi(t_0)$ iff it is locally covered by $F \circ h^{-1}$ from $h_{\xi^0} \equiv h(\xi^0)$ (and so the range of F is folded at $F(\xi^0)$ from ξ^0 iff the range of $F \circ h^{-1}$ is folded at $F(\xi^0)$ from $h(\xi^0)$). Since ξ^0 is a nondegenerate singular point of F such that $F(\xi^0)$ is not a cusp, *i.e.*, such that the tangent vector $\chi'(t_0)$ does not belong to the nullspace $\mathrm{D}F(\xi^0)$, it will follow from computations given below that h_{ξ^0} is a nondegenerate singular point of $F \circ h^{-1}$ for which $F \circ h^{-1}(h_{\xi^0})$ is not a cusp. Consequently, we may apply Theorem 4.1 to formulate a test of the behavior of $F \circ h^{-1} : h(U) \to \mathbb{R}^2$ at its singular point h_{ξ^0} and use the results as a test of the behavior of F at its singular point $\xi^0 = \chi(t_0)$.

To set up this application, we must study the quadratic form given in (4.4) that is appropriate for $F \circ h^{-1}$. To this end, we first generate a basis for the null space $\mathcal{N}\big(\mathsf{D}\big(F \circ h^{-1}\big)(h_{\xi^0})\big) \subset \mathbb{R}^N$ from the given basis $\{\nu_0^{(j)}\}_{j=1}^{N-1}$ for $\mathcal{N}\big(\mathsf{D}F(\xi^0)\big) \subset \mathsf{T}_{\xi^0}\mathcal{M}$. We have $\mathsf{D}F\big(\chi(t_0)\big)\nu_0^{(j)} = 0$, or

$$\sum_{q=1}^{N} \big(F_l \circ h^{-1}\big)_{,q}(h_{\xi^0})\nu_0^{(j)q} = 0, \qquad \text{for} \quad l = 1,\, 2, \qquad \text{for} \quad j = 1, \ldots, N-1,$$

from which we infer that a basis for $\mathcal{N}\big(\mathsf{D}\big(F \circ h^{-1}\big)(h_{\xi^0})\big)$ is given by $\big\{\tilde{\nu}_0^{(j)} := \sum_{q=1}^{N} \nu_0^{(j)q}\mathrm{e}^{(q)}\big\}_{j=1}^{N-1}$; observe that this basis is expressed in terms of the standard unit-basis vectors $\mathrm{e}^{(q)}$ for $\mathbb{R}^N$, as required in (4.4). Now, choosing any integer m_0 in $\{1, \ldots, N\}$ such that $\big[\big(F_1 \circ h^{-1}\big)_{,m_0}(\xi^0)\big]^2 + \big[\big(F_2 \circ h^{-1}\big)_{,m_0}(\xi^0)\big]^2 > 0$, with the elements prepared we construct the version $\widetilde{\mathcal{Q}}_{h_{\xi^0}}$ of the quadratic form on $\mathbb{R}^{N-1}$ given in (4.4) that is appropriate for the application of Theorem 4.1 to $F \circ h^{-1}$ (recall that $h_{\xi^0} \equiv h(\xi^0) = h(\chi(t_0))$):

$$\begin{aligned}
\widetilde{\mathcal{Q}}_{h_{\xi^0}}(\zeta) &:= \big(F_2 \circ h^{-1}\big)_{,m_0}\big(h_{\xi^0}\big)\,\mathrm{d}^2\big(F_1 \circ h^{-1}\big)\left(h_{\xi^0}; \sum_{j=1}^{N-1} \zeta_j \tilde{\nu}_0^{(j)}\right) \\
&\qquad - \big(F_1 \circ h^{-1}\big)_{,m_0}\big(h_{\xi^0}\big)\,\mathrm{d}^2\big(F_2 \circ h^{-1}\big)\left(h_{\xi^0}; \sum_{j=1}^{N-1} \zeta_j \tilde{\nu}_0^{(j)}\right) \\
&= \sum_{j,k=1}^{N-1} \left\{ \sum_{p,q=1}^{N} \Big[\big(F_2 \circ h^{-1}\big)_{,m_0}\big(h_{\xi^0}\big)\big(F_1 \circ h^{-1}\big)_{,pq}\big(h_{\xi^0}\big) \right. \\
&\qquad \left. - \big(F_1 \circ h^{-1}\big)_{,m_0}\big(h_{\xi^0}\big)\big(F_2 \circ h^{-1}\big)_{,pq}\big(h_{\xi^0}\big)\Big]\nu_0^{(j)p}\nu_0^{(k)q} \right\} \zeta_j \zeta_k, \\
&\qquad \text{for each } \zeta \equiv (\zeta_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}.
\end{aligned} \tag{5.2}$$

Now we want to exploit the hypothesis that ξ^0 is a rank-1 singular point of F to show that, in fact, we have $\mathcal{Q}_{\xi^0} = \alpha\widetilde{\mathcal{Q}}_{h_{\xi^0}}$ for some nonzero α. To see this, we observe first that, since the matrix $\Big\{\big(F_l \circ h^{-1}\big)_{,m}(h_{\xi^0})\Big\}_{2\times N}$ has rank 1, each of its 2×2 minors must vanish; in particular, the $N-1$ minors involving the ${m_0}^{\text{th}}$ column vanish, so we have

$$\begin{aligned}
\big(F_1 \circ h^{-1}\big)_{,m}(h_{\xi^0})\big(F_2 \circ h^{-1}\big)_{,m_0}(h_{\xi^0}) - \big(F_2 \circ h^{-1}\big)_{,m}(h_{\xi^0})\big(F_1 \circ h^{-1}\big)_{,m_0}(h_{\xi^0}) = 0, \\
\text{for} \quad m = 1, \ldots, N.
\end{aligned} \tag{5.3}$$

Consequently, by setting

$$\alpha_m := \begin{cases} \dfrac{\big(F_1 \circ h^{-1}\big)_{,m}(h_{\xi^0})}{\big(F_1 \circ h^{-1}\big)_{,m_0}(h_{\xi^0})} & \text{if} \quad \big(F_1 \circ h^{-1}\big)_{,m_0}(h_{\xi^0}) \neq 0, \\[2ex] \dfrac{\big(F_2 \circ h^{-1}\big)_{,m}(h_{\xi^0})}{\big(F_2 \circ h^{-1}\big)_{,m_0}(h_{\xi^0})} & \text{if} \quad \big(F_1 \circ h^{-1}\big)_{,m_0}(h_{\xi^0}) = 0, \end{cases}$$

we can write

$$\left.\begin{aligned} \big(F_1 \circ h^{-1}\big)_{,m}(h_{\xi^0}) &= \alpha_m \big(F_1 \circ h^{-1}\big)_{,m_0}(h_{\xi^0}) \\ \big(F_2 \circ h^{-1}\big)_{,m}(h_{\xi^0}) &= \alpha_m \big(F_2 \circ h^{-1}\big)_{,m_0}(h_{\xi^0}) \end{aligned}\right\}, \qquad \text{for} \quad m = 1, \ldots, N. \tag{5.4}$$

Clearly, we can use (5.3) to rewrite first the expressions appearing within the innermost square brackets on the right in (5.2) as

$$\Big[(F_2\circ h^{-1}),_{m_0}(h_{\xi^0})(F_1\circ h^{-1}),_{pq}(h_{\xi^0}) - (F_1\circ h^{-1}),_{m_0}(h_{\xi^0})(F_2\circ h^{-1}),_{pq}(h_{\xi^0})\Big]$$

$$= \Big[(F_2\circ h^{-1}),_{m_0}(h_{\xi^0})\Big]\Big[(F_1\circ h^{-1}),_{pq}(h_{\xi^0}) - \sum_{m=1}^{N}\Gamma^m_{pq}(h_{\xi^0})(F_1\circ h^{-1}),_m(h_{\xi^0})\Big]$$

$$- \Big[(F_1\circ h^{-1}),_{m_0}(h_{\xi^0})\Big]\Big[(F_2\circ h^{-1}),_{pq}(h_{\xi^0}) - \sum_{m=1}^{N}\Gamma^m_{pq}(h_{\xi^0})(F_2\circ h^{-1}),_m(h_{\xi^0})\Big];$$

finally, we can use (5.4) to show that the latter is, in turn, a nonzero multiple of the expression

$$\Big[\mathrm{grad}_{\mathcal{M}}F_2\big(\chi(t_0)\big)\cdot\tau_0\Big]\Big[(F_1\circ h^{-1}),_{pq}(h_{\xi^0}) - \sum_{m=1}^{N}\Gamma^m_{pq}(h_{\xi^0})(F_1\circ h^{-1}),_m(h_{\xi^0})\Big]$$

$$-\Big[\mathrm{grad}_{\mathcal{M}}F_1\big(\chi(t_0)\big)\cdot\tau_0\Big]\Big[(F_2\circ h^{-1}),_{pq}(h_{\xi^0}) - \sum_{m=1}^{N}\Gamma^m_{pq}(h_{\xi^0})(F_2\circ h^{-1}),_m(h_{\xi^0})\Big],$$

all for p and $q = 1,\ldots,N$. In fact, (5.4) allows us to write

$$\begin{aligned}\mathrm{grad}_{\mathcal{M}}F_l\big(\chi(t_0)\big)\cdot\tau_0 &= \sum_{p=1}^{N}(F_l\circ h^{-1}),_p(h_{\xi^0})\tau_0^p\\ &= \Big\{\sum_{p=1}^{N}\alpha_p\tau_0^p\Big\}(F_l\circ h^{-1}),_{m_0}(h_{\xi^0}), \qquad \text{for} \quad l=1,\,2,\end{aligned}$$

so that, by setting $\alpha := \sum_{p=1}^{N}\alpha_p\tau_0^p$, it is easy to see that we do indeed arrive at the claimed equality $\mathcal{Q}_{\xi^0} = \alpha\widetilde{\mathcal{Q}}_{h_{\xi^0}}$ on $\mathbb{R}^{N-1}$. Moreover, α does not vanish, since at least one of $\mathrm{grad}_{\mathcal{M}}F_1(\xi^0)\cdot\tau_0$ and $\mathrm{grad}_{\mathcal{M}}F_2(\xi^0)\cdot\tau_0$ is nonzero. Thus, $\mathcal{Q}_{\xi^0}$ and $\widetilde{\mathcal{Q}}_{h_{\xi^0}}$ have the same definiteness properties.

Now, completion of the reasoning is a matter of simple checking. If $\mathcal{Q}_{\xi^0}$ is definite, then the same is true of $\widetilde{\mathcal{Q}}_{h_{\xi^0}}$, so that the range of $F\circ h^{-1}$ is folded at $F(\xi^0)$ with respect to h_{ξ^0}, whence the range of F is folded at $F(\xi^0)$ with respect to ξ^0, as well. If $\mathcal{Q}_{\xi^0}$ is indefinite, a similar argument leads to the conclusion that $F(\xi^0)$ is locally covered by F from ξ^0, provided F and $\mathcal{M}$ have the additional smoothness required in statement (ii), so that the corresponding assertion of Theorem 4.1 can be invoked. Finally, $\mathcal{Q}_{\xi^0}$ is semidefinite iff the same is true of $\widetilde{\mathcal{Q}}_{h_{\xi^0}}$; in case of semidefiniteness, it is easy to manufacture examples like those given in Theorem 4.1 to show that either possibility may still occur. $\square$

6. Examples.

All of the examples are gathered in this section.

The examples are intended to reinforce all of the definitions and developments of the previous sections, such as local covering of a value and folding of the range, not just the procedure for computation of Pareto-minimal points. Examination of a variety of mappings not only brings out the breadth of behavior possible even in this most elementary setting with $n = 2$, but also places in perspective the results established here by indicating the cases that remain to be studied, involving, *e.g.*, strongly degenerate singular points and semidefinite quadratic forms (cases not covered in Theorems 4.1 and 5.1). Since it seems best to demonstrate an idea or result within the simplest possible setting, fairly detailed developments are provided for some of the most elementary mappings.

We depict the ranges of the mappings in the various examples by computing and plotting the images of grids of points in the domain, taking sufficiently many points that the patterns are clear and we may extrapolate from the indicated trends with reasonable certainty. All of the plots have been generated with the computer-algebra system Maple. Naturally, a tool such as Maple or Mathematica proves to be an indispensable aid in studying the properties of mappings between euclidean spaces.

The first two examples involve the germs of the only two generic singularities possible for mappings from $\mathbb{R}^2$ into itself taking the origin to itself; *cf.*, *e.g.*, MARTINET [14].

Example 6.1 (The Whitney fold). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\left.\begin{aligned} F_1(\xi) &:= \xi_1 \\ F_2(\xi) &:= \xi_2^2 \end{aligned}\right\}, \qquad \text{for} \quad \xi \equiv (\xi_1, \xi_2) \in \mathbb{R}^2.$$

The mapping action is completely transparent here: the range $\mathcal{R}_F = \{ (x_1, x_2) \mid x_2 \geq 0 \}$, the closed upper half-plane formed by the “folding” (and vertical stretching) of the domain-plane along the ξ_1-axis. The matrix of the differential $\mathrm{D}F(\xi)$ is given for each ξ by $\begin{pmatrix} 1 & 0 \\ 0 & 2\xi_2 \end{pmatrix}$, whence it follows easily that the singular points are all rank-1 and comprise the ξ_1-axis, which is mapped bijectively onto itself by F. Thus, the set of singular values here is the x_1-axis, which is clearly the included boundary of the range (coinciding here with the boundary of the range). Each singular point is completely singular, since no singular value has a regular-point preimage. This mapping sends none of its singular points to the interior of its range, and so generates no spurious boundary in its range. It should be noted that the singular points form a 1-manifold, while the null space of the differential at each singular point is just the span of the basis-vector $\varepsilon^{(2)} = (0, 1)$, which is orthogonal to the tangent space; this is in accord with the observation that the image $F(\Sigma F)$ of the singular points is smooth, *i.e.*, exhibits no cusps. The mapping is so simple that an illustrative plot is unnecessary.

Example 6.2 (The Whitney cusp). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\left.\begin{aligned} F_1(\xi) &:= \xi_1 \\ F_2(\xi) &:= \xi_1 \xi_2 + \xi_2^3 \end{aligned}\right\}, \qquad \text{for} \quad \xi \equiv (\xi_1, \xi_2) \in \mathbb{R}^2.$$

After a bit of study, the mapping action here becomes almost as evident as that of the Whitney fold. We observe first that the range is now all of $\mathbb{R}^2$. In fact, for each $\xi_1 \in \mathbb{R}$, the range of the $\mathbb{R}$-valued function $\xi_2 \mapsto f_{\xi_1}(\xi_2) := \xi_2^3 + \xi_1 \xi_2$ is all of $\mathbb{R}$, whence the assertion follows immediately, since it is now clear that any given full vertical line has a full vertical-line preimage in the domain.

In particular, the (global) boundary of the range is void. Examination of the same "vertical-line" functions also reveals the finer structure of the range. Indeed, if $\xi_1 \geq 0$, then f_{ξ_1} takes $\mathbb{R}$ bijectively onto itself. At $\xi_1 = 0$ a transition occurs, so that for $\xi_1 < 0$, f_{ξ_1} has one local maximum and one local minimum, at $\mp\sqrt{-\xi_1/3}$, respectively, and the points (ξ_1, x_2) in the range with $|x_2| < 2(-\xi_1/3)^{\frac{3}{2}}$ have three preimages, while those with $x_2 = \pm 2(-\xi_1/3)^{\frac{3}{2}}$ have two and all others have one. The formation of geometric "folds" in the range leading out from the origin is practically visible here algebraically. Graphically, all of this is reflected in Figure 1, which depicts the images of domain-points lying in a certain square uniform grid. In particular, it is apparent that a cusp is formed at the origin, where the folds meet. (The blurring visible along the positive x_1-axis near the origin does not indicate that points there have more than a single preimage, but arises from the finite size of the plotted "point-symbols.")

Now, let us compare the previous picture for agreement with that obtained by studying the singular points and values of F. The matrix of the differential $\mathrm{D}F(\xi)$ is given for each ξ by $\begin{pmatrix} 1 & 0 \\ \xi_2 & \xi_1 + 3\xi_2^2 \end{pmatrix}$. It follows that the singular points are all rank-1 and determined by the condition $\xi_1 + 3\xi_2^2 = 0$, so that they form the parabola $\{\, (-3t^2, t) \mid t \in \mathbb{R} \,\}$, which is shown in red and magenta in Figure 2. The set of singular values, the image of this parabola under F, is then described parametrically by $\{\, (-3t^2, -2t^3) \mid t \in \mathbb{R} \,\}$; this curve appears in Figure 3 superimposed on the plotted range values, where it evidently falls on the "folds" in the range formed by the mapping action already described (the colors in Figure 3 correspond to the respective colors of the preimages in Figure 2).

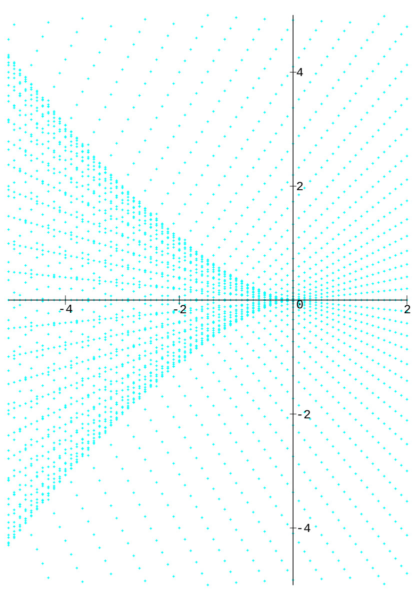

Fig. 1. The range of the Whitney cusp.

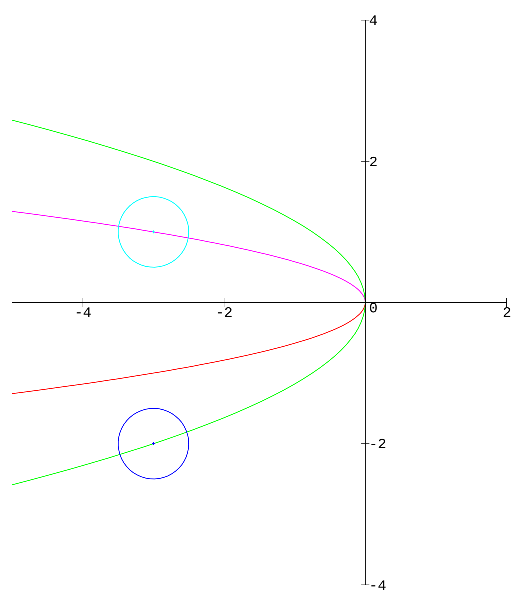

Fig. 2. Singular points and preimages of singular values of the Whitney cusp; discs to be mapped.

None of the singular points except $(0,0)$ is completely singular. Indeed, we find that each of the singular values except the cusp at the origin has a regular preimage, as well as its singular-point preimage; it is easy to check that these regular preimages are described parametrically by $\{\, (-3t^2, -2t) \,|\, t \in \mathbb{R} \,\}$ (in which the parameter has the same meaning as in the previous two parametric descriptions), and so form the green parabola shown in Figure 2. Consequently, each singular value except $(0,0)$ is locally covered from a regular-point preimage and must belong to the interior of the range, as we have already decided directly. In Figure 2 appear two circles of radius $1/2$, one (in cyan) centered on the parabola of singular points at $(-3,1)$ and one (in blue) centered on the parabola of regular preimages of singular values at $(-3,-2)$; both centers correspond to $t=1$ in the parametric descriptions and are mapped by F to the singular value $(-3,-2)$. The respective images of the interiors of these circles under F are shown in Figure 4 and Figure 5, as collections of mapped points in the same respective colors. In Figure 4, the image of the interior of the cyan circle has apparently been "folded" along the curve of singular values, with $(-3,-2)$ placed on the boundary of the image, while in Figure 5 one observes the apparent full image-neighborhood providing a local covering of $(-3,-2)$ in the range. These figures can be viewed as paradigms for "folding" and "local covering," respectively. The origin of the *three* preimages for range-points lying "inside the cusp" is explained again here: two of the preimages come from neighborhoods of the singular points and the folding action of the mapping, while one preimage comes from the simple local covering associated with any regular point. Similarly, one can regard the singular values on the fold as the borderline cases possessing but two preimages, range-points "outside the cusp" being images of single regular points.

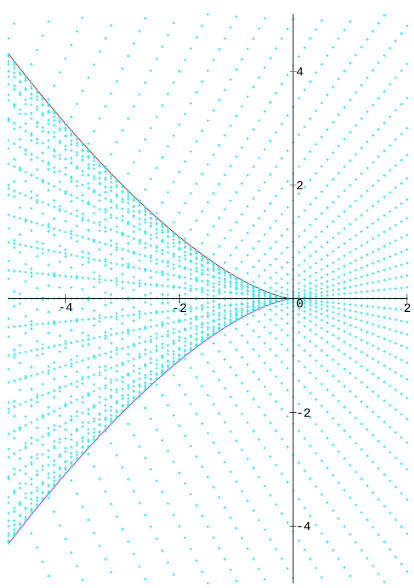

Fig. 3. Singular values of the Whitney cusp.

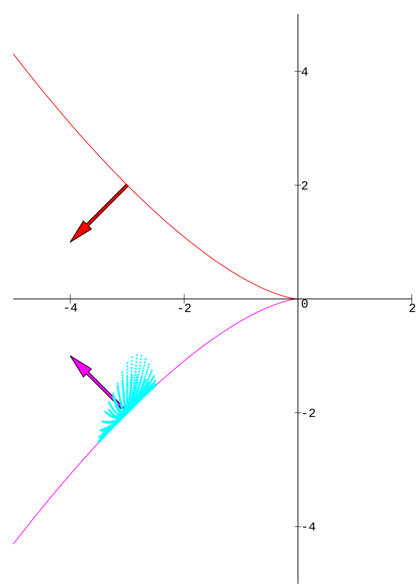

Fig. 4. Image of disk about singular point and mapped-side normals.

Let us show that Theorem 4.1.i can be applied to verify that each singular value except $(0,0)$ does belong to $\partial^{\star}_{\mathrm{loc}}\mathcal{R}_F$, *i.e.*, that the range is folded at each such singular value from its singular-point preimage in the analytical sense, as well as in the geometric sense. We adhere to the notation of Theorem 4.1. The curve of singular points is given here by $t \mapsto \chi(t) = (-3t^2, t)$. When ξ is a singular point $\mathrm{D}F(\xi)$, becomes $\begin{pmatrix} 1 & 0 \\ \xi_2 & 0 \end{pmatrix}$, whence it is evident that we always have $\mathcal{N}\big(\mathrm{D}F(\xi)\big) = \mathrm{sp}\,\{\mathrm{e}^{(2)} := (0,1)\}$. Of course, the tangent space $\mathrm{sp}\,\{\chi'(0)\}$ to the parabola of singular points at the origin is precisely $\mathcal{N}\big(\mathrm{D}F(0)\big)$, so that we do indeed have a cusp in the singular values corresponding to the parameter-value $t_0 = 0$, in the sense of the definition of Section 2; that value is therefore not covered under Theorem 4.1. Then, for t_0 nonzero, we must take $m_0 = 1$; we find $F_{1\,,\,1}\big(\chi(t_0)\big) = 1$ and $F_{2\,,\,1}\big(\chi(t_0)\big) = t_0$. Since the Hessian matrices are just $\Big(F_{1\,,\,lm}(\xi)\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\Big(F_{2\,,\,lm}(\xi)\Big) = \begin{pmatrix} 0 & 1 \\ 1 & 6\xi_2 \end{pmatrix}$, a short computation shows that $\mathrm{d}^2F_1\Big(\chi(t_0);\zeta\mathrm{e}^{(2)}\Big) = 0$ and $\mathrm{d}^2F_2\Big(\chi(t_0);\zeta\mathrm{e}^{(2)}\Big) = 6t_0\zeta^2$ for every $\zeta \in \mathbb{R}$. This implies that $(0,0)$ is the only strongly degenerate singular point and gives, with $\xi^0 := \chi(t_0)$,

$$\begin{aligned}\mathcal{Q}_{\xi^0}(\zeta) &:= F_{2\,,\,1}\big(\chi(t_0)\big)\,\mathrm{d}^2F_1\Big(\chi(t_0);\zeta\mathrm{e}^{(2)}\Big) - F_{1\,,\,1}\big(\chi(t_0)\big)\,\mathrm{d}^2F_2\Big(\chi(t_0);\zeta\mathrm{e}^{(2)}\Big)\\ &= \big(t_0\big)\big(0\big) - \big(1\big)\big(6t_0\zeta^2\big) = -6t_0\zeta^2.\end{aligned}$$

We conclude that $\mathcal{Q}_{\xi^0}$ is definite for every nonzero t_0, more precisely, positive-definite for $t_0 < 0$ and negative-definite for $t_0 > 0$. Therefore, Theorem 4.1.i says that the range is folded at every singular value $F\big(\chi(t_0)\big)$ with respect to $\chi(t_0)$, for $t_0 \neq 0$. To sum up, we can assert that all of the nonzero singular values, shown in Figure 3, do indeed belong to the local included boundary of the range, but all are spurious boundary-points, since each is locally covered from its regular-point preimage and so lies in the interior ${\mathcal{R}_F}^{\circ}$. In particular, we conclude by this alternate route that the global included boundary of the range must be empty (in agreement with the direct algebraic reasoning, which says that the global boundary itself is empty).

Finally, we shall compute a mapped-side normal at each point of the local included boundary of the range (except the origin) and check—for agreement with the previous reasoning—that such a normal always points "into the cusp." Applying the rule based on the definiteness of $\mathcal{Q}_{\xi^0}$ that is derived in Section 4, we find that a mapped-side normal at the singular value $F\big(\chi(t_0)\big)$ corresponding to the nonzero parameter value t_0 is

$$-\mathrm{sgn}\big(t_0\big)\Big\{F_{2\,,\,1}\big(\chi(t_0)\big)\varepsilon^{(1)} - F_{1\,,\,1}\big(\chi(t_0)\big)\varepsilon^{(2)}\Big\} = -\mathrm{sgn}\big(t_0\big)\Big\{t_0\varepsilon^{(1)} - \varepsilon^{(2)}\Big\}.$$

It is easy to see that these do in fact point in the directions anticipated, since the top [bottom] branch of the singular-value curve corresponds to $t_0 < 0$ [to $t_0 > 0$]. Two such mapped-side normals are shown in Figure 4.

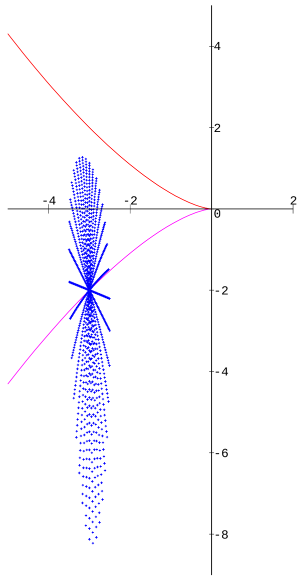

Fig. 5. Image of disk about regular point.

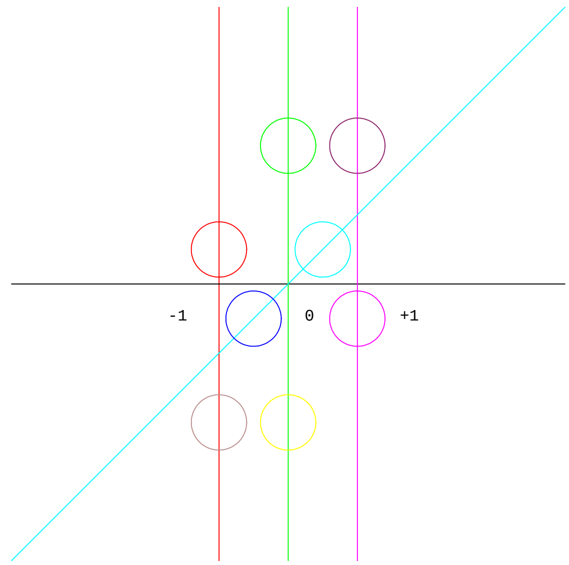

Fig. 6. Singular points of Example 3; discs to be mapped.

The computations here demonstrate the sequence of operations to be followed in the more difficult cases, as well.

The next example illustrates two "pathological" ways in which points of the local included boundary can be contained in the interior of the range.

Example 6.3. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\left.\begin{aligned} F_1(\xi) &:= \left(\xi_1^2 - 1\right)^3 \\ F_2(\xi) &:= \xi_1\left(\xi_2 - \xi_1\right)^2 \end{aligned}\right\}, \qquad \text{for} \quad \xi \equiv (\xi_1, \xi_2) \in \mathbb{R}^2 .$$

The range is the union of the point $(-1, 0)$ and the open half-plane lying to the right of the line described by $x_1 = -1$: $\mathcal{R}_F = \left\{ (x_1, x_2) \,\middle|\, x_1 > -1 \right\} \bigcup \left\{(-1, 0)\right\}$. In fact, the entire ξ_2-axis is mapped to the point $(-1, 0)$, while $F_1(\xi) > -1$ whenever $\xi_1 \neq 0$. Moreover, given any $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 > -1$, there are two nonzero ξ_1, of opposite sign, satisfying $\left(F_1(\xi_1, \xi_2) =\right) \left(\xi_1^2 - 1\right)^3 = x_1$; if one of these, call it $\xi_1^\star$, is chosen with $\xi_1^\star x_2 \geq 0$, then there is at least one ξ_2 such that $F_2(\xi_1^\star, \xi_2) = x_2$.

For each ξ, the matrix of the differential $\mathrm{D}F(\xi)$ is $\begin{pmatrix} 6\xi_1(\xi_1^2 - 1)^2 & 0 \\ (\xi_2 - 3\xi_1)(\xi_2 - \xi_1) & 2\xi_1(\xi_2 - \xi_1) \end{pmatrix}$. The singular points of F are therefore the solutions of $\xi_1^2\left(\xi_1^2 - 1\right)^2\left(\xi_2 - \xi_1\right) = 0$ and so comprise the four lines described by $\xi_1 = 0$, $\xi_1 = \pm 1$, and $\xi_2 = \xi_1$. The three points $(0, 0)$, $(-1, -1)$, and $(1, 1)$ are clearly rank-0 singularities, while all others are rank-1. The singular values are then produced as follows: the line described by $\xi_1 = 0$ is mapped to the one value $(-1, 0)$; the line given by $\xi_1 = -1$ [by $\xi_1 = +1$] is "folded over" at the point $(-1, -1)$ [at the point $(1, 1)$] and both halves are mapped onto the ray $\left\{ (x_1, x_2) \,\middle|\, x_1 = 0,\ x_2 \leq 0\ [x_2 \geq 0] \right\}$; the line given by $\xi_2 = \xi_1$ is "folded over" at the point $(0, 0)$ and both halves are mapped onto the ray $\left\{ (x_1, x_2) \,\middle|\, x_1 \geq -1,\ x_2 = 0 \right\}$. The lines of singular points are shown in Figure 6 along with eight circles of radius 0.4 centered at various points

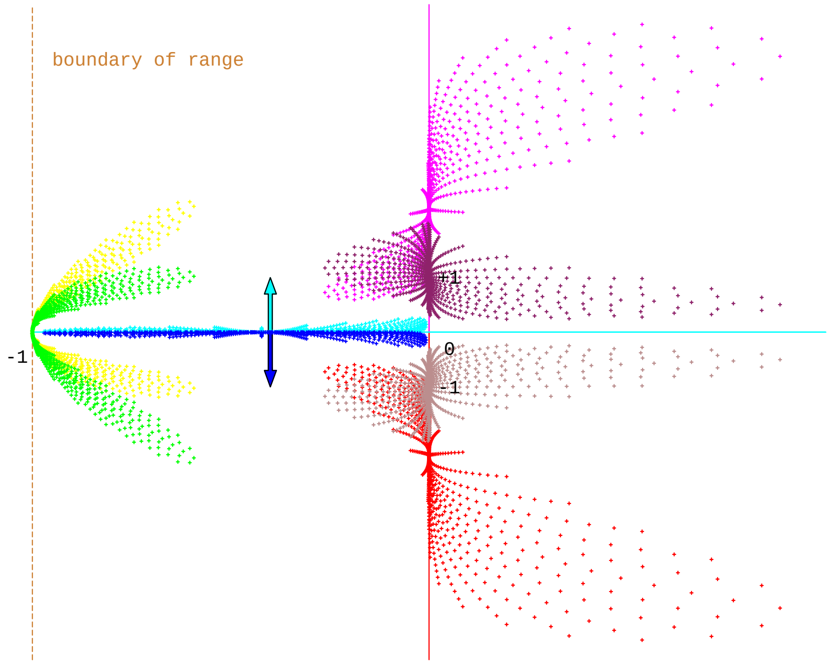

Fig. 7. Singular values of Example 3 and images of disks.

on those lines; the corresponding rays of singular values appear in Figure 7, along with "discrete images" of the discs interior to the circles (the domain-points and image-values are color-matched).

We consider in turn each of the lines of singular points, summarizing its properties and the relation of the corresponding set of singular values to the range. We record first the Hessian matrices of the components of F:

$$\Big(F_{1\,,\,lm}(\xi)\Big) = 6\begin{pmatrix} \left(\xi_1^2-1\right)\left(5\xi_1^2-1\right) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Big(F_{2\,,\,lm}(\xi)\Big) = 2\begin{pmatrix} 3\xi_1 - 2\xi_2 & -2\xi_1+\xi_2 \\ -2\xi_1+\xi_2 & \xi_1 \end{pmatrix}.$$

► $\{\,(\xi_1,\xi_2)\,|\,\xi_1 = 0\,\}$: The matrix of the differential $\mathrm{D}F(0,\xi_2)$ is just $\begin{pmatrix} 0 & 0 \\ \xi_2^2 & 0 \end{pmatrix}$, so the nullspace of $\mathrm{D}F(0,\xi_2)$ is $\mathrm{sp}\,\{\mathrm{e}^{(2)}\}$ except for $\xi_1 = 0$, which gives the rank-0 point $(0,0)$. The Hessian matrices displayed above reduce, respectively, to $6\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $2\begin{pmatrix} -2\xi_2 & \xi_2 \\ \xi_2 & 0 \end{pmatrix}$. With these expressions, one quickly finds that $\mathrm{D}^2F\Big((0,\xi_2);\zeta\mathrm{e}^{(2)}\Big) = 0$ for every real ξ_2 and ζ; thus, every singular point $(0,\xi_2)$ is strongly degenerate, and Theorem 4.1 has nothing to say about them. It is not so surprising to discover this degeneracy when one recalls that all of the singular points here are mapped to $(-1,0)$.

► $\{\,(\xi_1,\xi_2)\,|\,\xi_1 = \pm1\,\}$: The matrix of the differential $\mathrm{D}F(\pm1,\xi_2)$ is $(\xi_2\mp1)\begin{pmatrix} 0 & 0 \\ \xi_2\mp3 & -2 \end{pmatrix}$, showing that the nullspace of $\mathrm{D}F(\pm1,\xi_2)$ is $\mathrm{sp}\,\{\big(1,\mp(\xi_2\mp3)/2\big)\}$ except for the case $\xi_2 = \pm1$, which gives the rank-0 point $(\pm1,\pm1)$. The Hessian matrices now appear as, respectively, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and $2\begin{pmatrix} -4\xi_2 \pm 6 & 2\xi_2 \mp 4 \\ 2\xi_2 \mp 4 & \pm 2 \end{pmatrix}$, from which we find that $\mathrm{d}^2F_1\Big((\pm 1, \xi_2); \mp\zeta(\xi_2 \mp 1)/2\Big) = 0$ and $\mathrm{d}^2F_2\Big((\pm 1, \xi_2); \mp\zeta(\xi_2 \mp 1)/2\Big) = \mp(3/2)\zeta^2\big(\xi_2 \mp 1\big)^2$ for every real ζ and $\xi_2 \neq \pm 1$. This shows first that every such singular point $(\pm 1, \xi_2)$ is strongly nondegenerate, so now we should determine which of the cases of Theorem 4.1 obtains. The curves of singular points are given simply by $t \mapsto \chi^{\pm}(t) := \big(\pm 1, t\big)$. With any $t_0 \neq \pm 1$, we find that we may take $m_0 = 2$, since $F_{1,2}\big(\chi^{\pm}(t_0)\big) = 0$, $F_{2,2}\big(\chi^{\pm}(t_0)\big) = \pm 2(t_0 \mp 1)$. By combining these expressions, we come to $\mathcal{Q}^{\pm}_{\xi^0}(\zeta) = 0$ for every real ζ, with $\xi^0 := \chi(t_0)$. Therefore, the quadratic forms are semidefinite-trivial, and we are in case (*iii*) of Theorem 4.1. The graphical evidence in Figure 7 seems to indicate that the singular values corresponding to rank-1 singular points in the present case are, however, locally covered from their singular-point preimages. This is verified (analytically) in Example 4.1.2 for the particular singular point $(1, 2)$ and the corresponding singular value $(0, 1)$.

► $\{\, (\xi_1, \xi_2) \,|\, \xi_2 = \xi_1 \,\}$: All three of the rank-0 singular points $(-1, -1)$, $(0, 0)$, and $(1, 1)$ belong to this collection; we eliminate these from consideration, without further explicit reminders. The matrix of the differential $\mathrm{D}F(\xi_1, \xi_1)$ appears as $6\xi_1(\xi_1^2 - 1)^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so we find the nullspace of $\mathrm{D}F(\xi_1, \xi_1)$ to be $\mathrm{sp}\,\{\mathrm{e}^{(2)}\}$. The Hessian matrices have the respective forms $6\xi_1(\xi_1^2 - 1)^2(5\xi_1^2 - 1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $2\xi_1\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Now we get $\mathrm{d}^2F_1\Big((\xi_1, \xi_1); \zeta\mathrm{e}^{(2)}\Big) = 0$ and $\mathrm{d}^2F_2\Big((\xi_1, \xi_1); \zeta\mathrm{e}^{(2)}\Big) = 2\xi_1\zeta^2$ for every real ζ; in particular, every pertinent singular point is strongly nondegenerate. With $\chi(t) := (t, t)$ here, we choose for t_0 any value except -1, 0, and 1 and set up the corresponding quadratic form $\mathcal{Q}_{\xi^0}$. We find that we must now take $m_0 = 1$, and compute $F_{1,1}\big(\chi(t_0)\big) = 6t_0\big(t_0^2 - 1\big)^2$ and $F_{2,1}\big(\chi(t_0)\big) = 0$. An easy manipulation then leads to $\mathcal{Q}_{\xi^0}(\zeta) = -12t_0^2\big(t_0^2 - 1\big)^2\zeta^2$ for every real ζ and all t_0 except the three excluded; we conclude that $\mathcal{Q}_{\xi^0}$ is always negative-definite, so that the range is folded at all of the corresponding singular values with respect to their singular-point preimages, by Theorem 4.1.*i*. Finally, we compute mapped-side normals to the line of singular values, for each t_0: because of the persistent negative-definiteness, we may take such a normal in every case to be $-\big\{F_{2,1}\big(\chi(t_0)\big)\varepsilon^{(1)} - F_{1,1}\big(\chi(t_0)\big)\varepsilon^{(2)}\big\} = 6t_0\big(t_0^2 - 1\big)^2\varepsilon^{(2)}$, *i.e.*, the mapped-side normal points *down* when $t_0 < 0$ (corresponding to the color blue in Figure 6 and Figure 7) but points *up* when $t_0 > 0$ (indicated by the color cyan in the figures).

These conclusions agree with Figure 7, in which we observe the folds in the range falling atop one another along the horizontal axis for $x_1 > -1$, but having opposite mapped-side normals, typical examples of which are shown in the figure; the result is that all of the singular values in this case belong to the interior of the range, but none of them is locally covered from any preimage. This demonstrates the second, "pathological," means by which a singular value may be in the interior of the range, cited in the discussion of Section 3.

One should contrast a feature of the next example with the previous one: again, we find a manifold of singular values falling on top of another, but now the local mapping directions coincide, producing a part of the *genuine* boundary instead of a part of the *spurious* boundary:

Example 6.4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\left.\begin{aligned} F_1(\xi) &:= \big(\xi_1 - 3\big)^2 - \big(\xi_2 - 4\big)^2 \\ F_2(\xi) &:= \big(\xi_1 - 3\big)^2 + \big(\xi_2 - 1\big)^2 \end{aligned}\right\}, \qquad \text{for} \quad \xi \equiv (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Figure 8 gives some indication of a piece of the range of F. The singular points of this mapping are found in Example 1.1 (where the matrix of $\mathrm{D}F(\xi)$ is displayed) to comprise the two lines $\{\, \xi \in \mathbb{R}^2 \mid \xi_1 = 3 \,\}$ and $\{\, \xi \in \mathbb{R}^2 \mid \xi_2 = 5/2 \,\}$, which appear in Figure 9 along with some other curves that will be explained. We recall also that all of the singular points are rank-1. The singular values of F, *i.e.*, the images of the lines of singular points, are shown in Figures 10, 11, and 12, which are color-coördinated with Figure 9. Thus, the line $\{\, \xi \in \mathbb{R}^2 \mid \xi_1 = 3 \,\}$ of singular points, in red and magenta, is mapped to the red and magenta parabola of singular values, while the blue and maroon line $\{\, \xi \in \mathbb{R}^2 \mid \xi_2 = 5/2 \,\}$ of singular values is mapped to the blue line of singular values, the line being folded in half under the map, with one half-image falling atop the other. The point $(3, 5/2)$, at which the singular-point lines intersect, is taken over to the point $(-9/4, 9/4)$ at which all of the singular-value images meet. Since the matrix of $\mathrm{D}F(3, 5/2)$ is just $2\begin{pmatrix} 0 & 3/2 \\ 0 & 3/2 \end{pmatrix}$, it is clear that the cusp condition, *viz.*, that the tangent space to the singular-point curve be contained in the nullspace of the differential, is fulfilled at the point $(3, 5/2)$ relative to the singular-point line given by $\xi_2 = 5/2$, but *not* relative to the singular-point line given by $\xi_1 = 3$; this explains the appearance of the singular-value curves at and near $(-9/4, 9/4)$, where the (smooth) parabola touches the cusp of the folded line. Figure 10 shows how the singular values fit with the piece of the range appearing in Figure 8; Figure 11 gives a view of the singular values alone, magnified relative to the preceding one.

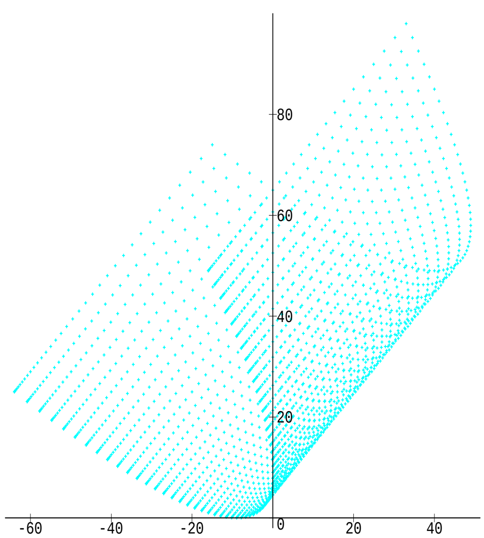

Fig. 8. Range of F in Example 4.

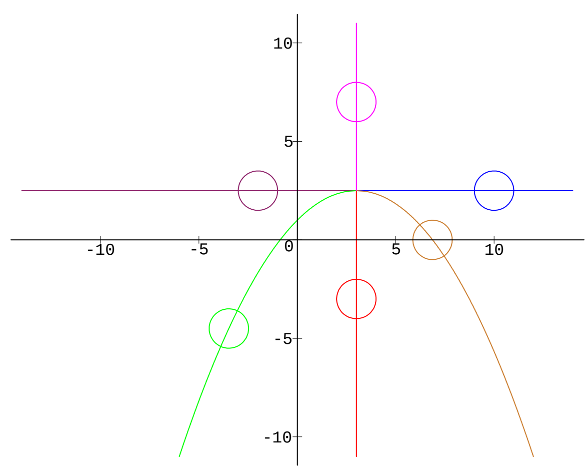

Fig. 9. Singular points and preimages of singular values, Example 4, showing discs to be mapped.

A search yields regular-point preimages only for each of the singular values lying on the magenta ("upper") half-parabola; in fact, each of these values has two regular-point preimages, one on the green half-parabola shown in Figure 9 and another on the gold half-parabola. In particular, each

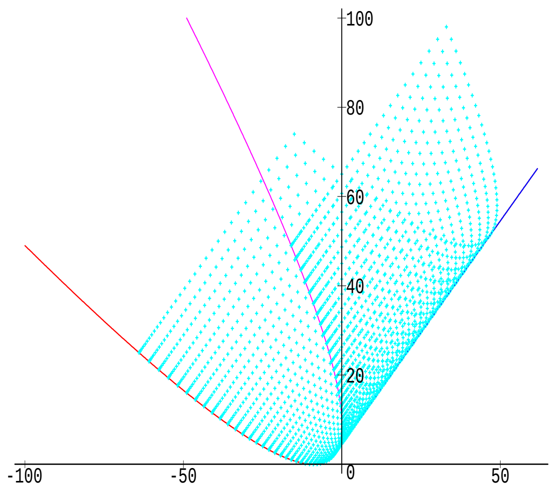

Fig. 10. Singular values and the range of F in Example 4.

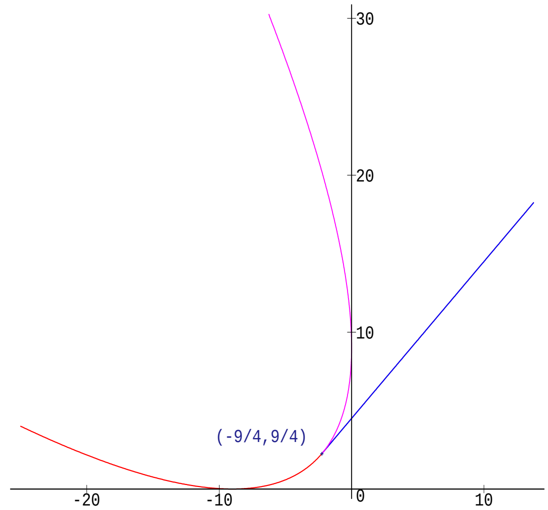

Fig. 11. Magnified view of the singular values of Example 4.

point on the upper half-parabola of singular values must then lie in the interior of the range (as already indicated graphically).

One can apply the test of Theorem 4.1.*i* to find that *all* of the singular values—including those on the magenta half-parabola just noted as possessing regular preimages—are on the local boundary of the range. That is, one discovers that the appropriate quadratic form is definite at each singular point. This is to be anticipated on the basis of the graphical evidence provided in Figure 12, which shows color-matched discrete images of the interiors of the circles appearing in Figure 9, which are centered at typical points on the singular- and regular-point preimage-curves. Specifically, in Figure 12 we see the "full-neighborhood" images of the interiors of the green and gold circles, while the image of each of the discs centered on a singular-point line is folded along the respective singular-value curve; at each of the latter four images is shown a normal vector pointing in the local mapping direction there, computed according to the rule described. Since the maroon and blue arrows point in the same direction, the blue line of singular values falls not in the interior of the range, but on the boundary. As indicated, this should be compared with the behavior exhibited in the previous example.

Apropos of the procedure described in Section 3 under step (A.I.0), observe that we could have computed the mapped-side normals shown in Figure 12 prior to searching for regular preimages for the singular values. On the basis of that calculation, it is clear that we could have eliminated all of the maroon and blue singular points as prospective Pareto minima along with the red and magenta singular points that are preimages of the piece of the singular-value parabola lying between the points of tangency with the axes (most easily seen in Figure 11); we would have avoided in this way the global search for preimages conducted for the corresponding singular values. While this is not so important for the present example, such a finding would constitute a significant saving in a more complex situation.

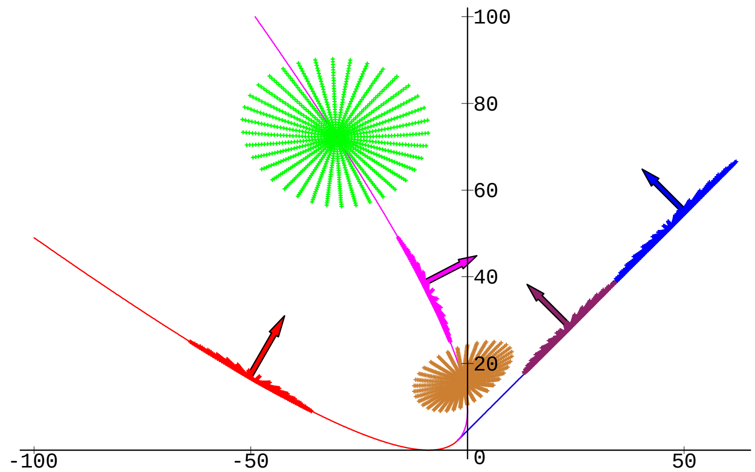

Fig. 12. Images of the discs of Fig. 9 under the map F of Example 4.

Finally, we can conclude that the ray $\{\,(3,\xi_2)\,|\,-\infty<\xi_2\le 1\,\}$ forms the set of Pareto minima for this F of Example 4, since the red lower-half parabola of singular values is tangent to the 1-axis at the image of the singular point $(3,1)$.

Remark. Upon inspecting Figure 8, in which a discrete representation of the range has been generated by mapping points on a grid with lines parallel to the coördinate axes in $\mathbb{R}^2$, one is led to conjecture for the case $N=n=2$ that the included boundary of the range lies in the *envelope* formed by the images of an appropriate one-parameter family of curves in the domain. A statement of this sort can be formulated and proven without too much difficulty. We shall report on this in a note elsewhere.

When N is increased from 2 to 3, so that the domain-space becomes $\mathbb{R}^3$, the image of the (two-dimensional) null space of the gradient at a rank-1 singularity under the biquadratic map given by the second quadratic differential is a convex cone with generally nonempty interior. In this case, the full statement of Theorem 4.1 comes into play. The next example demonstrates this and also features singular values that do not lie on the local included boundary of the range, *i.e.*, singular values that are locally covered from their (only) singular-point preimages.

Example 6.5. Let $F:\mathbb{R}^3\to\mathbb{R}^2$ be defined according to

$$\left.\begin{aligned} F_1(\xi) &:= (\xi_1-1)^2+(\xi_2-4)^2+\xi_3^2 \\ F_2(\xi) &:= (\xi_1-3)^2+(\xi_2-1)^2-(\xi_3-2)^2 \end{aligned}\right\}, \qquad \text{for}\quad \xi\equiv(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3.$$

The matrix of $\mathrm{D}F(\xi)$ is easily found to be $2\begin{pmatrix}\xi_1-1 & \xi_2-4 & \xi_3\\ \xi_1-3 & \xi_1-1 & -\xi_3+2\end{pmatrix}$. From this it follows that all singular points of F are rank-1 and comprise the hyperbola described parametrically (using the 3-coördinate) by $\left\{\,\left(\frac{2z-1}{z-1},\frac{5z-8}{2(z-1)},z\right)\,\middle|\,z\ne 1\,\right\}$. A view of this hyperbola appears in Figure 13, with segments in red, magenta, blue, and cyan, for reference; the color breaks are chosen—with hindsight—to correspond to the parameter-values $z_p:=0$ (red/magenta) and $z_c:=\frac{1}{2}(26)^{1/3}+1$ (blue/cyan), since the corresponding singular values turn out to be geometrically significant. Also

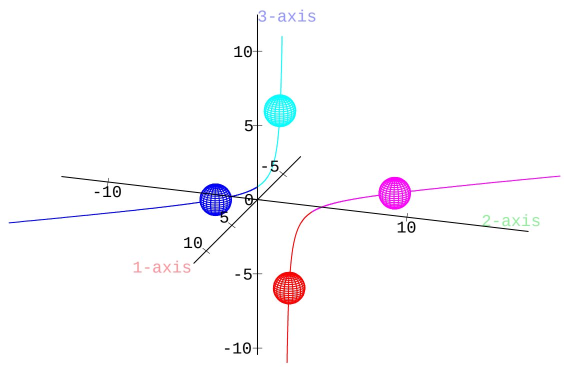

Fig. 13. Singular points of F in Example 5, and balls to be mapped

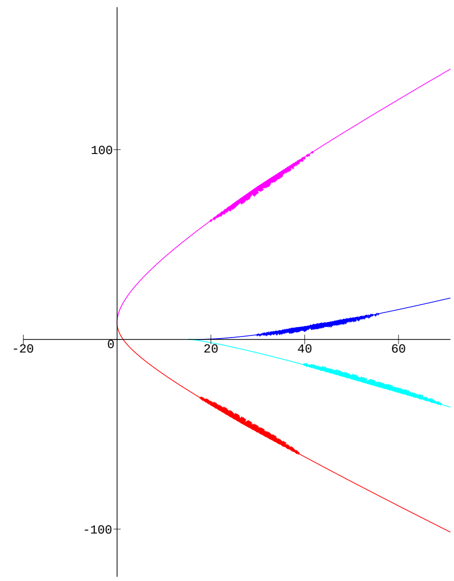

Fig. 14. Singular values of F and images of balls in Figure 13, Example 5.

shown in Figure 13 are four balls of unit radius, one per segment, in the same respective colors, and centered at the singular points corresponding to the parameter-values $z_1 := -6$ (red), $z_2 := \frac{3}{4}$ (magenta), $z_3 := \frac{7}{5}$ (blue), and $z_4 := 7$ (cyan). The images of the singular points, *i.e.*, the singular values, and the images of the balls in $\mathbb{R}^2$ are shown in Figure 14. The extents of the blue and cyan images relative to the corresponding curve-segments of singular values are not very clear in Figure 14, so a magnified view of the images of balls with the same centers but doubled radius 2 is provided in Figure 15. From these representations of the ball-images, one suspects that all of the singular values are fold points of the range, *i.e.*, belong to the local boundary of the range, except those of the blue curve-segment, which appear to be locally covered from their singular-point preimages.

A magnified view of the behavior of the blue and cyan singular values near the 1-axis in the plane reveals the presence of a cusp. It can be checked that the cusp occurs at the image of the singular point corresponding to the parameter-value z_c, or approximately at $(15.276031, 0.111457)$, by finding the singular point at which the tangent to the singular-point curve belongs to the nullspace of the differential. Evidently, the cusp forms the transition-point between the locally covered and the local-boundary singular points on the blue/cyan curve.

Let us apply the local quadratic-form tests of Theorem 4.1 to each of the four (typical) singular points already selected. For example, for $z_2 := \frac{3}{4}$ (magenta) the singular point is $\xi^1 := \left(-2, \frac{17}{2}, \frac{3}{4}\right)$ and the corresponding singular value is found as $\left(\frac{477}{16}, \frac{1275}{16}\right)$. The matrix of the differential $\mathrm{D}F(\xi^1)$ is computed as $\begin{pmatrix} -6 & 9 & 3/2 \\ -10 & 15 & 5/2 \end{pmatrix}$, so a basis for the nullspace $\mathcal{N}\big(\mathrm{D}F(\xi^1)\big)$ can be taken as $\big\{(-3/2, 0, -6), (0, -3/2, 9)\big\}$. The Hessian matrices for F_1 and F_2 are constant here, at $2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, respectively. By combining these results, we find the matrix of the quadratic form $\mathcal{Q}_{\xi^1}$ to be $\begin{pmatrix} 585/2 & -432 \\ -432 & 1305/2 \end{pmatrix}$, from which the eigenvalues of $\mathcal{Q}_{\xi^1}$ are found

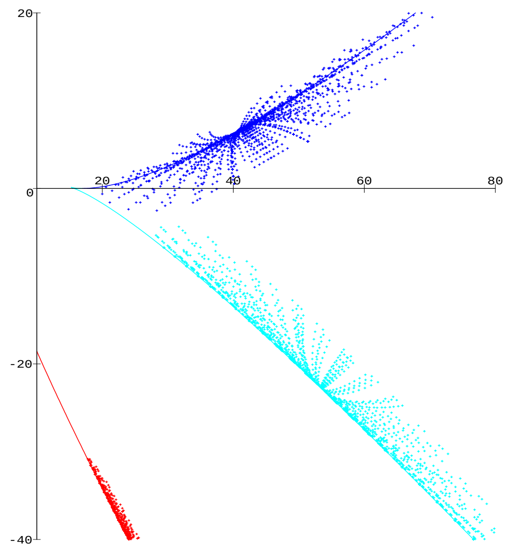

Fig. 15. Magnified view of images of balls as in Figure 13 (of twice the radius).

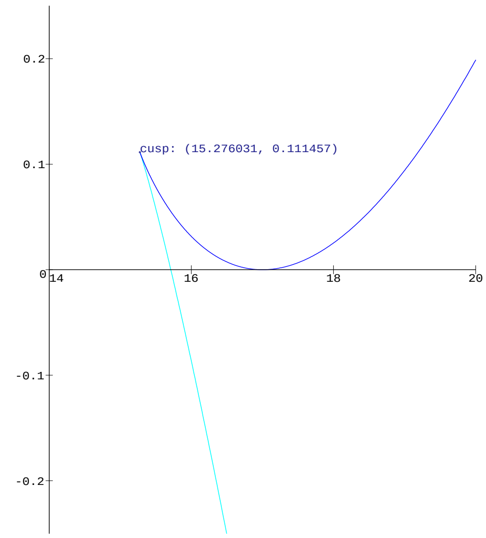

Fig. 16. Magnified view of singular values of F near cusp, Example 5.

as $\left\{\frac{9}{2}, \frac{1881}{2}\right\}$. By performing the similar computations for the other three sample singular points, one finds the eigenvalues $\left\{8064, \frac{39880}{49}\right\}$ for $z_1 := -6$ (red), $\left\{-\frac{3136}{125}, \frac{156114}{125}\right\}$ for $z_3 := \frac{7}{5}$ (blue), and $\left\{-9408, -\frac{83398}{9}\right\}$ for $z_4 := 7$ (cyan). Consequently, the quadratic forms corresponding to the red, magenta, and cyan singular points are definite, while the form is indefinite at the blue singular point. This agrees with the plot of Figure 17, which depicts discretely the (translated) convex-cone image of the nullspace of the differential $\mathrm{D}F(\xi^p)$ under the second-quadratic-differential map, just as in the statement of Theorem 4.1, at each of the four selected singular values $\xi^1, \ldots, \xi^4$. Clearly, only the blue cone contains a ray on the line tangent to the singular-value curve at the cone vertex, while each of the red, magenta, and cyan cones touches its singular-value curve-segment only at the cone vertex. Therefore, following along the program outlined in Section 3, we can discard the blue singular values from further consideration, since they must belong to the interior of the range. All of the other singular values lie on the local boundary of the range, and so remain in contention as possible genuine boundary points.

It makes sense to compute the mapped-side normals at the red, magenta, and cyan singular values. Such a computation at this point would clearly allow us to discard all of the magenta singular points, but none of the red and cyan ones.

Let us next seek regular-point preimages for the singular values; even though the magenta and blue values have been eliminated, we shall include them for the present purposes of illustration. We find regular preimages for neither the red nor the magenta singular values, but such preimages are found for the cyan singular values; the latter are therefore also in the interior of the range and so must be eliminated from further consideration. For example, Figure 18 shows, for the cyan singular value corresponding to $z_4 := 7$, the one (original) singular-point preimage at $(13/6, 9/4, 7)$ along with an entire closed curve of regular-point preimages. One finds regular preimages of the blue singular values, as well. Thus, *e.g.*, Figure 19 shows an entire closed curve of preimages of the blue singular value corresponding to $z_3 := 7/5$, one of which is the original singular-point preimage, the remainder

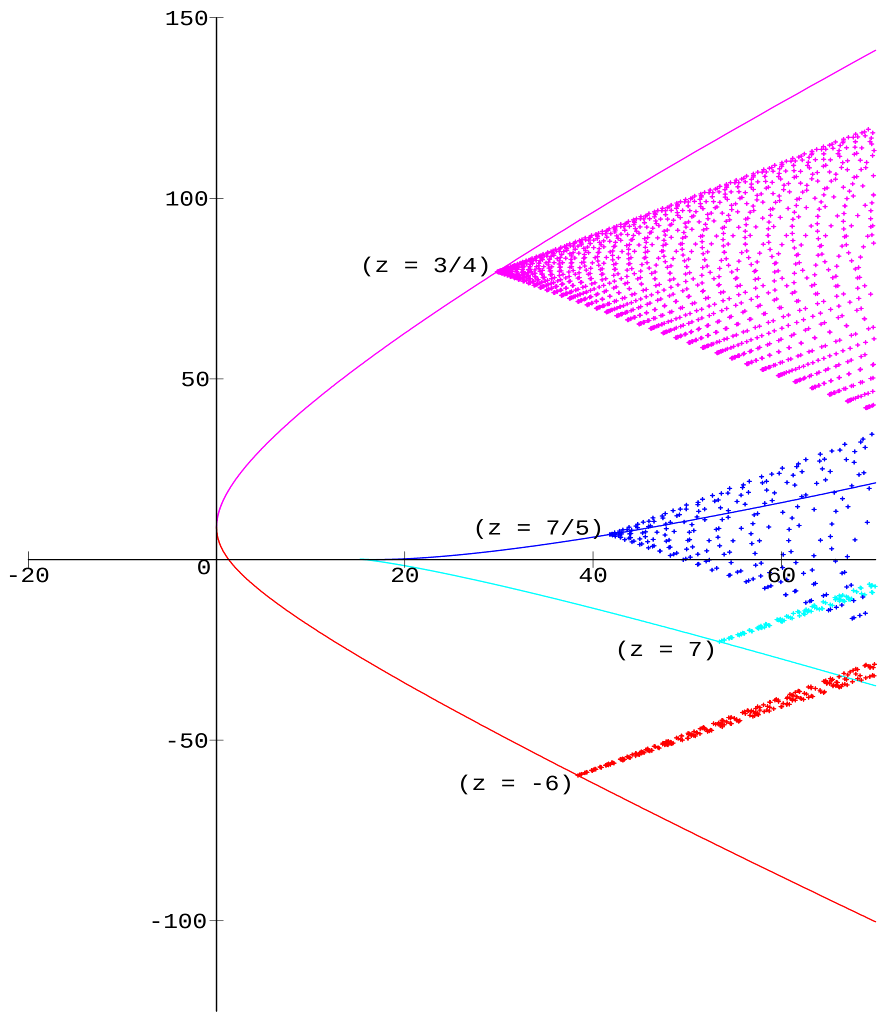

Fig. 17. Convex-cone image of nullspace of first differential at selected singular points, Example 5.

being regular-point preimages; a second view of these is given in Figure 20. Another view of all of these preimages appears in Figure 21, along with two balls, one gold and one orange, centered at typical *regular* points chosen on the respective preimage curves. As a further check, discrete images of the latter balls are shown in Figure 22, where they certainly appear to cover the singular-value images of the respective centers of the balls.

Since each of the red and magenta singular values has but one singular preimage, while hypothesis ($\star$) of Proposition 3.1 clearly holds here (since the map F has appropriate polynomial components), we can apply that Proposition to conclude that the (global) included boundary of the range of this F is exactly the red/magenta 1-manifold of singular values. Figure 23, showing a discrete depiction of a portion of the range, *i.e.*, the images of many points under F, supports this conclusion. Finally, we decide that the set of Pareto minima for this F coincides with the red quarter-hyperbola of (completely) singular points, shown in Figure 13.

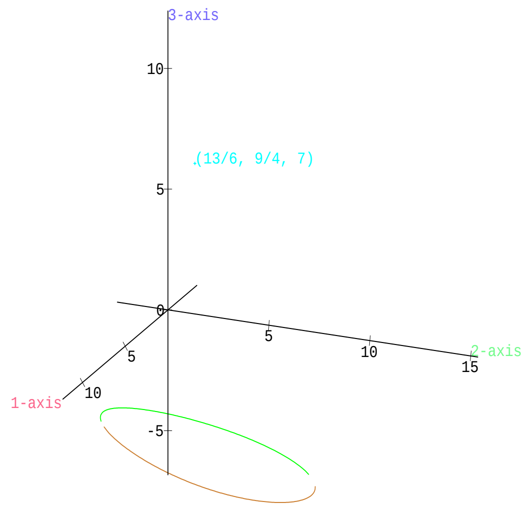

Fig. 18. Preimages of the typical cyan singular value, Example 5.

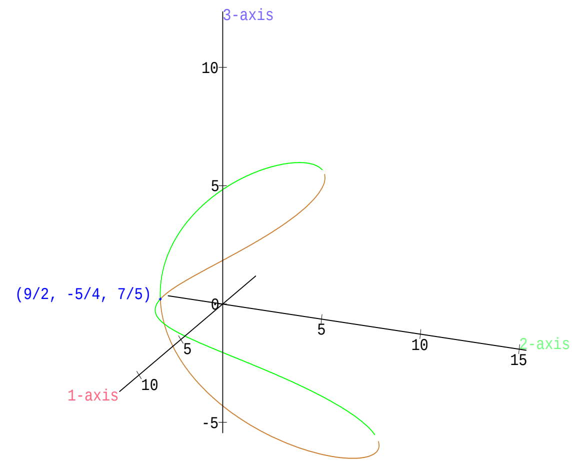

Fig. 19. Preimages of the typical blue singular value, Example 5.

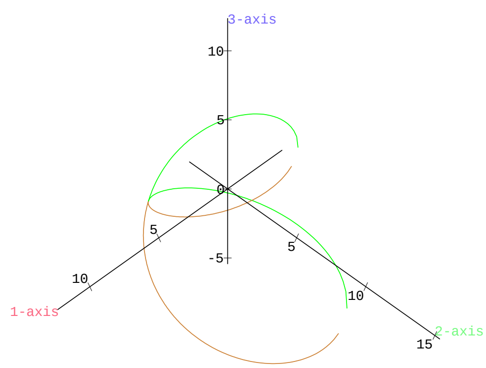

Fig. 20. Preimages of the typical blue singular value, Example 5.

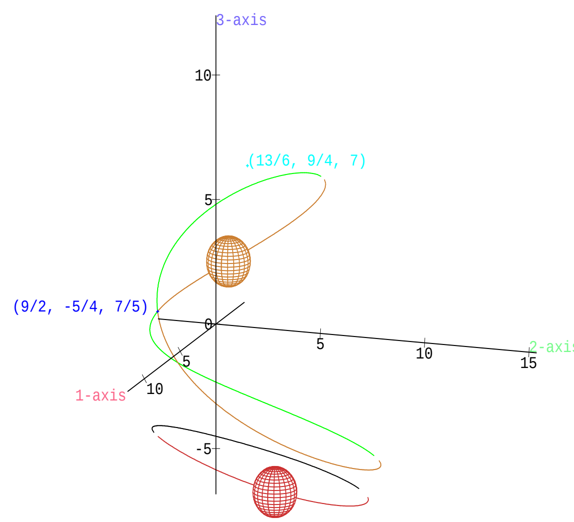

Fig. 21. Preimages of singular values of F and balls, Example 5.

Our final example, falling under case (B) of Section 3, illustrates the use of Theorem 2.2, Theorem 5.1, and Proposition 3.2 in finding the boundary of the range and the Pareto minima of an $\mathbb{R}^2$-valued map defined on the closure of a ball in $\mathbb{R}^N$.

Example 6.6. Let $N \geq 2$ and $\varrho > 0$. Here is the definition of a map $G : \overline{\mathrm{B}_\varrho^N(0)} \to \mathbb{R}^2$ that we shall

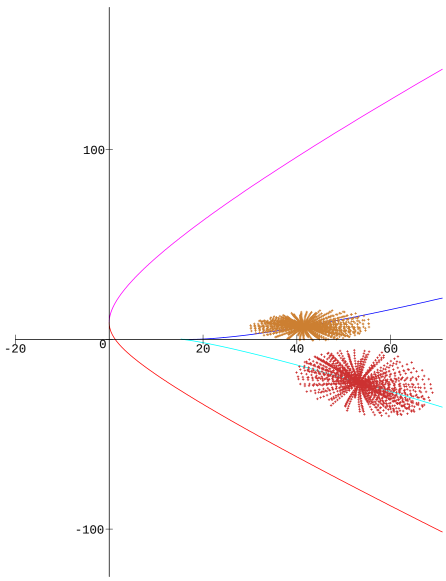

Fig. 22. Images of balls in Figure 21 under F of Example 5.

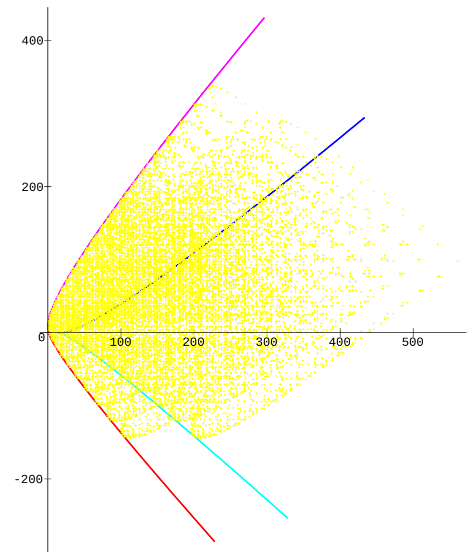

Fig. 23. Range of F in Example 5.

call the Dolph-AKK map:

$$\left.\begin{aligned} G_1(\xi) &:= -\beta \cdot \xi \\ G_2(\xi) &:= \sqrt{(W\xi)\cdot \xi} \end{aligned}\right\}, \qquad \text{for} \quad |\xi|_N \le \varrho. \tag{6.1}$$

In (6.1), W is the operator induced on $\mathbb{R}^N$ by the Gram-matrix with elements $\{W_{pq}\}_{N\times N}$ and $\beta \equiv (\beta_p)_{p=1}^N$, where

$$W_{pq} := \int_0^{\theta_0} w_p(\theta) w_q(\theta)\, d\theta, \qquad \text{and} \qquad \beta_p := w_p\big(\frac{\pi}{2}\big),$$

and the functions w_p are given by

$$w_p(\theta) := \cos\big(p\kappa\delta\cos\theta\big) - \cos\big(p\kappa\delta\cos\theta_0\big), \qquad p, q = 1, \ldots, N,$$

for some $\theta_0 \in (0, \pi/2)$ and positive κ and δ. The Dolph-AKK map G is developed in [1] in a study of the optimization of the radiation pattern of a symmetric, broadside, line array of $2N+1$ electromagnetic dipoles. The arguments ξ_k, $k = 1, \ldots, N$, represent the "excitation coefficients" of $2N$ of the paired, symmetrically arranged dipole-sources, while the coefficient of the center dipole is to be adjusted so that the main lobe of the radiation pattern covers the angular sector $[-\theta_0, \theta_0]$ (with nulls at the endpoints), for a chosen angle θ_0. The number δ is the dipole separation distance, and κ is the wavenumber.

Specifically, for a selected value of $\varrho > 0$, it is desired to determine all sets of excitation coefficients ξ that are constrained in magnitude by $|\xi|_N \le \varrho$ and realize Pareto-optimal values for two costs: (1) the peak value of the radiation pattern (to achieve a maximal value) and (2) the radiated power owing to the side-lobes (to obtain a minimal value). The respective components of $G(\xi)$ essentially give these properties of the radiation pattern corresponding to the argument ξ.

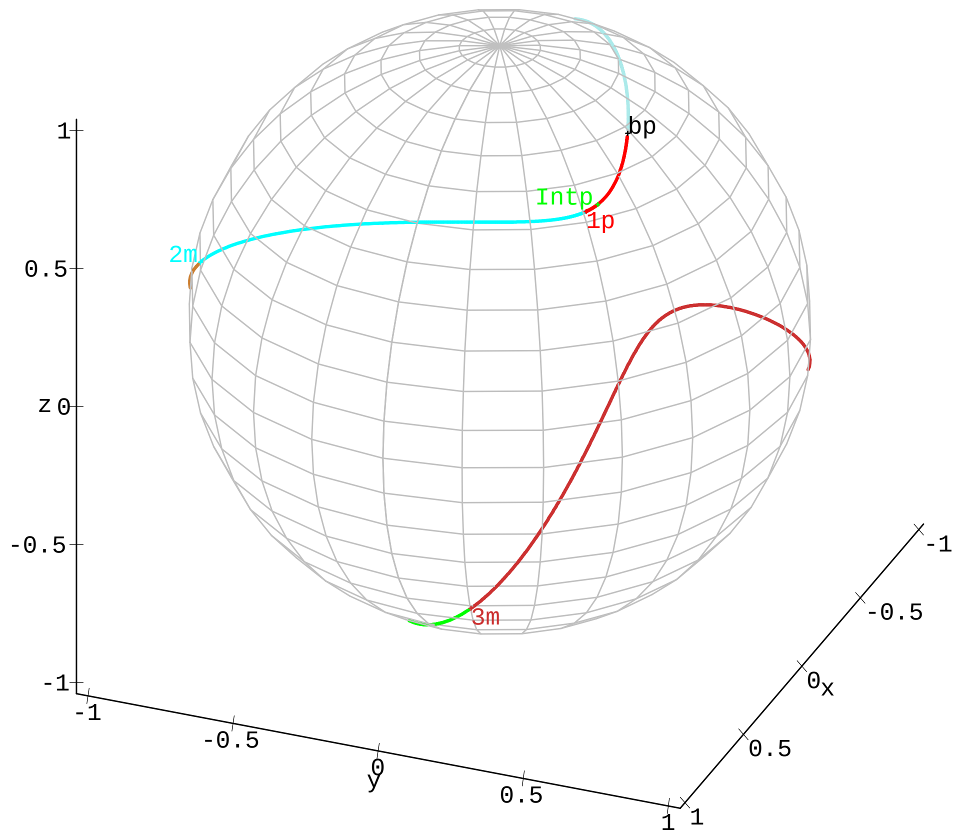

Fig. 24. Singular points of Dolph-AKK map for $N = 3$, $\kappa\delta = 5.0$, $\theta_0 = 9\pi/20$, $\varrho = 1$.

Strictly, however, the first component of $G(\xi)$ should be defined to be $-\left|\beta \cdot \xi\right|$ instead of just $-\beta \cdot \xi$, but the absolute value can be dropped on the basis of a symmetry argument, provided it is then kept in mind that the range of the "actual" mapping is obtained by "folding" the range of the map G defined above along the vertical axis so that the righthand half falls atop the lefthand half; in particular, this doubles the number of preimages of each value by including the reflection in the origin of each original preimage. The "actual" Pareto minima are then to be reckoned accordingly, which requires that one include also the singular points obtained by reflecting in the origin of $\mathbb{R}^N$ all of the Pareto minima found for the map G defined here.

The formulation in [1] is a modification of the original set-up of C.L. Dolph [8], who worked to minimize the maximum value in the side lobes, for a given main-beam width and peak power; this explains the name given here to the map G.

Guided by the outline of Section 3 for case (B), we begin by seeking all of the singular points of the restriction of G to the interior $\mathrm{B}^N_\varrho(0)$ of the domain. G is not differentiable at 0; at any $\xi \neq 0$ we

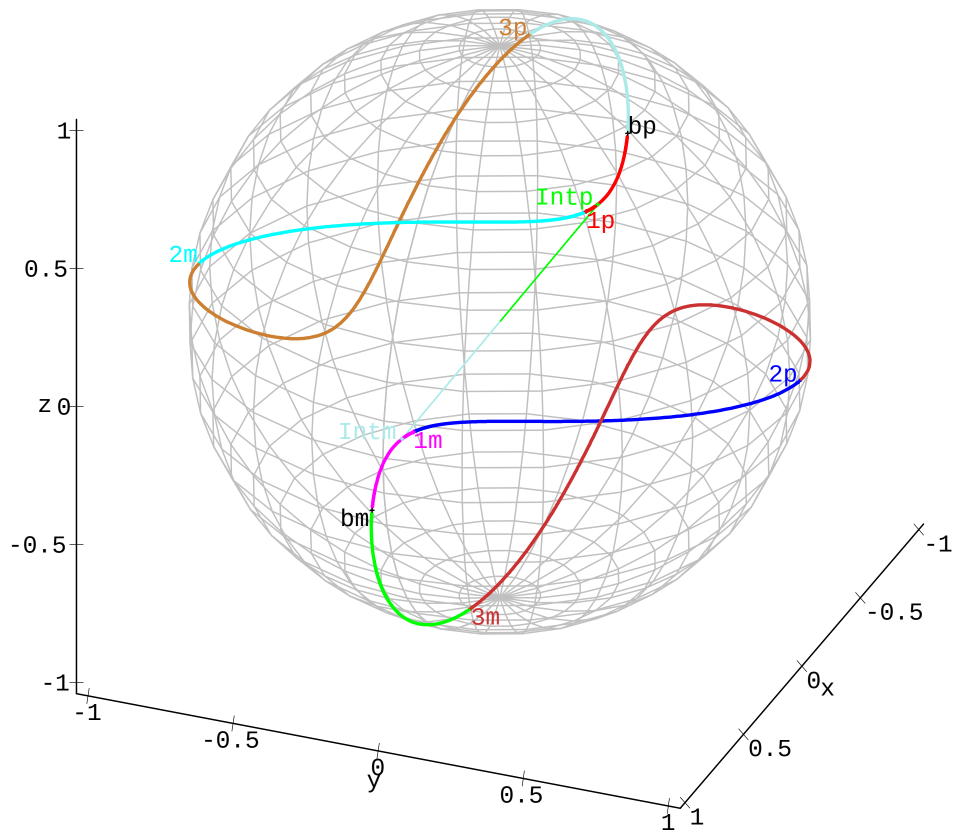

Fig. 25. Singular points of Dolph-AKK map for $N = 3$; a second view.

find $\operatorname{grad} G_1(\xi) = -\beta$ and $\operatorname{grad} G_2(\xi) = \{G_2(\xi)\}^{-1} W\xi$. Clearly, neither gradient vanishes, so we find that the singular points in the interior of the domain are rank-1 and comprise the nonzero elements of $\operatorname{sp}\{W^{-1}\beta\}$ having magnitude less than ϱ. It follows that the corresponding singular values in $\mathbb{R}^2$ are given by the set of points of the form $\left(-t\mu_\beta, |t|\mu_\beta^{1/2}\right)$ for $0 < |t| < \varrho/\left|W^{-1}\beta\right|_N$, in which $\mu_\beta := \left(W^{-1}\beta\right)\cdot\beta$, *i.e.*, by the two line segments joining the origin and the points $\left\{\varrho/\left|W^{-1}\beta\right|_N\right\}\left(\pm\mu_\beta, \mu_\beta^{1/2}\right)$. It is easy to check analytically that each of these singular values has no other preimages, regular or singular, even when we consider G as defined on all of $\mathbb{R}^N$. Moreover, the application of the test of Theorem 4.1 reveals that the quadratic form $\mathcal{Q}_\xi$ is definite for every singular point ξ, so each of the singular values belongs at least to the local boundary of the range of G. We can therefore conclude already that all of these singular values belong in fact to the global boundary of the range of G, by applying Proposition 3.1 to the restriction of the map G to the ball $\mathsf{B}_\varrho^N(0)$ with the origin removed. This assertion can be corroborated by using the generalized Cauchy-Schwarz inequality to check that the inequality $G_2(\xi) \geq \mu_\beta^{-1/2}\left|G_1(\xi)\right|$ holds for all ξ.

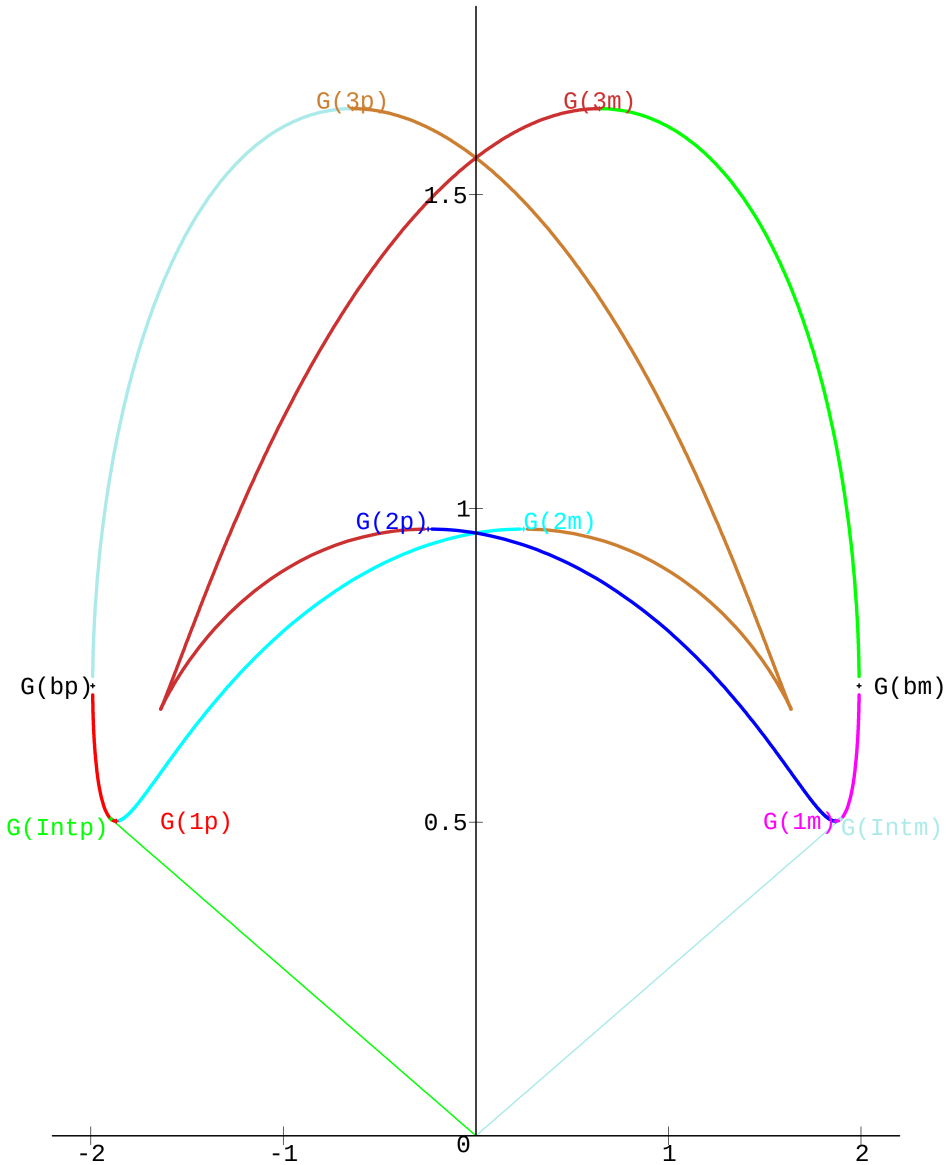

Fig. 26. Singular values of Dolph-AKK map for $N = 3$, $\kappa\delta = 5.0$, $\theta_0 = 9\pi/20$, $\varrho = 1$.

Next, we must carry out the corresponding singular-point analysis for the restriction $G^\star := G \,\big|\, \mathsf{S}^N_\varrho(0)$ of G to the boundary of the domain. It is easy to compute here the surface gradients of the components $G^\star_1$ and $G^\star_2$ of $G^\star$. That is, at a point $\xi \in \mathsf{S}^N_\varrho(0)$ we need only find the tangential component of the gradient of G (regarded as extended to all of $\mathbb{R}^N$ in the obvious manner) at ξ. Accordingly, with $\hat{\xi} := |\xi|^{-1}_N \xi$, we get

$$\mathrm{grad}_{\mathcal{M}} G^\star_1(\xi) = -\beta + \big(\beta \cdot \hat{\xi}\big)\hat{\xi} \qquad \text{and} \qquad \mathrm{grad}_{\mathcal{M}} G^\star_2(\xi) = \frac{1}{G_2(\xi)}\Big\{W\xi - \big((W\xi)\cdot\hat{\xi}\big)\hat{\xi}\Big\} \qquad \text{for} \quad |\xi|_N = \varrho.$$

Clearly, $\mathrm{grad}_{\mathcal{M}} G^\star_1(\xi) = 0$ iff $\xi = \pm\beta_\varrho$, with $\beta_\varrho := \varrho|\beta|^{-1}_N \beta$, while $\mathrm{grad}_{\mathcal{M}} G^\star_2(\xi) = 0$ iff $\xi \in \mathsf{S}^N_\varrho(0)$ is an eigenvector of W. Thus, $G^\star$ has a rank-0 singular point at β_ϱ iff β_ϱ is an eigenvector of W, in which case $-\beta_\varrho$ is also a rank-0 singular point; otherwise, $\pm\beta_\varrho$ are rank-1 singular points (and a curve of singular values in $\mathbb{R}^2$ passing through the image of either of these two points under $G^\star$ will have a vertical tangent at that image, if it has a tangent there at all).

Let us suppose from this point that W has only simple eigenvalues and that β_ϱ is not an eigenvector of W. Then, denoting by $\lambda_1, \ldots, \lambda_N$ the (positive) eigenvalues of W arranged in, say, increasing order, and by $\{\hat{\zeta}_1, \ldots, \hat{\zeta}_N\}$ an ordered collection of corresponding eigenvectors forming an orthonormal set in $\mathbb{R}^N$, in addition to the two singular points at $\pm\beta_\varrho$, $G^\star$ also has rank-1 singular points at $\pm\varrho\hat{\zeta}_1, \ldots, \pm\varrho\hat{\zeta}_N$ on $\mathsf{S}^N_\varrho(0)$. (Note that a curve of singular values in $\mathbb{R}^2$ passing through the image under $G^\star$ of any of the latter $2N$ points will have a horizontal tangent at that image, if it has a tangent there.) These are all of the singular points at which one of the surface gradients of the components of $G^\star$ vanishes.

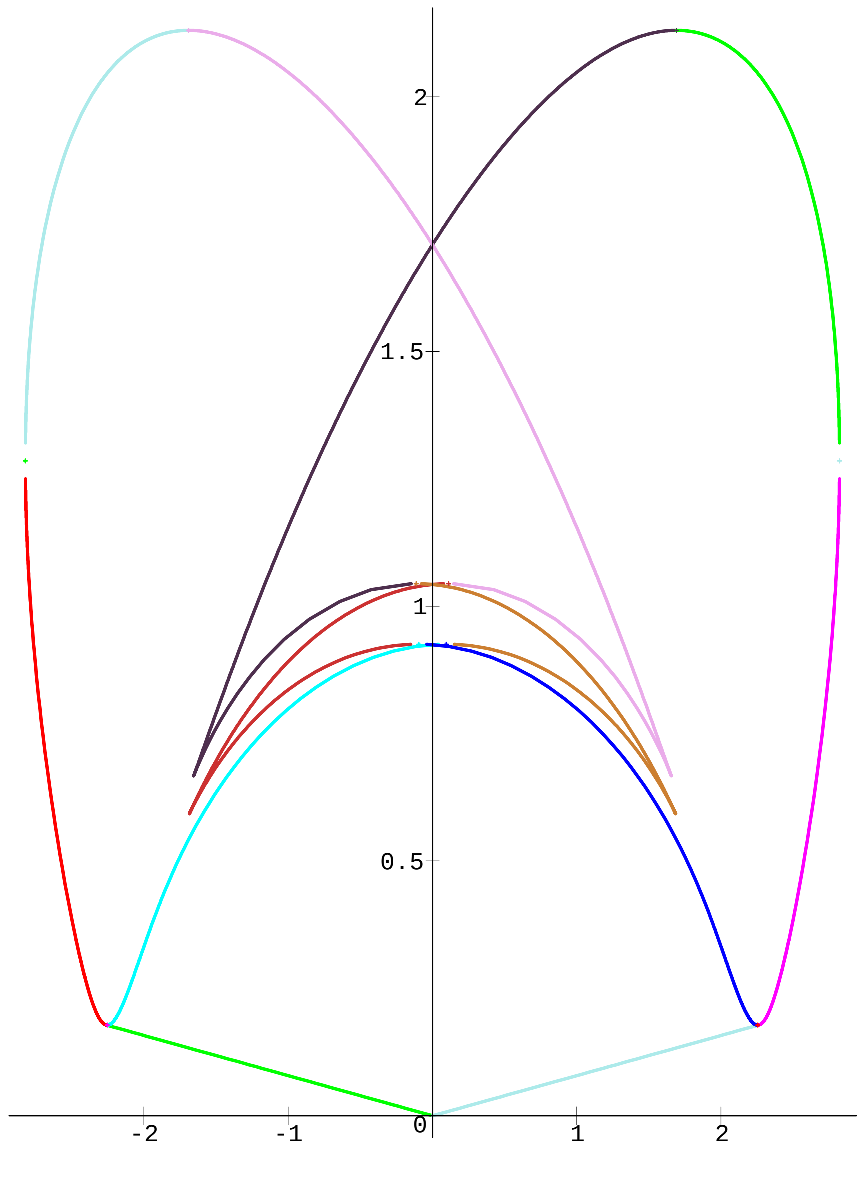

Fig. 27. Singular values of the Dolph-AKK map for $N = 4$, $\kappa\delta = 5.0$, $\theta_0 = 9\pi/20$, $\varrho = 1$.

More generally, if $\xi \in \mathsf{S}_\varrho^N(0)$ and $\xi \neq \pm\beta_\varrho$, one finds that ξ is a singular point of $G^\star$ iff there exist λ and $b \in \mathbb{R}$ such that $W\xi = \lambda\xi + b\beta$. Besides encompassing the eigenvectors already introduced, this condition yields, corresponding to each λ that is not an eigenvalue of W, two additional rank-1 singular points as $\pm\alpha_\lambda$, with

$$\alpha_\lambda := \frac{\varrho(W-\lambda I)^{-1}\beta}{\left|(W-\lambda I)^{-1}\beta\right|_N} \qquad (\lambda \text{ not an eigenvalue of } W).$$

The α_λ can be computed easily by, for example, using the spectral representation of $(W-\lambda I)^{-1}\beta$.

For $k = 1, \ldots, N$, easy manipulations produce the limiting values $\lim_{\lambda\to\pm\infty}\alpha_\lambda = \mp\beta_\varrho$ and $\lim_{\lambda\to\lambda_k^\pm}\alpha_\lambda = \mp\operatorname{sgn}(\beta\cdot\hat{\zeta}_k)\varrho\hat{\zeta}_k$, the latter provided that $\beta\cdot\hat{\zeta}_k \neq 0$; if $\beta\cdot\hat{\zeta}_k = 0$, the second pair of limits are easy to work out, but somewhat more involved to display and describe. For simplicity, we shall suppose that $\beta\cdot\hat{\zeta}_k \neq 0$ for each k. Consequently, on the boundary $\mathsf{S}_\varrho^N(0)$ of the domain of G, the curves $\lambda \mapsto \pm\alpha_\lambda$, for λ running through each of the $N+1$ open intervals in $\mathbb{R}$ determined by the N eigenvalues of W, comprise curves of singular points of $G^\star$, symmetric with respect to the origin and connecting the points $\pm\beta_\varrho, \pm\varrho\hat{\zeta}_1, \ldots, \pm\varrho\hat{\zeta}_N$ in the indicated manner, *i.e.*, *not* continuously as the parameter λ increases. However, the union of the images, along with the limiting endpoints, will form two connected paths on the sphere. Observe also that the curves of singular points on the surface include $\pm\alpha_0$, which are the points at which the line segments of interior singular points, already found above, emerge to meet the boundary of the domain. Thus, the curves of singular points on the surface of the sphere are connected by the line of interior singular points passing through the center of the sphere. Recall that these results hold under the hypothesis that W has only simple eigenvalues, while $\beta\cdot\hat{\zeta}_k \neq 0$ for $k = 1, \ldots, N$; the situation becomes much more complicated if these conditions are not fulfilled.

The accompanying figures illustrate the main features of this geometrical situation for $N = 3$ and $N = 4$, with $\varrho = 1$, $\kappa\delta = 5.0$, and $\theta_0 = 9\pi/20$ in both cases. Thus, in Figure 24 and Figure 25 we show two views of the singular points for $N = 3$, the first a "hidden-line" view in which the sphere is opaque, the second a "wire-frame" view in which the sphere is transparent. The "plus-minus" pairs of eigenvectors are labelled as `1p`, `1m`,..., `3m`, the points $\pm\beta_\varrho$ are indicated by `bp` and `bm`, and the endpoints of the segment of interior singular points by `Intp` and `Intm`. The colored curve-segments are just those already described, each connecting an eigenvector with either another eigenvector or one of the points $\pm\beta_\varrho$. In Figure 25, the line of interior singular points is visible, half in green and half in turquoise, divided by the center of the unit ball.

The singular values, *i.e.*, the images of the singular points shown in Figure 24 and Figure 25, appear in Figure 26. The three figures illustrating the case $N = 3$ are color-keyed, red being mapped to red, *etc.* The images of the eigenvectors, of $\pm\beta_\varrho$, and of the endpoints of the interior singular

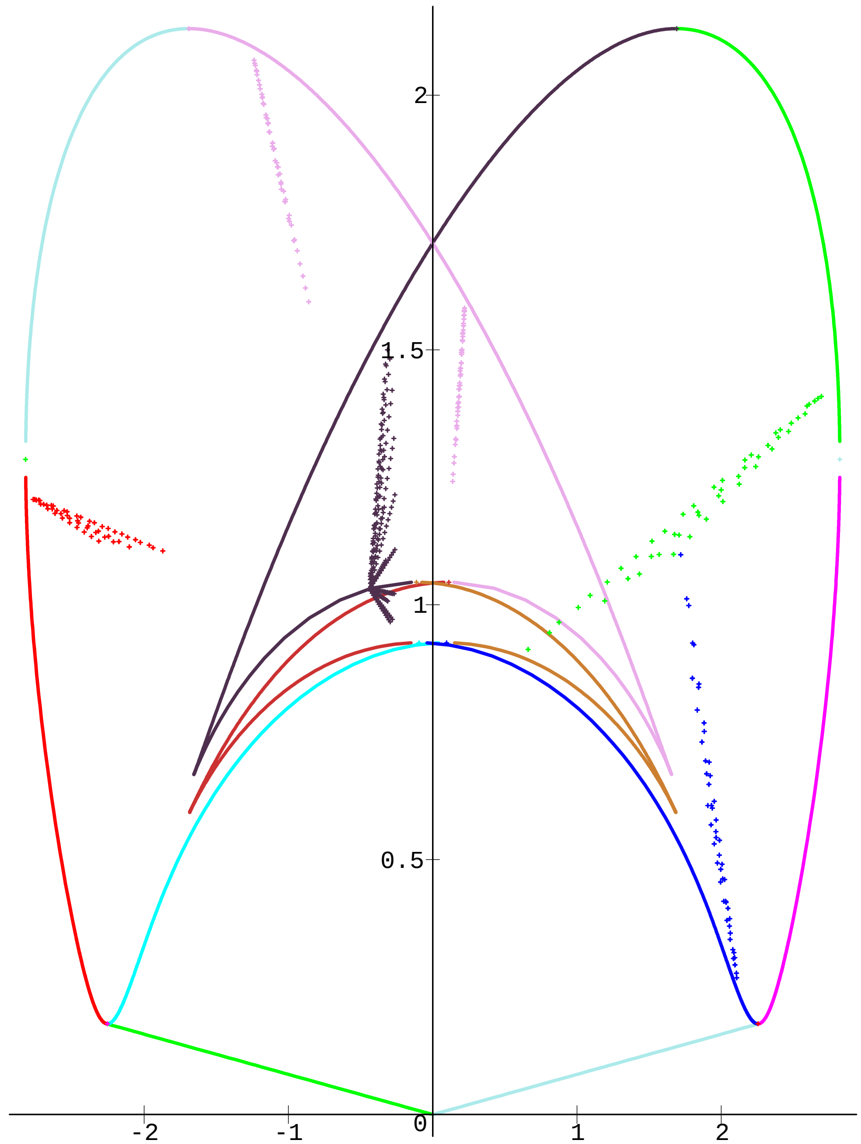

Fig. 28. Applying Theorem 5.1 to the Dolph-AKK map; $N = 4$, $\kappa\delta = 5.0$, $\theta_0 = 9\pi/20$, $\varrho = 1$.

values are indicated in Figure 26. In particular, the points of horizontal or vertical tangents are as predicted.

Since the singular points cannot be depicted when $N = 4$, only the singular values for that case (with the other parameters retaining their values), are plotted, in Figure 27. The same general features are observed there, the major difference being the greater number of cusps relative to the case $N = 3$, so the remaining observations on this Example 6 will be made in reference to $N = 4$. From Figure 27, it is easy to guess the locations of the boundaries of the ranges of G and $G^\star$, as well as the images of the Pareto minima of G. However, we want to know whether our analytical predictions agree with these obvious conjectures. We have already shown that the two lower line segments, in green and turquoise, the images of the interior singular points, form part of the global boundary of the range of G. We should next apply Theorem 5.1 to examine the other singular values. Here, we shall merely display a few of the results of this application, passing over the computationally essential aspects such as the determination of, and construction of bases for, the nullspace of the differential of $G^{\star}$, the calculation of the second quadratic differential, *etc.* Thus, the convex cones in Figure 28 were generated from the recipe of Theorem 5.1, to demonstrate the testing of a few singular values (those located at the vertices of the cones). That is, just as in Figure 17 for Example 5, the situation of the convex cone at each singular value is a graphical indication of the definiteness property of the pertinent quadratic form corresponding to the singular-point preimage for that value; we are not providing here the eigenvalues of the quadratic forms—which, of course, are all that one need determine in an ordinary computation. Clearly, each of the tests depicted in Figure 28 and indicated by the red, navy-blue, green, and the two violet cones, indicates a definite quadratic form, hence a singular value on the local boundary of the range; the one test indicating a singular value locally covered from the singular preimage, and so belonging to the interior of the range, is that for the black cone near the center of the figure, where the quadratic form is revealed as indefinite. Repeated trials lead to the conclusion that only the singular values contained *between cusps* have corresponding indefinite forms, and so are implied by Theorem 5.1 to belong to the interior of the range of $G^{\star}$, hence of G; these values can therefore be discarded. All of the other singular values have definite quadratic forms, and so are fold points of the range. To decide about these latter fold points, we are reduced to seeking other preimages, in particular, regular preimages. After carrying out a number of such searches, it appears that we can find regular-point preimages under $G^{\star}$ for (1) the black and violet singular values lying below the point of intersection of those curves on the vertical axis, near $(0.0, 1.7)$, (2) the gold and orange singular values farther down (including those already disqualified by the quadratic-form testing), and (3) the (barely visible) cyan and navy-blue singular values lying above the point of intersection of those curves on the vertical axis, near $(0.0, 0.9)$. For none of the remaining singular values of $G^{\star}$ can we find any preimages other than its original singular preimage; on the strength of Proposition 3.1 we conjecture, therefore, that these form the boundary of the range of $G^{\star}$, with the latter range itself occupying the "tooth-shaped" area lying on and within those remaining curves of singular values, *i.e.*, listing the pieces of its boundary in clockwise order starting from the lower left, that region bounded by the red, the turquoise, part of the violet, part of the black, the green, the magenta, most of the navy-blue, and most of the cyan curves in Figure 28.

According to the third step in the search procedure for case (B), we must find and exclude those points of the boundary of the range of $G^{\star}$ belonging to the image of the ball $\mathrm{B}_1^4(0)$ under G itself. Many numerical trials indicate that these are precisely the remaining cyan and navy-blue singular values forming the "bottom" portion of the boundary of the range of $G^{\star}$, *i.e.*, of the remaining singular values, we have been able to find for only those points regular preimages under G. When these singular values are discarded, the remaining points comprise just the original "obvious" guess for the boundary of the range of G; our contention remains a conjecture, of course. In any event, this example indicates the extent to which we can presently conduct a definitive and completely conclusive search in those cases for which we can make no firm conclusions concerning the outcomes of the global searches required.

With the same *caveat*, we could provide an analytical description of the "apparent" Pareto minima for the map G, since we can completely specify the preimages of the conjectured Pareto-point images here. Note that one must take some care in this near the intersection of the red curve and the green line of singular values, since not all of the red values there are Pareto-point images. Finally, apropos of the original (physical-model) problem, one should now recall our initial remark, to the effect that we must also include as Pareto minima for the "actual" Dolph-AKK map the preimages of the reflections in the vertical axis of the Pareto images just described for G, *i.e.*, one must include the reflections in the origin of the Pareto minima found for G.

This completes our presentation of examples illustrating the general developments of Sections 1–6.

7. Conclusion.

We have attacked the problem of locating the Pareto-optimal points of a differentiable mapping $F : \{\mathcal{D}_F \subset \mathbb{R}^{N_0}\} \to \mathbb{R}^n$ by seeking procedures for first finding the included boundary of the range of such a map, then looking in the preimage of that set for the Pareto optima. For the basic setting, in which the domain $\mathcal{D}_F$ is an N-submanifold-without-boundary of class C^1, we have given the fundamental necessary condition in Theorem 2.2: only singular points of F can be mapped by F to the boundary of its range, so that all Pareto optima of F belong to its collection of (completely) singular points. But then it is discovered that, in general, F will map many of its singular points to the interior of its range, not to the boundary of its range. This latter circumstance forces us to seek means for discriminating between those singular values belonging to the interior of the range and those singular values lying on the boundary of the range. For this testing, we have given some sufficient conditions in Theorem 4.1 (for the open-set domain) and Theorem 5.1 (for the general manifold-domain). We have outlined a systematic search procedure based on results such as these.

We also indicated a search procedure for the commonly occurring case in which the domain $\mathcal{D}_F$ is the closure of a bounded, regular open set in $\mathbb{R}^N$. To handle that setting, we naturally proposed to break the domain down into interior-manifold and boundary-manifold(s), analyze the action of F on each piece by using the tests developed for the manifold domain, then combine the results appropriately.

We illustrated all of the developments and ideas with simple first examples. In some of these examples, we could locate all of the Pareto minima systematically and completely by following the steps of the proposed procedures, *i.e.*, we could do this even without the "crutch" provided by the visualization tools in the Maple computer-algebra system. In the harder example discussed, Example 6.6, we relied after a certain point on numerical trials and plausibility arguments coupled with visual inspection to "complete" the determination of the Pareto minima.

Generally, the examples indicate the areas in which our testing results are lacking. Thus, we propose some lines along which future work should progress:

(1.) Results analogous to those of Theorems 4.1 and 5.1 must be developed for the cases in which $n > 2$. This is especially important, since one commonly finds applications involving three or more cost functions.

(2.) A deeper analysis is required to handle the case of indeterminacy in Theorems 4.1 and 5.1, when the quadratic form turns out to be semidefinite.

(3.) Theorems 4.1 and 5.1 should be extended to cover singular points that are strongly degenerate.

(4.) The gaps remaining in the search procedures outlined in Section 3 should be eliminated, in particular, by discerning more completely the ways in which a value can belong to the interior of the range when it is not locally covered from some preimage.

Finally, we list two further projects that are presently under way:

(5.) We must begin the accumulation of a body of experience in applying the theorems proven here to the study of physical applications. As an initial step in this direction, by employing the present procedures one can rework studies that have already been analyzed in the literature by some other means; *cf.*, *e.g.*, [1], [5], and the works listed in [17].

(6.) There are numerous applications that are properly formulated only when the domain $\mathcal{D}_F$ is taken to be some type of set in an infinite-dimensional Banach or Hilbert space or even, more generally, a linear topological space (while the range of F remains in some $\mathbb{R}^n$). It is therefore important to find out the forms of the extensions to such settings of the results that have been developed here in the finite-dimensional-domain case. Once those extensions are better understood, one can begin to study the matter of approximating Pareto optima of a map defined on an infinite-dimensional space by solving the corresponding problem posed for the restriction of the map to some finite-dimensional subspace, since the latter may offer a more tractable—indeed, indispensable—approach for the actual numerical computations.

Appendix A. The range of a biquadratic map.

Let $Q \equiv \big(Q_1, Q_2\big) : \mathbb{R}^N \to \mathbb{R}^2$ be a mapping with (symmetric, homogeneous) quadratic-form components, *i.e.*, with

$$Q_l(\xi) := \sum_{j,k=1}^{N} \mathcal{A}^{(l)}_{jk} \xi_j \xi_k \qquad \text{for each} \quad \xi \in \mathbb{R}^N, \qquad \text{for} \quad l = 1 \text{ and } 2, \tag{A.1}$$

in which $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are real symmetric $N \times N$ matrices. Naturally, we call such a map "biquadratic." In searching for the boundary of the range of a *general* $\mathbb{R}^2$-valued map, one needs some information about the range of a biquadratic map; in that case, the matrices $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ will be, respectively, the Hessian matrices of real C^2-functions F_1 and F_2, evaluated at some point. We develop the needed results in this Appendix.

The most important *qualitative* fact about the range of a biquadratic map is contained in

Proposition A.1. *Let $Q \equiv \big(Q_1, Q_2\big) : \mathbb{R}^N \to \mathbb{R}^2$ be a biquadratic map. The range $\mathcal{R}_Q := Q(\mathbb{R}^N)$ of Q is a convex cone with vertex at the origin in $\mathbb{R}^2$.*

Proof. It is clear that whenever z is in the range and $\alpha \geq 0$, then αz is also in the range. Therefore, the range $\mathcal{R}_Q$ is certainly a cone with vertex at the origin. To verify that $\mathcal{R}_Q$ is convex, we must show that the (perhaps degenerate) line segment joining $Q(\xi^1)$ and $Q(\xi^2)$ is in $\mathcal{R}_Q$ whenever ξ^1 and $\xi^2 \in \mathbb{R}^N$; we may suppose here that both ξ^1 and ξ^2 are nonzero. For that, it *suffices* to show that the line segment joining $Q(\xi^1)$ and $Q(\xi^2)$ lies in the range of the *restriction* of Q to the subspace spanned by ξ^1 and ξ^2. But the latter restriction can be recast as a biquadratic map on $\mathbb{R}^2$, so it suffices, in turn, to check that the convexity obtains just when $N = 2$—which is easily carried out directly.

Indeed, now let $\mathcal{Q} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\mathcal{Q}(x) := \begin{pmatrix} a x_1^2 + 2b x_1 x_2 + c x_2^2 \\ \\ d x_1^2 + 2e x_1 x_2 + f x_2^2 \end{pmatrix}, \qquad \text{for} \quad x \equiv (x_1, x_2) \in \mathbb{R}^2. \tag{A.2}$$

By using the coördinate-transformation relations under a simple translation and rotation, it is an easy exercise to verify that the image of the unit circle under this $\mathcal{Q}$ is a proper ellipse when $(a-c)e - (d-f)b \neq 0$ and a degenerate ellipse, *i.e.*, a closed line segment (perhaps even passing through the origin) or a point, when $(a-c)e - (d-f)b = 0$. Thus, we see that in every case a convex cone results by constructing all rays emanating from the origin and passing through the image of the unit circle; since that construction also produces exactly the full range $\mathcal{R}_{\mathcal{Q}}$, the assertion of the Proposition follows.

(This strategy of proof was lifted from Kato [12], where it is employed to show that the numerical range of an operator in a complex Hilbert space is a convex subset of the complex plane.) ∎

Remarks. (1.) Thus, the range of a nontrivial biquadratic map is either the entire plane or a straight line through the origin or a "wedge" with vertex at the origin and vertex angle not exceeding π radians; in the latter case, the wedge may be degenerate, *i.e.*, it may be a ray emanating from the origin. Evidently, the wedge may not include its collinear bounding rays when the vertex angle is π radians. For example, in the setting $\mathcal{Q} : \mathbb{R}^2 \to \mathbb{R}^2$ used in the proof, when the image of the unit circle is a proper ellipse passing through the origin, then the range $\mathcal{R}_{\mathcal{Q}}$ will be the union of the origin and the open half-plane containing the remainder of the ellipse.

(2.) It is clear from the proof that Proposition A.1 is also correct when the domain $\mathbb{R}^N$ is replaced by any real inner-product space (finite- or infinite-dimensional) and the components Q_l are given by quadratic forms on the space.

(3.) In passing, we point out that we can get *quantitative* information on the location of the convex-cone range of a biquadratic map by either of two approaches, both of which lead to the study of a generalized eigenvalue problem constructed from the matrices figuring in the two quadratic-form components of the biquadratic map. Indeed, the basic result in the main text says that we are to seek the *included* boundary of the range by looking amongst the singular values. With $Q : \mathbb{R}^N \to \mathbb{R}^2$ a biquadratic map having components Q_1 and Q_2 given by (A.1), since $\operatorname{grad} Q_l(\xi) = 2\mathcal{A}^{(l)}\xi$ for $l = 1$, 2, we see that $\xi \in \mathbb{R}^N$ is a singular point of Q iff the set $\{\mathcal{A}^{(1)}\xi, \mathcal{A}^{(2)}\xi\}$ is linearly dependent. Clearly, the linear dependence obtains iff either

$$\mathcal{A}^{(1)}\xi = 0 \qquad \text{or} \qquad \mathcal{A}^{(2)}\xi - \lambda\mathcal{A}^{(1)}\xi = 0 \qquad \text{for some} \quad \lambda \in \mathbb{R} \tag{A.3}$$

(or both); a nonzero ξ satisfying (A.3) then reveals a ray of singular values in the range, of undefined slope if $Q_1(\xi) = 0$ and of slope λ if $Q_1(\xi) \neq 0$. All points of the included boundary of the range must lie on the rays identified in this manner. Alternately, we can seek the extreme values of the slopes of the lines comprising the convex-cone range, by looking amongst the critical values of the ratio

$$\mu(\xi) := \frac{Q_2(\xi)}{Q_1(\xi)} \qquad \text{for} \quad \xi \in \mathbb{R}^N \setminus Q_1^{-1}\{0\}.$$

It is easy to check that ξ is a critical point of the real function μ iff $Q_1(\xi) \neq 0$ and $\mathcal{A}^{(2)}\xi - \mu(\xi)\mathcal{A}^{(1)}\xi = 0$, *i.e.*, iff $\mu(\xi)$ is a *real* eigenvalue of the same generalized problem that appears in (A.3), with corresponding eigenvector ξ; in that case, a corresponding ray of singular values of Q is determined by $Q(\xi)$, which will have slope $\mu(\xi)$. One must seek separately rays of singular values of undefined slope. But such computations alone will not in general serve to identify the range completely, since they reveal only the possible locations of the *included* boundary. For example, if we find that the loci of singular values reduces to the origin alone, we still cannot conclude that the range of Q must be all of $\mathbb{R}^2$. We will not pursue these questions further here, since they are somewhat of a digression.

The "second-derivative test" developed in the main text derives essentially from the simple observations made in the next proposition, concerning the determination of the position of a line through the origin in $\mathbb{R}^2$ relative to the range of a biquadratic map. For the statement it is convenient to introduce the idea of a *strongly nondegenerate* biquadratic map (*cf.* the definition of *strongly nondegenerate singular point*, given in Section 4):

Definition. A biquadratic map $Q : \mathbb{R}^N \to \mathbb{R}^2$ is *strongly nondegenerate* iff $Q(\xi) \neq 0$ for each nonzero $\xi \in \mathbb{R}^N$.

Proposition A.2. *Let* $Q \equiv (Q_1, Q_2) : \mathbb{R}^N \to \mathbb{R}^2$ *be a biquadratic map. Let* $x \equiv (x_1, x_2) \in \mathbb{R}^2$ *with* $x \neq 0$. *Denote by* $\mathfrak{l}_x$ *the line in* $\mathbb{R}^2$ *through* 0 *and* x, *and by* $\mathfrak{l}_x^+ := \{(\alpha x_1, \alpha x_2) \mid \alpha > 0\}$ *and* $\mathfrak{l}_x^- := \{(\alpha x_1, \alpha x_2) \mid \alpha < 0\}$ *the two rays emanating from* 0 *(excluding* 0*) and contained in* $\mathfrak{l}_x$. *Introduce a quadratic form* q_x *on* $\mathbb{R}^N$, *corresponding to* Q *and* x, *by setting*

$$q_x(\xi) := x_2 Q_1(\xi) - x_1 Q_2(\xi) \qquad \textit{for} \quad \xi \in \mathbb{R}^N.$$

(*i.*) *Let us say that the form* q_x *is weakly positive-definite [weakly negative-definite] iff* $q_x(\xi) > 0$ *[*< 0*] whenever* $Q(\xi) \neq 0$ *and weakly definite iff the form has either of these properties.*

Then q_x is weakly definite iff $\mathfrak{l}_x$ meets the range $\mathcal{R}_Q$ only at 0. If the map Q is strongly nondegenerate, then q_x is definite iff $\mathfrak{l}_x$ meets the range $\mathcal{R}_Q$ only at 0.

(*ii.*) *Let the convex-cone range $\mathcal{R}_Q$ be other than a line through the origin; for this it is sufficient that Q be strongly nondegenerate. Then the form q_x is indefinite iff at least one of $\mathfrak{l}_x^+$ and $\mathfrak{l}_x^-$ lies in the interior $\{\mathcal{R}_Q\}^\circ$ of the range.*

(*iii.*) *Let us say that the form q_x is weakly positive-semidefinite [weakly negative-semidefinite] iff q_x is nonnegative [nonpositive] and there exists some $\xi \in \mathbb{R}^N$ with $q_x(\xi) = 0$ but $Q(\xi) \neq 0$, and weakly semidefinite iff the form has either of these properties.*

Then q_x is weakly semidefinite iff at least one of $\mathfrak{l}_x^+$ and $\mathfrak{l}_x^-$ lies in the included boundary $\mathcal{R}_Q \cap \partial\mathcal{R}_Q$ of the range of Q. If the map Q is strongly nondegenerate, then q_x is semidefinite iff at least one of $\mathfrak{l}_x^+$ and $\mathfrak{l}_x^-$ lies in the included boundary $\mathcal{R}_Q \cap \partial\mathcal{R}_Q$ of the range of Q.

Remarks. (1.) A necessary condition for definiteness of the quadratic form q_x in the Proposition is the strong nondegeneracy of the biquadratic map Q.

(2.) With notation as in the Proposition, if we already know that the point x lies in the range $\mathcal{R}_Q$, then assertions (*ii*) and (*iii*) provide conditions that can be used to decide whether x belongs to the interior or the boundary of the range, respectively, based on the indefiniteness or weak semidefiniteness of the quadratic form q_x. If we also know that Q is strongly nondegenerate, then the conditions are much simplified.

Proof of Proposition A.2. At the outset, note that the line $\mathfrak{l}_x$ and the two disjoint, open half-planes Π_x^+ and Π_x^- whose union is the complement of $\mathfrak{l}_x$ can be described as the sets of $(y_1, y_2) \in \mathbb{R}^2$ for which $x_2 y_1 - x_1 y_2$ is zero, positive, and negative, respectively.

(*i*). In view of the characterizations just cited, we see that q_x is weakly positive-definite [weakly negative-definite], *i.e.*, that $q_x(\xi) > 0$ $[< 0]$ whenever $\xi \in \mathbb{R}^N$ and $Q(\xi) \neq 0$, iff $Q(\xi)$ lies in Π_x^+ [in Π_x^-] whenever $\xi \in \mathbb{R}^N$ and $Q(\xi)$ is not the origin in $\mathbb{R}^2$, *i.e.*, iff $\mathcal{R}_Q \setminus \{0\}$ is contained in Π_x^+ [in Π_x^-]. This establishes the first statement of (*i*). But if Q is strongly nondegenerate, then it is easy to check that q_x is weakly definite iff it is definite, so the second assertion of (*i*) follows immediately from the first.

(*ii*). Assume that $\mathcal{R}_Q$ is not a line through the origin. Suppose first that q_x is indefinite. Let ξ^-, $\xi^+ \in \mathbb{R}^N$ with $q_x(\xi^-) < 0$ and $q_x(\xi^+) > 0$. Since $Q(\xi^-) \in \Pi_x^-$ and $Q(\xi^+) \in \Pi_x^+$ are then nonzero, there are two rays emanating from $0 \in \mathbb{R}^2$ (excluding 0) passing through these two points and lying in Π_x^- and Π_x^+, respectively; these rays also lie in the range of the biquadratic map Q. The two rays (with 0) bound two cones in $\mathbb{R}^2$, one of which contains $\mathfrak{l}_x^-$ in its interior while the other contains $\mathfrak{l}_x^+$ in its interior. If one of the vertex angles of the cones is less than π radians, then the corresponding cone is convex and so must be contained in the convex cone $\mathcal{R}_Q$ (since that cone's boundary is in $\mathcal{R}_Q$). Therefore, one of $\mathfrak{l}_x^-$ and $\mathfrak{l}_x^+$ belongs, in this first case, to $\{\mathcal{R}_Q\}^\circ$. On the other hand, if each cone has vertex angle π radians, *i.e.*, if each is a closed half-plane, then the range must be one of those half-planes or the full plane, since it is, by hypothesis, not the line formed by the two rays in this case. In either of the latter cases, again we conclude that at least one of $\mathfrak{l}_x^-$ and $\mathfrak{l}_x^+$ is in $\{\mathcal{R}_Q\}^\circ$. Conversely, suppose that at least one of the rays $\mathfrak{l}_x^-$ and $\mathfrak{l}_x^+$ is contained in $\{\mathcal{R}_Q\}^\circ$. Let z lie on this ray (so $z \neq 0$) and let the disc $\mathrm{B}_\delta^2(z)$ be contained in $\mathcal{R}_Q$; this disc certainly meets both Π_x^- and Π_x^+. But then, for $\xi^- \in \mathbb{R}^2$ such that $Q(\xi^-) \in \mathrm{B}_\delta^2(z) \cap \Pi_x^-$, we have $q_x(\xi^-) < 0$, while for $\xi^+ \in \mathbb{R}^2$ such that $Q(\xi^+) \in \mathrm{B}_\delta^2(z) \cap \Pi_x^+$, we have $q_x(\xi^+) > 0$, and we conclude that q_x is indefinite.

To complete the proof of (ii), we assume that Q is strongly nondegenerate and show that its range cannot be a line through the origin. Suppose, to the contrary, that $\mathcal{R}_Q$ is just a line through the origin, say, that line determined by 0 and the nonzero point $z \equiv \big(z_1, z_2\big)$. Then we have

$$z_2 Q_1(\xi) - z_1 Q_2(\xi) = 0 \qquad \text{for every} \quad \xi \in \mathbb{R}^N \tag{A.4}$$

(which just says that the range lies *in* the line). Now, if $z_1 = 0$, then Q_1 is the zero-form and Q_2 must be indefinite; we find that $Q(\xi) = 0$ for any nonzero ξ such that $Q_2(\xi) = 0$, which is impossible, since it violates the strong nondegeneracy of Q. Thus, z_1 can only be nonzero, so that the slope of $\mathcal{R}_Q$ must be defined and Q_1 must be indefinite. But then we get $Q(\xi) = 0$ for any nonzero ξ such that $Q_1(\xi) = 0$, since (A.4) shows that also $Q_2(\xi) = \big(z_2/z_1\big)Q_1(\xi) = 0$ for such a ξ, so we are again forced to a violation of the assumed strong nondegeneracy of Q. Therefore, the range of a strongly nondegenerate biquadratic map cannot be a line.

(iii). Assume first that the form q_x is weakly positive-semidefinite. That is, suppose that $q_x(\xi) \geq 0$ for every $\xi \in \mathbb{R}^N$, so that the range $\mathcal{R}_Q$ lies in the closure $\overline{\Pi_x^+}$, while there is some $\xi^0 \in \mathbb{R}^N$ with $q_x(\xi^0) = 0$ but $Q(\xi^0) \neq 0$. Clearly, $Q(\xi^0)$ belongs then to one of the rays $\mathfrak{l}_x^-$ and $\mathfrak{l}_x^+$, so that entire ray is as well contained in the range. Moreover, the latter ray must also be in the boundary $\partial\mathcal{R}_Q$, since the range itself is contained in $\overline{\Pi_x^+}$. Thus, the implication claimed in (iii) holds in this case; the argument is not essentially different when q_x is negative-semidefinite. Conversely, suppose that at least one of $\mathfrak{l}_x^-$ and $\mathfrak{l}_x^+$ lies in $\mathcal{R}_Q \cap \partial\mathcal{R}_Q$. Then $q_x(\xi^0) = 0$ for some ξ^0 with $Q(\xi^0)$ contained in either $\mathfrak{l}_x^-$ or $\mathfrak{l}_x^+$; obviously, $Q(\xi^0) \neq 0$ for any such ξ^0. Further, since $\mathcal{R}_Q$ is a convex cone with vertex at the origin, it must lie in one of the closed half-planes $\overline{\Pi_x^-}$ and $\overline{\Pi_x^+}$, so we also have either $q_x(\xi) \geq 0$ for every $\xi \in \mathbb{R}^N$ or $q_x(\xi) \leq 0$ for every $\xi \in \mathbb{R}^N$, and we conclude that q_x is indeed weakly semidefinite. This establishes the first statement of (iii). For the second statement, we argue just as in the proof of (i): if Q is strongly nondegenerate, then q_x is weakly semidefinite iff it is semidefinite, so that the second assertion of (iii) is just a consequence of the first. $\square$

In general, singular values of a map from a subset of one euclidean space to another need not lie on the boundary of the range. However, there is at least one special case of interest in which we are assured that "almost every" singular value belongs to the boundary of the range—and the interior of the range comprises regular values with perhaps one exception:

Proposition A.3. *Let $\mathcal{Q} : \mathbb{R}^2 \to \mathbb{R}^2$ be a biquadratic map; denote the matrices of the components of $\mathcal{Q}$ by $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, so that*

$$\mathcal{Q}(\xi) = \begin{pmatrix} \mathcal{Q}_1(\xi) = \big(\mathcal{A}^{(1)}\xi\big)\cdot\xi \\ \\ \mathcal{Q}_2(\xi) = \big(\mathcal{A}^{(2)}\xi\big)\cdot\xi \end{pmatrix}, \qquad \text{for} \quad \xi \in \mathbb{R}^2 .$$

- $(i.)$ *Let $\xi^0 \in \mathbb{R}^2$ with $\mathcal{Q}(\xi^0) \neq 0$ and suppose that $\mathcal{Q}(\xi^0)$ lies in the interior $\big\{\mathcal{R}_\mathcal{Q}\big\}^\circ$ of the range. Then ξ^0 is not a singular point of $\mathcal{Q}$.*
- $(ii.)$ *Let the quadratic form q on $\mathbb{R}^2$ be defined by*

$$q(\xi) := \big(\mathcal{A}^{(1)}\xi\big)_1\big(\mathcal{A}^{(2)}\xi\big)_2 - \big(\mathcal{A}^{(1)}\xi\big)_2\big(\mathcal{A}^{(2)}\xi\big)_1, \qquad \text{for} \quad \xi \in \mathbb{R}^2 .$$

Then $q(\xi^0) = 0$ for some $\xi^0 \in \mathbb{R}^2$ iff ξ^0 is a singular point of $\mathcal{Q}$. Therefore, if $\xi^0 \in \mathbb{R}^2$ with $\mathcal{Q}(\xi^0) \neq 0$, then $q(\xi^0) = 0$ iff $\mathcal{Q}(\xi^0)$ belongs to the boundary $\partial\mathcal{R}_\mathcal{Q}$ of the range of $\mathcal{Q}$.

Proof. (i). Let $\xi^0 \in \mathbb{R}^2$ have the properties listed. Since $\operatorname{grad} \mathcal{Q}_l(\xi) = 2\mathcal{A}^{(l)}\xi$ for $l = 1, 2$, we see that ξ^0 is a singular point of $\mathcal{Q}$ iff the set $\{\mathcal{A}^{(1)}\xi^0, \mathcal{A}^{(2)}\xi^0\}$ is linearly dependent. $\mathcal{Q}(\xi^0)$ is contained in $\{\mathcal{R}_\mathcal{Q}\}^\circ$, so the range of $\mathcal{Q}$ is not a line through the origin. Therefore, since $\mathcal{Q}(\xi^0) \neq 0$, we can apply Proposition A.2.*ii* with $x = \mathcal{Q}(\xi^0)$ to conclude that the quadratic form $\xi \to \mathcal{Q}_2(\xi^0)\mathcal{Q}_1(\xi) - \mathcal{Q}_1(\xi^0)\mathcal{Q}_2(\xi)$ on $\mathbb{R}^2$ is indefinite. Then the (2×2) matrix of this form, which is just $\mathcal{Q}_2(\xi^0)\mathcal{A}^{(1)} - \mathcal{Q}_1(\xi^0)\mathcal{A}^{(2)}$, has one positive and one negative eigenvalue, so it cannot have eigenvalue zero, *i.e.*, must be nonsingular:

$$\mathcal{Q}_2(\xi^0)\mathcal{A}^{(1)}\xi - \mathcal{Q}_1(\xi^0)\mathcal{A}^{(2)}\xi \neq 0 \qquad \text{for each} \quad \xi \in \mathbb{R}^2, \quad \xi \neq 0.$$

Now suppose that $c_1\mathcal{A}^{(1)}\xi^0 + c_2\mathcal{A}^{(2)}\xi^0 = 0$ for some real c_1 and c_2. Then also $c_1\mathcal{Q}_1(\xi^0) + c_2\mathcal{Q}_2(\xi^0) = 0$, but we know that at least one of $\mathcal{Q}_1(\xi^0)$ and $\mathcal{Q}_2(\xi^0)$ is nonzero. Assume first that $\mathcal{Q}_2(\xi^0) \neq 0$: then $c_2 = -c_1\mathcal{Q}_1(\xi^0)/\mathcal{Q}_2(\xi^0)$, whence $c_1\Big(\mathcal{Q}_2(\xi^0)\mathcal{A}^{(1)}\xi^0 - \mathcal{Q}_1(\xi^0)\mathcal{A}^{(2)}\xi^0\Big) = 0$, which implies that $c_1 = 0$. But then $c_2\mathcal{Q}_2(\xi^0) = 0$, so also $c_2 = 0$. In a similar fashion, the assumption that $\mathcal{Q}_1(\xi^0) \neq 0$ again forces the conclusion that $c_1 = c_2 = 0$. The set $\{\mathcal{A}^{(1)}\xi^0, \mathcal{A}^{(2)}\xi^0\}$ is then in fact linearly independent, so ξ^0 is not a singular point of $\mathcal{Q}$.

(ii). Upon noting that $4q(\xi)$ is just the determinant of the matrix of the derivative $\mathrm{D}\mathcal{Q}(\xi)$ with respect to the standard bases, the first assertion of (ii) follows. Now suppose that $\xi^0 \in \mathbb{R}^2$ with $\mathcal{Q}(\xi^0) \neq 0$: if $\mathcal{Q}(\xi^0)$ belongs to $\partial\mathcal{R}_\mathcal{Q}$, then ξ^0 must be a singular point of $\mathcal{Q}$, so $q(\xi^0) = 0$; conversely, if the latter equality holds, then ξ^0 is a singular point of $\mathcal{Q}$ and statement (i) implies that $\mathcal{Q}(\xi^0)$ cannot belong to $\{\mathcal{R}_\mathcal{Q}\}^\circ$, so we must have $\mathcal{Q}(\xi^0) \in \partial\mathcal{R}_\mathcal{Q}$. □

Finally, we verify a simple result that is needed in the proof of Theorem 4.1.

Proposition A.4. *For each t in an open interval $I \subset \mathbb{R}$ let the biquadratic map $\mathcal{Q}_t$ on $\mathbb{R}^2$ be given by*

$$\mathcal{Q}_t(x) := \begin{pmatrix} a_t x_1^2 + 2b_t x_1 x_2 + c_t x_2^2 \\ \\ d_t x_1^2 + 2e_t x_1 x_2 + f_t x_2^2 \end{pmatrix}, \qquad \text{for} \quad x \equiv (x_1, x_2) \in \mathbb{R}^2,$$

in which the real coefficient-functions $t \mapsto a_t$, $t \mapsto b_t$, $\ldots$, $t \mapsto f_t$, are continuous on I. Then the set $\{\, t \in I \,|\, \mathcal{R}_{\mathcal{Q}_t} = \mathbb{R}^2 \,\}$ of values t in I at which the range of $\mathcal{Q}_t$ is all of $\mathbb{R}^2$ is open.

Proof. First consider any $t \in I$. As noted in the proof of Proposition A.1, the image of the unit circle in $\mathbb{R}^2$ under $\mathcal{Q}_t$ is either a proper ellipse, which obtains iff

$$(a_t - c_t)e_t - (d_t - f_t)b_t \neq 0, \tag{A.5}$$

or a degenerate ellipse, which occurs precisely when the expression on the left in (A.5) vanishes. It follows that the range $\mathcal{R}_{\mathcal{Q}_t}$ is all of $\mathbb{R}^2$ iff the image of the unit circle is a proper ellipse enclosing the origin. By supposing that (A.5) holds and transforming the equation of the image of the unit circle to the standard form

$$\tilde{A}\tilde{y}_1^2 + \tilde{C}\tilde{y}_2^2 = 1,$$

by translation and/or rotation of the original y_1, y_2-coördinate system to a $\tilde{y}_1, \tilde{y}_2$-system, one derives the necessary and sufficient condition for the ellipse to enclose the origin as

$$\tilde{A}\tilde{z}_1^2 + \tilde{C}\tilde{z}_2^2 < 1, \tag{A.6}$$

in which $(\tilde{z}_1, \tilde{z}_2)$ are the coördinates of the original origin in the transformed system. A bit of algebra produces the condition (A.6) as

$$\big(c_t d_t - a_t f_t\big)^2 + 4e_t^2 a_t c_t + 4b_t^2 d_t f_t - 4b_t e_t\big(a_t f_t + c_t d_t\big) < 0. \tag{A.7}$$

Thus, $\mathcal{R}_{\mathcal{Q}_t} = \mathbb{R}^2$ iff both (A.5) and (A.7) hold. But the set of $t \in I$ for which both (A.5) and (A.7) are satisfied is open, by the continuity of the coefficient-functions. □

Appendix B. Submanifolds of euclidean spaces.

To fix the definitions and recall the fundamental facts for the convenience of the reader, we include here a very compressed review of some basic developments and results concerning submanifolds of euclidean spaces. The presentation can be kept at the simplest level, since we do not require the general construction of differentiable manifolds. We consider here only manifolds without boundary; throughout, the term "manifold" should be understood to mean "manifold-without-boundary" (and similarly for "submanifold"). Essentially, we have augmented the presentation of FLEMING [9] with the most elementary information about mappings of one manifold into another.

First definitions. Let $1 \leq N < N_0$ and $q \geq 1$: a nonvoid subset $\mathcal{M}^N \subset \mathbb{R}^{N_0}$ is a *submanifold of class C^q and dimension N* iff for each $\xi \in \mathcal{M}^N$ there is an open $\mathbb{R}^{N_0}$-neighborhood U_ξ of ξ and a C^q-map $\Phi^\xi : U_\xi \to \mathbb{R}^{N_0 - N}$ such that the differential $\mathrm{D}\Phi^\xi(y) : \mathbb{R}^{N_0} \to \mathbb{R}^{N_0-N}$ has (maximal) rank $N_0 - N$ for each $y \in U_\xi$ and $\mathcal{M}^N \cap U_\xi = \{\, y \in U_\xi \mid \Phi^\xi(y) = 0 \,\}$. By an *$N_0$-dimensional submanifold of* $\mathbb{R}^{N_0}$ is meant simply an open subset of $\mathbb{R}^{N_0}$.

In the remainder of this Appendix, $\mathcal{M}^N$ denotes an N-dimensional submanifold of $\mathbb{R}^{N_0}$ of class (at least) C^1; for brevity, we shall say simply "N-manifold (in $\mathbb{R}^{N_0}$)." We may write just "$\mathcal{M}$" in place of "$\mathcal{M}^N$." By ξ we always denote a point of $\mathcal{M}$.

Tangent spaces. A nonzero N_0-vector τ is a *tangent vector to $\mathcal{M}^N$ at ξ* iff it is the derivative $\tau = \psi'(0)$ of a C^1-smooth map $\psi : (-\epsilon, \epsilon) \to \mathcal{M}^N$ with $\psi(0) = \xi$ (*i.e.*, of a "parametrized curve in $\mathcal{M}$" passing through ξ). The *tangent space $\mathsf{T}_\xi\mathcal{M}$ to $\mathcal{M}$ at ξ* is the collection of all tangent vectors to $\mathcal{M}$ at ξ, with the zero-vector adjoined. Then, with the neighborhood U_ξ and the map Φ^ξ as in the fundamental definition of "submanifold," it follows that $\mathsf{T}_\xi\mathcal{M}$ is just the null space of the differential $\mathrm{D}\Phi^\xi(\xi)$, and so forms an N-dimensional subspace of $\mathbb{R}^{N_0}$. The $(N_0 - N)$-dimensional orthogonal complement of $\mathsf{T}_\xi\mathcal{M}$ is termed *the normal space to $\mathcal{M}$ at ξ*; it is the linear span of the gradients $\{\operatorname{grad} \Phi_1^\xi(\xi), \ldots, \operatorname{grad} \Phi_{N_0-N}^\xi(\xi)\}$.

Coördinate systems. Let $\mathcal{M}^N$ be of class C^q $(q \geq 1)$. A *coördinate system in* $\mathcal{M}$ is a pair (U, h), with U a relatively open subset of $\mathcal{M}$ and $h : U \to \mathbb{R}^N$ a homeomorphism of U onto an open set $h(U) \subset \mathbb{R}^N$ having inverse $h^{-1} : h(U) \to U$ of class C^q with the differential $\mathrm{D}h^{-1}(z)$ of rank N for every $z \in h(U)$. When (U, h) is a coördinate system for $\mathcal{M}$, U is termed *a coördinate patch on* $\mathcal{M}$ and h *a coördinate function*; the coördinates of a point $\xi \in U$ relative to this coördinate system are just the numbers $h_1(\xi), h_2(\xi), \ldots, h_N(\xi)$. It follows that $\mathcal{M}$ is covered with coördinate systems, *i.e.*, that each point of $\mathcal{M}$ is contained in a coördinate patch. Moreover, whenever the patches of two systems (U, h) and $(\widetilde{U}, \tilde{h})$ overlap, the composition $\tilde{h} \circ h^{-1} : h(\widetilde{U} \cap U) \to \tilde{h}(\widetilde{U} \cap U)$ is a C^q-diffeomorphism. Finally, it is important to recall that, if $\xi \in \mathcal{M}$ and (U, h) is a coördinate system about ξ, *i.e.*, with ξ contained in the patch U, then the differential $\mathrm{D}h^{-1}(h(\xi))$ is a linear-space isomorphism of $\mathbb{R}^N$ onto the tangent space $\mathsf{T}_\xi\mathcal{M}$; in particular, the partial derivatives $\{h^{-1}_{,1}(h(\xi)), h^{-1}_{,2}(h(\xi)), \ldots, h^{-1}_{,N}(h(\xi))\}$ form a basis for $\mathsf{T}_\xi\mathcal{M}$.

Differentiable maps between manifolds. Differentiable manifolds provide all of the underlying structure necessary for introducing a reasonable property of "smoothness" for a map from one such manifold into another. Let $F : \mathcal{M}^N \to \mathcal{N}^n$ be a mapping carrying the N-dimensional submanifold $\mathcal{M}^N$ of $\mathbb{R}^{N_0}$ into the n-dimensional submanifold $\mathcal{N}^n$ of $\mathbb{R}^{n_0}$; both submanifolds are supposed to be of class C^q. F is said to be *of class C^q* iff whenever (U, h) is a coördinate system in $\mathcal{M}^N$ and (V, k) is a coördinate system in $\mathcal{N}^n$, the composition $k \circ F \circ h^{-1}$ is of class C^q where it is defined. Supposing that F is of class C^1, the *differential* $\mathrm{D}F(\xi)$ *of F at ξ* is defined to be the linear operator $\mathrm{D}F(\xi) : \mathsf{T}_\xi\mathcal{M} \to \mathsf{T}_{F(\xi)}\mathcal{N}$ acting between the indicated tangent spaces according to the following rule:

let $\tau \in \mathsf{T}_\xi\mathcal{M}$ and choose any coördinate system (U,h) about ξ for $\mathcal{M}$ and any coördinate system (V,k) about $F(\xi)$ for $\mathcal{N}$; then we can write $\tau = \sum_{p=1}^N \tau^p h^{-1}_{\,,\,p}\big(h(\xi)\big)$, with uniquely determined expansion coefficients $\{\tau^p\}_{p=1}^N$ relative to the basis $\{h^{-1}_{\,,\,p}\big(h(\xi)\big)\}_{p=1}^N$ for $\mathsf{T}_\xi\mathcal{M}$ (the so-called "contravariant components" of τ), and we define $\mathrm{D}F(\xi)\tau \in \mathsf{T}_{F(\xi)}\mathcal{N}$ by

$$\mathrm{D}F(\xi)\tau := \sum_{l=1}^n \Bigg\{ \sum_{q=1}^N \big(k\circ F\circ h^{-1}\big)_{l\,,\,q}\big(h(\xi)\big)\tau^q \Bigg\} k^{-1}_{\,,\,l}\Big(k\big(F(\xi)\big)\Big).$$

One can show that the differential is independent of the particular coördinate systems (U,h) and (V,k) about ξ and $F(\xi)$, respectively. The *rank* of the C^1-map $F : \mathcal{M}^N \to \mathcal{N}^n$ at $\xi \in \mathcal{M}^N$ is naturally defined to be the rank of the linear operator $\mathrm{D}F(\xi) : \mathsf{T}_\xi\mathcal{M}^N \to \mathsf{T}_{F(\xi)}\mathcal{N}^n$; from the definition given above, it is clear that this coincides with the rank of the differential $\mathrm{D}\big(k\circ F\circ h^{-1}\big)\big(h(\xi)\big)$, *i.e.*, with the rank of the $n\times N$ matrix with elements $\big(k\circ F\circ h^{-1}\big)_{l\,,\,q}\big(h(\xi)\big)$, independently of the coördinate systems selected.

Of special importance for present purposes in the study of Pareto optima is the setting $F : \mathcal{M}^N \to \mathbb{R}^n$, *i.e.*, when the codomain n-manifold is just $\mathbb{R}^n$. That case is also simpler to treat, since we can use the coördinate system comprising $\mathbb{R}^n$ itself as coördinate patch along with the identity map as coördinate function. It is useful to introduce first the *manifold gradient* $\mathrm{grad}_{\mathcal{M}} f$ for a *real-valued* function $f : \mathcal{M} \to \mathbb{R}$ of class C^1. Supposing that f is such a function and $\xi \in \mathcal{M}$, then the $\mathbb{R}^{N_0}$-vector $\mathrm{grad}_{\mathcal{M}} f(\xi)$ is determined by the requirement that the differential $\mathrm{d}f(\xi) \equiv \mathrm{D}f(\xi) : \mathsf{T}_\xi\mathcal{M} \to \mathbb{R}$ be given by

$$\mathrm{d}f(\xi)\tau = \mathrm{grad}_{\mathcal{M}} f(\xi)\cdot\tau \qquad \text{for each} \quad \tau \in \mathsf{T}_\xi\mathcal{M};$$

the inner product here takes place in the containing euclidean space $\mathbb{R}^{N_0}$. A short computation produces the expression

$$\mathrm{grad}_{\mathcal{M}} f(\xi) = \sum_{j,l=1}^N g^{jl}\big(h(\xi)\big)\big(f\circ h^{-1}\big)_{,\,j}\big(h(\xi)\big)h^{-1}_{\,,\,l}\big(h(\xi)\big),$$

in which (U,h) is any coördinate system about ξ for $\mathcal{M}$ and the matrix $\{g^{jl}\big(h(\xi)\big)\}_{N\times N}$ is the inverse of the matrix $\{g_{jl}\big(h(\xi)\big) := h^{-1}_{\,,\,j}\big(h(\xi)\big)\cdot h^{-1}_{\,,\,l}\big(h(\xi)\big)\}_{N\times N}$ (the first fundamental, or metric, tensor of the manifold $\mathcal{M}$ at ξ). One can also show that the manifold gradient can be computed as the orthogonal projection in $\mathbb{R}^{N_0}$ onto $\mathsf{T}_\xi\mathcal{M}$ of the ordinary gradient $\mathrm{grad}\,\widetilde{f}(\xi)$, in which $\widetilde{f}$ is any C^1-extension of f from a $\mathsf{T}_\xi\mathcal{M}$-neighborhood of ξ to a full $\mathbb{R}^{N_0}$-neighborhood of ξ.

Returning to a C^1-map $F \equiv \big(F_1,\dots,F_n\big) : \mathcal{M}^N \to \mathbb{R}^n$, we obtain a representation for $\mathrm{D}F(\xi)$ in terms of the manifold gradients of the components of F. In fact, let $\tau \in \mathsf{T}_\xi\mathcal{M}$, choose any coördinate system (U,h) for $\mathcal{M}$ about ξ, and expand $\tau = \sum_{p=1}^N \tau^p h^{-1}_{\,,\,p}\big(h(\xi)\big)$; denoting by $\varepsilon^{(l)}$, $l = 1,\dots,n$, unit basis vectors in $\mathbb{R}^n$, we get

$$\mathrm{D}F(\xi)\tau = \sum_{l=1}^n \Bigg\{ \sum_{q=1}^N \big(F_l\circ h^{-1}\big)_{,\,q}\big(h(\xi)\big)\tau^q \Bigg\}\varepsilon^{(l)} = \sum_{l=1}^n \Big\{ \mathrm{d}F_l(\xi)\tau \Big\}\varepsilon^{(l)} = \sum_{l=1}^n \Big\{ \mathrm{grad}_{\mathcal{M}} F_l(\xi)\cdot\tau \Big\}\varepsilon^{(l)}.$$

From these expressions, one can show that the rank of the operator $\mathrm{D}F(\xi)$, *i.e.*, the rank of F at ξ, is also given either by the rank of $F\circ h^{-1}$ at $h(\xi)$, *i.e.*, by the rank of the matrix $\{\big(F_l\circ h^{-1}\big)_{,\,q}\big(h(\xi)\big)\}_{n\times N}$, or by the maximal number of linearly independent manifold gradients in $\{\mathrm{grad}_{\mathcal{M}} F_l(\xi)\}_{l=1}^n$, or by the rank of the Gram matrix $\{\mathrm{grad}_{\mathcal{M}} F_j(\xi)\cdot\mathrm{grad}_{\mathcal{M}} F_l(\xi)\}_{n\times n}$ of the manifold gradients.

Finally, we recall that the chain rule holds in its familiar form for compositions of differentiable maps between manifolds. In the most important case for present considerations, let $F : \mathcal{M}^N \to \mathbb{R}^n$ and $\chi : (t_1, t_2) \to \mathcal{M}^N$ be C^1-maps. Then the composition $F \circ \chi : (t_1, t_2) \to \mathbb{R}^n$, a curve in $\mathbb{R}^n$, is of class C^1, with derivative given by

$$\big(F \circ \chi\big)'(t) = \mathsf{D}F\big(\chi(t)\big)\chi'(t) = \sum_{l=1}^{n} \Big\{ \operatorname{grad}_{\mathcal{M}} F_l\big(\chi(t)\big) \cdot \chi'(t) \Big\} \varepsilon^{(l)} \qquad \text{for} \quad t_1 < t < t_2 .$$

We observe that the curve is *smooth* at the point $\big(F \circ \chi\big)(t)$, *i.e.*, possesses a tangent vector at that point, iff $\chi'(t)$ does not belong to the nullspace $\mathcal{N}\big(\mathsf{D}F\big(\chi(t)\big)\big)$.

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