

**Quadratic Functionals
and Small Ball Probabilities
for the m-fold Integrated Brownian Motion**

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Abstract: Let the Gaussian process $X_m(t)$ be the m-fold integrated Brownian motion for positive integer m . The Laplace transform of the quadratic functional of $X_m(t)$ is found by using an appropriate self-adjoint integral operator. The result is then used to show the power of a general connection between small ball probabilities for Gaussian process. The connection is discovered by introducing an independent random shift. Various interplay between our results and principal eigenvalues for non-uniform elliptic generators on an unbounded domain are discussed.

Abbreviated title: The m-fold Integrated Brownian Motion

Keywords: The m-fold Integrated Brownian Motion, quadratic functionals, small ball probabilities, principal eigenvalues.

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1 Introduction

Let $W(t)$, $t \geq 0$, be the standard Brownian motion starting at 0. Denote by $X_0(t) = W(t)$ and

$$X_m(t) = \int_0^t X_{m-1}(s) ds, \quad t \geq 0, \quad m \geq 1.$$

the m -fold integrated Brownian motion for positive integer m . Upon integration by parts we also have the representation

$$X_m(t) = \frac{1}{m!} \int_0^t (t-s)^m dW(s), \quad m \geq 0.$$

The Gaussian process $X_m(t)$ has been studied from various points of view in Shepp (1966), Wahba (1978), Cheleyat-Maurel and Elie (1981), Lachal (1997a, 1997b, 2001) and Lin (2001). It is clear that the \mathbb{R}^{m+1} valued process $(X_0(t), X_1(t), \dots, X_m(t))$ is Markov and the transition density can be found easily by computing expectation and covariances of the Gaussian process involved, see McKean (1963) for $m = 1$ and Lachal (1997) for $m \geq 1$. Its generator is given in (5.1). In particular, when $m = 1$, the process $(X_0(t), X_1(t)) = (W(t), \int_0^t W(s) ds)$ is often called the Kolmogorov diffusion since its study was apparently initiated by Kolmogorov (1934), see Lachal (1998) and Groeneboom, Jongbloed and Wellner (1999) for other related works and references in this case.

Other significant use of X_m , as $m \rightarrow \infty$, can be found in Li and Shao (2002) where a limiting representation is given for the decay exponent of the probability that a random polynomial has no real root.

This work is concerned with the quadratic functionals of the real process $X_m(t)$ as well as the asymptotic behavior of small ball probability. In particular, we developed a method of estimating small ball probability such as $\mathbb{P}(\sup_{0 \leq t \leq 1} |X_m(t)| \leq \varepsilon)$ as $\varepsilon \rightarrow 0$ via the appropriated quadratic functionals. This is our primary motivation for studying the quadratic functionals for $X_m(t)$, beside its own importance and various connections discussed in Section 5.

Another motivation for studying the quadratic functionals and small ball probabilities for $X_m(t)$ originated in a study of Khoshnevisan and Shi (1998) for the liminf behavior of the integrated Brownian motion $X_1(t)$. In particular, they showed that

$$\mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^1 X_1^2(s) ds\right) = \left(\frac{2}{\cosh^2 \sqrt{\theta/2} + \cos^2 \sqrt{\theta/2}}\right)^{1/2} \quad (1.1)$$

and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{2/3} \log \mathbb{P}\left(\sup_{0 \leq t \leq 1} |X_1(t)| \leq \varepsilon\right) = -\kappa_1 \quad (1.2)$$

with

$$3/8 \leq \kappa_1 \leq 2c^2 \quad (1.3)$$

where the constant $c > 0$ satisfies

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/3} \log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |S(t)| \leq \varepsilon \right) = -c \quad (1.4)$$

and $S(t)$ is a symmetric (1/3)-stable process.

In this paper, we first find the Laplace transform of the quadratic functional of $X_m(t)$, i.e

$$\mathbb{E} \exp \left(-\lambda \int_0^1 X_m^2(t) dt \right), \quad m \geq 0. \quad (1.5)$$

for any $\lambda > 0$. This may look easy and seems possible to find it by using standard Karhunen-Loève expansion for Gaussian quadratic functionals. Indeed we can find (1.1) and the well known case $m = 0$,

$$\mathbb{E} \exp \left\{ -\lambda \int_0^1 W(s)^2 ds \right\} = \left(\cosh \sqrt{2\lambda} \right)^{-1/2}$$

by the Karhunen-Loève expansion. However, we could not carry out the detailed and complicated calculations along that line since the eigenvalues can not be found explicitly in a reasonable form. In fact, we see from our formula for (1.5) given in Theorem 2.1 that the two cases $m = 0$ and $m = 1$ are indeed very special and hence partially explain why the standard Karhunen-Loève expansion approach does not work well in the general case. On the other hand, the proof of (1.1) given in Khoshnevisan and Shi (1998) is based on a stochastic Fubini argument and then some special facts on the rotation invariance of Brownian motion together with the explicit formula

$$\mathbb{E} \exp \left\{ i\lambda \int_0^1 W(t)W(1-t) dt \right\} = \left(\frac{1+i}{\cosh \sqrt{\lambda} + i \cos \sqrt{\lambda}} \right)^{1/2} \quad (1.6)$$

given in Klyachko and Solodyannikov (1987). In our evaluation of (1.5) given in Theorem 2.1 in Section 2, we begin with a stochastic Fubini argument which we have learned from Donati-Martin and Yor (1991), see also Yor (1992) and Khoshnevisan and Shi (1998). Then we analyze an associated self-adjoint operator in a way different from the one used in Klyachko and Solodyannikov (1987) for (1.6). All the difficulties we encountered are due to the inherent difficulties of the problem.

Second, based on the exact formula for (1.5) given in Theorem 2.1, we obtain the following asymptotic result. The equivalence follows from de Bruijn's exponential Tauberian theorem given in Bingham et al. (1987), Theorem 4.12.9, see also Section 3.3 in the survey of Li and Shao (2001).

Theorem 1.1 *For each integer $m \geq 0$,*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1/(2m+2)} \log \mathbb{E} \exp \left\{ -\lambda \int_0^1 X_m^2(t) dt \right\} = -2^{-(2m+1)/(2m+2)} \left(\sin \frac{\pi}{2m+2} \right)^{-1} \quad (1.7)$$

and thus equivalently, as $\varepsilon \rightarrow 0$

$$\log \mathbb{P} \left(\int_0^1 X_m^2(t) dt \leq \varepsilon^2 \right) \sim -2^{-1}(2m+1) \left((2m+2) \sin \frac{\pi}{2m+2} \right)^{-(2m+2)/(2m+1)} \varepsilon^{-2/(2m+1)}. \quad (1.8)$$

Note that by the scaling relation $X_m(ct) = c^{(2m+1)/2}X_m(t)$ in law for any $c > 0$, we have for any $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ -\lambda \int_0^t X_m^2(s) ds \right\} = -2^{-(2m+1)/(2m+2)} \left(\sin \frac{\pi}{2m+2} \right)^{-1} \lambda^{1/(2m+2)}.$$

The estimates given in (1.8) are small ball probabilities for Gaussian process $X_m(t)$ under the L_2 -norm. More general, the small ball probability (or small deviation) studies the behavior of

$$\log \mathbb{P} (\|X\| \leq \varepsilon) \tag{1.9}$$

for a random process under various norm $\|\cdot\|$ as $\varepsilon \rightarrow 0$. In the last few years there have been considerable progress on the small ball estimate for Gaussian processes. As it was established in Kuelbs and Li (1993) (see also Li and Linde (1999) for further developments), the behavior of (1.9) for Gaussian random element X is determined up to a constant by the metric entropy of the unit ball of the reproducing kernel Hilbert space associated with X , and vice versa. For other connections and applications of small ball probabilities, we refer readers to a recent survey paper by Li and Shao (2001). And in addition, various connections and interplays between our results and principal eigenvalues are discussed in Section 5 for degenerated (non-uniform elliptic) generators on an unbounded domain. This provides a concrete and relative simple example that is not covered by general theory as far as we know. We hope this special example, a \mathbb{R}^{m+1} -valued Gaussian and Markov process, can attract some interests in the direction of this paper.

The third and probably the most important contribution in this paper is the following general connection between small ball probabilities. It can be used to estimate small ball probabilities under any norm via a relative easier L_2 -norm estimate.

Let X and Y be any two centered Gaussian random vectors in a separable Banach space E with norm $\|\cdot\|$. We recall the definition of the reproducing kernel Hilbert space of $\mu = \mathcal{L}(X)$. Since $\int f^2(x)\mu(dx) < +\infty$ for each $f \in E^*$, where E^* is the topological dual of E , one can define the map $\phi: E^* \rightarrow E$ by

$$\phi(f) = \int_E xf(x)\mu(dx) \quad f \in E^*$$

where the integral is defined as Bochner integral. The image space $\phi(E^*)$ becomes an inner product space under the inner product

$$\langle x, y \rangle_\mu = \int f(x)g(x)\mu(dx)$$

where $f, g \in E^*$ satisfy $x = \phi(f)$ and $y = \phi(g)$. We use $|\cdot|_\mu$ to denote the inner product norm and call the completion H_μ of $\phi(E^*)$ (under the norm $|\cdot|_\mu$) the reproducing kernel Hilbert space of $\mu = \mathcal{L}(X)$.

Theorem 1.2 For any $\lambda > 0$ and $\varepsilon > 0$,

$$\mathbb{P}(\|Y\| \leq \varepsilon) \geq \mathbb{P}(\|X\| \leq \lambda\varepsilon) \cdot \mathbb{E} \exp\{-2^{-1}\lambda^2|Y|_\mu^2\}. \quad (1.10)$$

Note that we need $Y \in H_\mu \subset E$ almost surely. Otherwise for $f \notin H_\mu$, $|f|_\mu = \infty$ and the result is trivial. Thus the result can also be stated as follows. Let $(H, |\cdot|_H)$ be a Hilbert space and Y be a Gaussian vector in H . Then for any linear operator $L : H \rightarrow E$ and the Gaussian vector X in E with covariance operator LL^*

$$\mathbb{P}(\|LY\| \leq \varepsilon) \geq \mathbb{P}(\|X\| \leq \lambda\varepsilon) \cdot \mathbb{E} \exp\{-2^{-1}\lambda^2|Y|_H^2\}$$

for any $\lambda > 0$ and $\varepsilon > 0$.

The proof of Theorem 1.2 is based on both directions of the well known shift inequalities (4.1). The key new idea is the use of an independent random shift. To see the power of Theorem 1.2, we have the following consequence of Theorem 1.1 and 1.2 for the sup-norm. Other norms such as Hölder norm and L_p -norm can also be considered similarly, but we omit the exact statements, see remarks on (4.5) in Section 4 for more details.

Theorem 1.3 We have for $m \geq 1$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |X_m(t)| \leq \varepsilon \right) = -\kappa_m \quad (1.11)$$

with

$$\frac{2m+1}{2} \left((2m+2) \sin \frac{\pi}{2m+2} \right)^{-\frac{2m+2}{2m+1}} \leq \kappa_m \leq \frac{2m+1}{2} \left(\frac{\pi}{2} \right)^{\frac{2}{2m+1}} \left(2m \sin \frac{\pi}{2m} \right)^{-\frac{2m}{2m+1}} \quad (1.12)$$

and $\lim_{m \rightarrow \infty} m^{-1}\kappa_m = \pi^{-1}$. In addition, the following upper bound holds for κ_m based on κ_{m-1} .

$$(4\kappa_m/(2m+1))^{2m+1} \leq (4\kappa_{m-1}/(2m-1))^{2m-1}. \quad (1.13)$$

The particular case of $m = 1$, or the so called integrated Brownian motion given in (1.2) and (1.3), was studied in Khoshnevisan and Shi (1998) by using local time techniques. Here our upper bound $\kappa_1 \leq (2\pi)^{2/3} \cdot (3/8)$ is explicit. In general, the exact values of κ_m in (1.11) is unknown for $m \geq 1$. The value $\kappa_0 = \pi^2/8$ is well known.

Finally, we mention the following easy application of Theorem 1.1 and 1.3 for the liminf behavior of $X_m(t)$. The proof is standard and similar to that for $X_1(t)$ outlined in Khoshnevisan and Shi (1998) and we omit the details in this paper.

Theorem 1.4 *We have with probability one*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \left(\frac{\log \log t}{t} \right)^{2m+1} \int_0^t |X_m(s)|^2 ds = \left(\frac{2m+1}{2} \right)^{2m+1} \left((2m+2) \sin \frac{\pi}{2m+2} \right)^{-(2m+2)}$$

and

$$\liminf_{t \rightarrow \infty} \left(\frac{\log \log t}{t} \right)^{(2m+1)/2} \sup_{s \leq t} |X_m(s)| = \kappa_m^{(2m+1)/2}$$

where κ_m is given in Theorem 1.3.

The rest of the paper is organized as follows. In Section 2, we find the exact formula for (1.5) given as Theorem 2.1. In Section 3, we present the proof of Theorem 1.1 which requires detailed asymptotic analysis. The proofs of Theorem 1.2 and Theorem 1.3 are presented in Section 4. Finally, we discuss some interplays between our results and principal eigenvalue problems in Section 5 and there is a clear need for further research.

2 Exact Laplace transform via a stochastic Fubini argument

To begin with, we set for real number $\theta > 0$

$$\phi(\theta) = \mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^1 X_m^2(t) dt\right). \quad (2.1)$$

Let us introduce the Cameron-Martin space

$$H = \left\{ f \in L^2(0, 1) : f' \in L^2(0, 1) \quad \text{and} \quad \forall t \in (0, 1), f(t) = \int_0^t f'(s) ds \right\}$$

together with the inner product defined for any $f, g \in H$ by $\langle f, g \rangle_H = \int_0^1 f'(t)g'(t)dt$. Set

$$f_m(t) = \int_0^t \frac{(t-s)^m}{m!} df(s) \quad \text{and} \quad g_m(t) = \int_0^t \frac{(t-s)^m}{m!} dg(s).$$

We have

$$\begin{aligned} \int_0^1 f_m(t)g_m(t)dt &= \int_0^1 \left(f_m(t) \int_0^t \frac{(t-s)^m}{m!} dg(s) \right) dt \\ &= \int_0^1 \left(\int_s^1 \frac{(t-s)^m}{m!} f_m(t) dt \right) dg(s) \\ &= \langle \mathcal{A}(f), g \rangle_H \end{aligned}$$

where

$$\mathcal{A}(f)(t) = \int_t^1 \frac{(u-t)^m}{m!} f_m(u) du = \int_0^1 k(s, t) df(s) \quad \text{and} \quad k(s, t) = \int_{s \vee t}^1 \frac{(u-s)^m}{m!} \frac{(u-t)^m}{m!} du.$$

Clearly, \mathcal{A} is a self-adjoint operator on H and

$$\phi(\theta) = \mathbb{E} \exp\left(-\frac{\theta^2}{2} \langle \mathcal{A}(W), W \rangle_H\right) = (\det(I + \theta^2 \mathcal{A}))^{-1/2} = \prod_{k \geq 1} (1 + \nu_k \theta^2)^{-1/2}$$

where the ν_k 's are the eigenvalues of the operator \mathcal{A} , defined by the integral equation $\mathcal{A}(\phi) = \nu\phi$. It is easily seen that this equation is equivalent to the boundary value problem, see Lachal (2001),

$$\begin{cases} \phi^{(2m+2)}(t) = \frac{(-1)^{m+1}}{\nu} \phi(t), & t \in (0, 1), \\ \phi^{(j)}(1) = 0, & 0 \leq j \leq m, \\ \phi^{(j)}(0) = 0, & m+1 \leq j \leq 2m+1. \end{cases}$$

It is difficult to evaluate explicitly the eigenvalues through this way for all m . Similarly, we can follow the approach studied in Chiang, Chow and Lee (1986, 1994) to the point of explicit evaluation of all eigenvalues above. To overcome the difficulties of finding all eigenvalues, we derive the Laplace transform by a different method.

To motivate and show how we find a different operator to work with, we introduce an auxiliary independent Brownian motion $\widetilde{W}(t)$. By the stochastic Fubini theorem mentioned in the introduction, we have

$$\begin{aligned} \phi(\theta) &= \mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^1 X_m^2(t) dt\right) = \mathbb{E} \exp\left(i\theta \int_0^1 X_m(t) d\widetilde{W}(t)\right) \\ &= \mathbb{E} \exp\left(i\theta \int_0^1 \int_0^t X_{m-1}(s) ds d\widetilde{W}(t)\right) \\ &= \mathbb{E} \exp\left(i\theta \int_0^1 X_{m-1}(s) (\widetilde{W}(1) - \widetilde{W}(s)) ds\right) \\ &= \mathbb{E} \exp\left(i\theta \int_0^1 X_{m-1}(s) \widetilde{W}(1-s) ds\right) \end{aligned}$$

since $\{\widetilde{W}(1) - \widetilde{W}(1-t); 0 \leq t \leq 1\}$ is Brownian motion independent of $\{X_{m-1}(t); 0 \leq t \leq 1\}$.

Next we can write the the convolution in the form

$$\begin{aligned} & \int_0^1 X_{m-1}(u) \widetilde{W}(1-u) du \\ &= \int_0^1 \int_0^u \frac{1}{(m-1)!} (u-s)^{m-1} dW(s) \int_0^{1-u} d\widetilde{W}(t) du \\ &= \int_0^1 \int_0^1 \frac{1}{(m-1)!} \left(\int_0^1 1_{(0,u)}(s) (u-s)^{m-1} 1_{(0,1-u)}(t) du \right) dW(s) d\widetilde{W}(t) \\ &= \int_0^1 \int_0^1 K(s, t) dW(s) d\widetilde{W}(t) \end{aligned}$$

where

$$K(s, t) = (m!)^{-1} \max(0, 1-s-t)^m = (m!)^{-1} (1-s-t)_+^m. \quad (2.2)$$

Now consider the compact continuous self-adjoint operator

$$A(f)(t) = \int_0^1 K(s, t)f(s)ds \quad (2.3)$$

and denote by λ_k and ϕ_k , $k \geq 1$ its eigenvalue and orthonormal eigenfunctions. Then

$$\int_0^1 \int_0^1 K(s, t)dW(s)d\widetilde{W}(t) = \sum_{k \geq 1} \lambda_k \xi_k \widetilde{\xi}_k$$

and

$$K(s, t) = \sum_{k \geq 1} \lambda_k \phi_k(s)\phi_k(t)$$

where $\xi_k = \int_0^1 \phi_k(t)dW(t)$ and $\widetilde{\xi}_k = \int_0^1 \phi_k(t)d\widetilde{W}(t)$ are independent standard normal random variables.

Thus we have by combining things together

$$\begin{aligned} \phi(\theta) &= \mathbb{E} \exp\left(i\theta \int_0^1 X_{m-1}(s)\widetilde{W}(1-s) ds\right) \\ &= \mathbb{E} \exp\left(i\theta \sum_{k \geq 1} \lambda_k \xi_k \widetilde{\xi}_k\right) \\ &= \prod_{k \geq 1} \mathbb{E} \exp\left(-2^{-1}\theta^2 \lambda_k^2 \xi_k^2\right) \\ &= \prod_{k \geq 1} (1 - i\theta \lambda_k)^{-1/2} (1 + i\theta \lambda_k)^{-1/2}. \end{aligned}$$

In fact, as pointed out by one of the referees, the operator A is exactly the square root of the operator \mathcal{A} and the above relation follows easily from $\nu_k = \lambda_k^2$, $\forall k \geq 1$. To be more precise, the kernel K_2 of the operator A^2 coincides with the kernel k of the operator \mathcal{A} , following from the relation

$$K_2(s, t) = \int_0^1 K(s, x)K(x, t)dx = k(s, t).$$

For the remaining of this section, we carry out the spectral study of K and the computations seem to be less ugly and more appropriate than that of k .

Let

$$\psi(\tau) = \prod_{k \geq 1} (1 - \tau \lambda_k).$$

Then

$$\frac{\phi'(\theta)}{\phi(\theta)} = -\frac{i}{2} \left(\frac{\psi'(i\theta)}{\psi(i\theta)} - \frac{\psi'(-i\theta)}{\psi(-i\theta)} \right) = \Im \left(\frac{\psi'(i\theta)}{\psi(i\theta)} \right) \quad (2.4)$$

where $\Im(\cdot)$ represents the imaginary part. On the other hand, $\psi(\tau)$ can be represented by the resolvent kernel

$$\Gamma(s, t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(s, t) = \sum_{k \geq 1} \frac{\lambda_k \phi_k(s)\phi_k(t)}{1 - \lambda \lambda_k} \quad (2.5)$$

of the operator A given in (2.3), where

$$K_n(s, t) = \int_0^1 \dots \int_0^1 K(s, x_1)K(x_1, x_2) \dots K(x_{n-1}, t)dx_1 \dots dx_{n-1}$$

are the iterated kernels of A . That is, we have

$$\frac{\psi'(\tau)}{\psi(\tau)} = - \sum_{k \geq 1} \frac{\lambda_k}{1 - \tau \lambda_k} = - \int_0^1 \Gamma(t, t; \tau) dt. \quad (2.6)$$

It remains for us to compute the resolvent kernel $\Gamma(s, t; \lambda)$ given in (2.5) and then use (2.6) to find $\psi(\tau)$. This indeed was the approach used to find the Laplace transform of $\int_0^1 W(t)W(1-t)dt$ in the paper by Klyachko and Solodyannikov (1987). But in our problem, we run into serious difficulties to find a reasonable expression for $\Gamma(s, t; \lambda)$ that is suitable to evaluate (2.5).

To overcome the difficulty, we observe from (2.6) that

$$\frac{\psi'(\tau)}{\psi(\tau)} = - \sum_{k \geq 1} \frac{\lambda_k}{1 - \tau \lambda_k} = - \sum_{n=0}^{\infty} \tau^n \int_0^1 K_{n+1}(t, t) dt. \quad (2.7)$$

We now compute the integrals on the right hand side. Recall that $K(s, t) = (m!)^{-1}(1-s-t)_+^m$ for $0 \leq s, t \leq 1$. It is convenient to make the substitution $y_i = 1/2 - u_i$ to obtain

$$\begin{aligned} & \int_0^1 K_{n+1}(u_0, u_0) du_0 \\ &= \int_0^1 \dots \int_0^1 K(u_0, u_1)K(u_1, u_2) \dots K(u_{n-1}, u_n)K(u_n, u_0) du_0 \dots du_n \\ &= (m!)^{-(n+1)} \int_D (y_0 + y_1)^m (y_1 + y_2)^m \dots (y_n + y_0)^m dy_0 dy_1 \dots dy_n \end{aligned}$$

where the region D is determined by the inequalities

$$|y_i| \leq 1/2, \quad y_i + y_{i+1} \geq 0, \quad 0 \leq i \leq n$$

with $y_{n+1} = y_0$. We can partition the region D into disjoint subsets $D_k, 0 \leq k \leq n$ so that $\max_{0 \leq i \leq n} y_i = y_k$ in D_k . That is, for $0 \leq k \leq n$

$$D_k = D \cap \{(y_1, \dots, y_n) : \max_{0 \leq i \leq n} y_i = y_k\}.$$

Note that the integral over all these regions are the same, and therefore it is sufficient to evaluate the integral over one of them, say D_0 . Using the substitutions $y_i = y_0 t_i, 1 \leq i \leq n$, we have

$$\begin{aligned} & \int_0^1 K_{n+1}(t, t) dt \\ &= (n+1)(m!)^{-(n+1)} \int_{D_0} (y_0 + y_1)^m (y_1 + y_2)^m \dots (y_n + y_0)^m dy_0 dy_1 \dots dy_n \end{aligned}$$

$$\begin{aligned}
&= (n+1)(m!)^{-(n+1)} \int_0^{1/2} y_0^{(m+1)(n+1)-1} dy_0 \cdot \\
&\quad \int_{|t_i| \leq 1; t_i + t_{i+1} \geq 0} (1+t_1)^m (t_1+t_2)^m \dots (t_n+1)^m dt_1 \dots dt_n \\
&= (m+1)^{-1} 2^{-(m+1)(n+1)} (m!)^{-(n+1)} P_{n+1}(1),
\end{aligned} \tag{2.8}$$

where $P_k(x)$, $k \geq 1$, is the polynomial defined by the recursion relations

$$P_1(x) = (1+x)^m, \quad P_{k+1}(x) = \int_{-x}^1 P_k(t)(t+x)^m dt. \tag{2.9}$$

Thus we need to find the generation function for the polynomials $P_k(x)$. It is given in the following Lemma along with other basic properties of polynomials $P_k(x)$, $k \geq 1$.

Lemma 2.1 1) $\deg P_k(x) = (m+1)k - 1$, $k \geq 1$.

$$2) P_{k+1}^{(l)}(x) = \frac{m!}{(m-l)!} \int_{-x}^1 P_k(t)(t+x)^{m-l} dt, \quad l = 0, 1, 2, \dots, m.$$

$$3) P_{k+1}^{(m+1)}(x) = m! P_k(-x), \quad k \geq 1.$$

4) $P_k^{(l)}(-1) = 0$, $l = 0, 1, \dots, m$. More generally, $P_k^{(q)}(-1) = 0$ for $q = 2j(m+1) + l$ and $P_k^{(q)}(1) = 0$ for $q = (2j+1)(m+1) + l$ where $0 \leq l \leq m$, $j \geq 0$.

5) The generation function for the polynomials $P_k(x)$ is

$$G(x, y) \equiv \sum_{k=1}^{\infty} P_k(x) y^{(m+1)k-1} = m! \sum_{l=0}^m (-1)^{m+l} F_l(xy) \cdot \frac{W(F_0, \dots, F_{l-1}, F_{l+1}, \dots, F_m)(-y)}{W(F_0, F_1, \dots, F_m)(-y)} \tag{2.10}$$

where the analytic functions $F_l(z)$ is given by

$$F_l(z) = \sum_{j=0}^{\infty} \frac{(-1)^{j+l+(m+1)(j-1)j/2} (m!)^j}{((m+1)j+l)!} z^{(m+1)j+l}, \quad 0 \leq l \leq m, \tag{2.11}$$

and the notation

$$W(f_0, f_1, \dots, f_m)(z) = \det \left(f_j^{(i)}(z) \right)_{0 \leq i, j \leq m}$$

is the Wronskian determinant for functions f_i , $0 \leq i \leq m$.

6) In particular, for any complex number z with $D(z) \neq 0$,

$$G(1, z) = \sum_{k=1}^{\infty} P_k(1) z^{(m+1)k-1} = m! \frac{\widehat{D}(z)}{D(z)}$$

where $D(z) = W(F_0, F_1, \dots, F_m)(-z)$ and, $\widehat{D}(z)$ is the determinant of the matrix obtained by replacing the last row of the matrix $\left(F_l^{(q)}(-z) \right)_{0 \leq q, l \leq m}$ with the row vector $((F_0(z), \dots, F_m(z)))$.

Proof: The first three properties follow from the recursive definition given in (2.9) by differentiation. Property 4) follows from Properties 2) and 3).

To show 5), we first need a recursive relation on the coefficients of $P_k(x)$ represented by

$$P_k(x) = \sum_{l=0}^{(m+1)k-1} a_{k,l} x^l = \sum_{j=0}^{k-1} \sum_{l=0}^m a_{k,(m+1)j+l} x^{(m+1)j+l}. \quad (2.12)$$

Substituting (2.12) into the relation of the property 3), we have after simplifying the expression and equating the coefficients,

$$a_{k+1,(m+1)(j+1)+l} = \frac{(-1)^{(m+1)j+l} ((m+1)j+l)! m!}{((m+1)(j+1)+l)!} a_{k,(m+1)j+l} \quad (2.13)$$

for any $k \geq 1$, $j \geq 0$ and $0 \leq l \leq m$.

Next note that

$$\begin{aligned} G(x, y) &= \sum_{k=1}^{\infty} P_k(x) y^{(m+1)k-1} \\ &= \sum_{l=0}^m \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} a_{k,(m+1)j+l} x^{(m+1)j+l} y^{(m+1)k-1} \\ &= \sum_{l=0}^m \sum_{j=0}^{\infty} Q_{j,l}(y) x^{(m+1)j+l} \end{aligned} \quad (2.14)$$

where

$$Q_{j,l}(y) = \sum_{k=j+1}^{\infty} a_{k,(m+1)j+l} y^{(m+1)k-1}.$$

Using the recursive relation in (2.13), we see that

$$\begin{aligned} Q_{j+1,l}(y) &= \sum_{k=j+1}^{\infty} a_{k+1,(m+1)(j+1)+l} y^{(m+1)(k+1)-1} \\ &= \sum_{k=j+1}^{\infty} \frac{(-1)^{(m+1)j+l} ((m+1)j+l)! m!}{((m+1)(j+1)+l)!} a_{k,(m+1)j+l} y^{(m+1)(k+1)-1} \\ &= \frac{(-1)^{(m+1)j+l} ((m+1)j+l)! m!}{((m+1)(j+1)+l)!} y^{m+1} \cdot Q_{j,l}(y) \end{aligned}$$

Thus by iterating the above relation, we obtain

$$Q_{j,l}(y) = \frac{(-1)^{jl+(m+1)(j-1)j/2} (m!)^j j!}{((m+1)j+l)!} y^{(m+1)j} \cdot Q_{0,l}(y). \quad (2.15)$$

Now back to the relation (2.14) for $G(x, y)$, we have

$$G(x, y) = \sum_{l=0}^m \sum_{j=0}^{\infty} \frac{(-1)^{jl+(m+1)(j-1)j/2} (m!)^j j!}{((m+1)j+l)!} y^{(m+1)j} Q_{0,l}(y) x^{(m+1)j+l}$$

$$\begin{aligned}
&= \sum_{l=0}^m F_l(xy) l! Q_{0,l}(y) \cdot y^{-l} \\
&= \sum_{l=0}^m F_l(xy) G_l(y)
\end{aligned} \tag{2.16}$$

where F_l is as defined in (2.11) and $G_l(y) = l! Q_{0,l}(y) y^{-l}$. Next we need to find $G_l(y)$, $0 \leq l \leq m$. It is not easy to find them via the generating function idea given above. But observe that we have from the definition of $G(x, y)$ in (2.10) and property 4),

$$\begin{cases} \frac{\partial^q}{\partial x^q} G(x, y) \Big|_{x=-1} = 0 & q = 0, 1, \dots, m-1 \\ \frac{\partial^m}{\partial x^m} G(x, y) \Big|_{x=-1} = m! y^m \end{cases}$$

which are from the relation in (2.16)

$$\begin{cases} \sum_{l=0}^m F_l^{(q)}(-y) \cdot G_l(y) = 0, & 0 \leq q \leq m-1 \\ \sum_{l=0}^m F_l^{(m)}(-y) G_l(y) = m!. \end{cases} \tag{2.17}$$

Since F_l are linear independent, we see that the Wronskian determinant $W(F_0, F_1, \dots, F_m)(-y) \neq 0$ and thus from (2.17)

$$G_l(y) = (-1)^{m+l} m! \cdot \frac{W(F_0, \dots, F_{l-1}, F_{l+1}, \dots, F_m)(-y)}{W(F_0, F_1, \dots, F_m)(-y)} \tag{2.18}$$

and we finishes the proof of the lemma.

Summarize what we have so far, (2.4), (2.7), (2.8) and Lemma 2.1, we obtain the following result for the Laplace transform of the quadratic functional of the m-fold integrated Brownian motion by observing

$$\frac{\phi'(\theta)}{\phi(\theta)} = \frac{1}{(m+1)\theta} \Im \left(i \sum_{k \geq 1} P_k(1) \left(\frac{i\theta}{m! 2^{m+1}} \right)^k \right)$$

and

$$\sum_{k \geq 1} P_k(x) y^k = y^{1/(m+1)} G(x, y^{1/(m+1)}).$$

Theorem 2.1 *Let $\phi(\theta)$ be given in (2.1) for $\theta > 0$ real. Then we have*

$$\phi'(\theta)/\phi(\theta) = (2m+2)^{-1} (m!/\theta)^{m/(m+1)} \Re \left\{ i^{1/(m+1)} \widehat{D} \left(2^{-1} (i\theta/m!)^{1/(m+1)} \right) / D \left(2^{-1} (i\theta/m!)^{1/(m+1)} \right) \right\}$$

where the notation $\Re(\cdot)$ is for the real part of a complex number, the functions $D(z)$ and $\widehat{D}(z)$ are defined in part 6) of Lemma 2.1.

Note that for the special case $m = 1$, we have $F_0(y) = \cosh y$, $F_1(y) = \sin y$, and by setting $a = 2^{-1} \sqrt{\theta/2}$,

$$\begin{aligned}
D(a(1+i)) &= (\cosh^2 a - \sin^2 a) - i(\cosh^2 a - \cos^2 a) \\
\widehat{D}(a(1+i)) &= \sin(2a) + i \sinh(2a).
\end{aligned}$$

Therefore Theorem 2.1 implies

$$\frac{\phi'(\theta)}{\phi(\theta)} = -\frac{1}{4\sqrt{2\theta}} \frac{\sinh \sqrt{2\theta} - \sin \sqrt{2\theta}}{\cosh^2 \sqrt{\theta/2} + \cos^2 \sqrt{\theta/2}}$$

from which (1.1) follows.

3 Proof of Theorem 1.1

It is clear that we need to obtain the asymptotic of $\frac{\phi'(\theta)}{\phi(\theta)}$ and hence $\log \phi(\theta)$ for $\theta \rightarrow \infty$. To this end, the following lemma for analytic functions over the complex with power series representation is critical in order to find the asymptotics of F_l , $W(F_0, F_1, \dots, F_m)(-y)$ and G_l . General results of this type can be found in Wright (1940).

Lemma 3.1 *For any integer $n \geq 0$ and q , and complex number z*

$$\sum_{jn+q \geq 0} \frac{1}{(jn+q)!} z^j = \frac{1}{n} \sum_{k=1}^n Z_k^{-q} e^{Z_k}$$

where

$$Z_k = |z|^{1/n} \exp \{i(2k\pi + \arg z)/n\} \quad k = 1, 2, \dots, n.$$

Consequently,

$$\sum_{jn+q \geq 0} \frac{1}{(jn+q)!} z^j = \frac{1}{n} \sum_{|(2k\pi + \arg z)/n| \leq \pi/2} Z_k^{-q} e^{Z_k} + o(\exp\{-\delta|z|^{1/n}\}) \quad (3.1)$$

as $|z| \rightarrow \infty$, where δ can be any positive number less than

$$\min \{ |\cos((2k\pi + \arg z)/n)|; \pi/2 < |(2k\pi + \arg z)/n| \leq \pi \}.$$

Proof. First note that for each integer j

$$\sum_{k=1}^n Z_k^j = \begin{cases} 0 & j \neq 0 \pmod{n} \\ nz^p & j = 0 \pmod{n} \text{ and } j = pn. \end{cases}$$

Hence

$$\sum_{k=1}^n Z_k^{-q} e^{Z_k} = \sum_{k=1}^n Z_k^{-q} \sum_{j=0}^{\infty} \frac{1}{j!} Z_k^j = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=1}^n Z_k^{j-q} = n \sum_{pn+q \geq 0} \frac{1}{(pn+q)!} z^p.$$

Proof of Theorem 1.1. By variable substitution and L'Hospital's principle, we only need to prove

$$\lim_{\theta \rightarrow \infty} \theta^{m/(m+1)} \frac{\phi'(\theta)}{\phi(\theta)} = - \left(2(m+1) \sin \frac{\pi}{2(m+1)} \right)^{-1}. \quad (3.2)$$

We apply Theorem 2.1. Note that for all $0 \leq q, l \leq m$,

$$F_l^{(q)}(z) = z^{l-q} \sum_{2(m+1)j+l-q \geq 0} \frac{(-1)^{j(m+1)}(m!)^{2j}}{[2(m+1)j+l-q]!} z^{2(m+1)j} \\ + (-1)^l z^{m+1+l-q} \sum_{2(m+1)j+m+1+l-q \geq 0} \frac{(-1)^{j(m+1)}(m!)^{2j+1}}{[2(m+1)j+m+1+l-q]!} z^{2(m+1)j}.$$

Let

$$I = \{k; |k + (1 - (-1)^m)/4| \leq (m+1)/2\}. \quad (3.3)$$

Then I contains exactly $m+1$ integers. Let

$$Z_k = \frac{1}{2} \theta^{1/(m+1)} \exp \left\{ i \frac{2k\pi + 2^{-1}(1 - (-1)^m)\pi}{2(m+1)} \right\} = \frac{1}{2} \theta^{1/(m+1)} \epsilon_m^k e^{i\pi(1 - (-1)^m)/4(m+1)} \quad k \in I$$

where

$$\epsilon_m = \exp \{ i\pi/(m+1) \}. \quad (3.4)$$

Applying (3.1) of Lemma 3.1 with $n = 2(m+1)$ and $z = (-1)^m(2^{-(m+1)}\theta)^2$ (hence $\arg z = (1 - (-1)^m)\pi/2$) we have after simplification that

$$F_l^{(q)}(-2^{-1}(i\theta/m!)^{1/(m+1)}) \\ = \frac{1}{2(m+1)} H_m^{q-l} \sum_{k \in I} \epsilon_m^{k(q-l)} [1 + (-1)^{m+k+l+1} i^{2^{-1}(1+(-1)^m)}] \exp\{Z_k\} + o(\exp\{-\delta\theta^{\frac{1}{m+1}}\}) \\ = f_{ql} + o(\exp\{-\delta\theta^{1/(m+1)}\}) \quad (\text{say})$$

as $\theta \rightarrow \infty$, where $\delta > 0$ is a constant and $H_m = -(m!)^{1/(m+1)} e^{-(1+(-1)^m)\pi/4(m+1)}$.

To estimate $D(2^{-1}(i\theta/m!)^{1/(m+1)})$, we break it into two terms according to the above decomposition. Since Z_k has a positive real part for each $k \in I$,

$$D(2^{-1}(i\theta/m!)^{1/(m+1)}) = \det(f_{ql})_{0 \leq q, l \leq m} + o\left(\exp\left\{\sum_{k \in I} Z_k\right\}\right).$$

We can also break each row in the determinant of the the first term on the right hand side, according to the decomposition

$$f_{ql} = \frac{1}{2(m+1)} H_m^{q-l} \sum_{k \in I} \epsilon_m^{k(q-l)} [1 + (-1)^{m+k+l+1} i^{2^{-1}(1+(-1)^m)}] \exp\{Z_k\}.$$

We thus obtain a sum of $(m+1)^{m+1}$ determinants of order $(m+1) \times (m+1)$. Each of these determinants who has two or more rows indexed by the same k vanishes (This can also be seen from a computation similar to the one given below). The others can be obtained by using the fact

$$\det\left([1 + (-1)^{m+k_q+l+1} i^{2^{-1}(1+(-1)^m)}] H_m^{q-l} \epsilon_m^{k_q(q-l)}\right)_{0 \leq q, l \leq m}$$

$$= \det \left([1 + (-1)^{m+k_q+l+1} i^{2^{-1}(1+(-1)^m)}] \epsilon_m^{k_q(q-l)} \right)_{0 \leq q, l \leq m}$$

where the equality follows from taking factor H_m^q from the q th row ($q = 0, 1, \dots, m$) and H_m^{-l} from the l th column ($l = 0, 1, \dots, m$), and $\{k_0, \dots, k_m\}$ is a permutation of the integers in I . Therefore,

$$D(2^{-1}(i\theta/m!)^{1/(m+1)}) = \exp \left\{ \sum_{k \in I} Z_k \right\} [(2(m+1))^{-(m+1)} \cdot \Delta + o(1)]$$

where

$$\Delta = \sum_{k_0, \dots, k_m} \det \left((1 + (-1)^{m+k_q+l+1} i^{2^{-1}(1+(-1)^m)}) \epsilon_m^{k_q(q-l)} \right)_{0 \leq q, l \leq m}.$$

Similarly,

$$D(2^{-1}(i\theta/m!)^{1/(m+1)}) = - \exp \left\{ \sum_{k \in I} Z_k \right\} \cdot [(2(m+1))^{-(m+1)} (m!)^{-m/(m+1)} e^{-i \frac{1+(-1)^m}{4(m+1)} \pi} \cdot \widehat{\Delta} + o(1)]$$

where

$$\widehat{\Delta} = \sum_{k_0, \dots, k_m} \epsilon_m^{k_m} \det \left((1 + (-1)^{m+k_q+l+1} i^{2^{-1}(1+(-1)^m)}) \epsilon_m^{k_q(q-l)} \right)_{0 \leq q, l \leq m}.$$

Let p_0, \dots, p_m be the integers in I with increasing order. Given a permutation $\{k_0, \dots, k_m\}$ of $\{p_0, \dots, p_m\}$, we see that

$$\begin{aligned} & \det \left((1 + (-1)^{m+k_q+l+1} i^{2^{-1}(1+(-1)^m)}) \epsilon_m^{k_q(q-l)} \right)_{0 \leq q, l \leq m} \\ &= \left(\prod_{n=0}^m \epsilon_m^{nk_n} \right) \cdot \det \left((1 + (-1)^{m+k_q+l+1} i^{2^{-1}(1+(-1)^m)}) \epsilon_m^{-k_q l} \right)_{0 \leq q, l \leq m} \\ &= (-1)^{a(k_0, \dots, k_m)} \left(\prod_{n=0}^m \epsilon_m^{nk_n} \right) \cdot \det \left((1 + (-1)^{m+p_q+l+1} i^{2^{-1}(1+(-1)^m)}) \epsilon_m^{-p_q l} \right)_{0 \leq q, l \leq m} \\ &= (-1)^{a(k_0, \dots, k_m)} \prod_{n=0}^m \epsilon_m^{nk_q} \cdot C \quad (\text{say}) \end{aligned}$$

where $a(k_0, \dots, k_m)$ is the permutation number of $\{k_0, \dots, k_m\}$ with respect to $\{p_0, \dots, p_m\}$. Here the determinant $C \neq 0$ is given in Lemma 3.2 at the end of this section. Thus by the definition of determinant,

$$\begin{aligned} \Delta &= C \cdot \sum_{k_0, \dots, k_m} (-1)^{a(k_0, \dots, k_m)} \prod_{n=0}^m \epsilon_m^{nk_q} \\ &= C \cdot \det \left(\epsilon_m^{q p_l} \right)_{0 \leq q, l \leq m} \\ &= C \cdot \prod_{k=1}^m \epsilon_m^{k p_0} \cdot \det \left(\epsilon_m^{q l} \right)_{0 \leq q, l \leq m} \\ &= C \cdot \prod_{k=1}^m \epsilon_m^{k p_0} \cdot \prod_{0 \leq j < k \leq m} (\epsilon_m^k - \epsilon_m^j) \end{aligned}$$

where the last step follows from the well known formula for Vandermonde's determinant.

Similarly,

$$\widehat{\Delta} = C \cdot \sum_{k_0, \dots, k_m} \epsilon_m^{k_m} (-1)^{a(k_0, \dots, k_m)} \prod_{n=0}^m \epsilon_m^{nk_n} = C \cdot \det (r_{ql})_{0 \leq q \leq m; 0 \leq l \leq m}$$

where

$$r_{ql} = \epsilon_m^{qp_l}, \quad 0 \leq q \leq m-1, l \leq m \quad \text{and} \quad r_{ml} = \epsilon_m^{(m+1)p_l}, \quad 0 \leq l \leq m.$$

By the formula for Vandermonde's determinant,

$$\widehat{\Delta} = C \cdot \prod_{k=1}^{m-1} \epsilon_m^{kp_0} \cdot \epsilon_m^{(m+1)p_0} \cdot \prod_{0 \leq j < k \leq m-1} (\epsilon_m^k - \epsilon_m^j) \cdot \prod_{j=0}^{m-1} (\epsilon_m^{m+1} - \epsilon_m^j)$$

Using the fact that $C \neq 0$ from Lemma 3.2, we see $\Delta \neq 0$ and $\widehat{\Delta} \neq 0$. Hence we have

$$\lim_{\theta \rightarrow \infty} \widehat{D}(2^{-1}(i\theta/m!)^{1/(m+1)}) / D(2^{-1}(i\theta/m!)^{1/(m+1)}) = -(m!)^{-\frac{m}{m+1}} e^{-i\frac{1+(-1)^m}{4(m+1)}\pi} \widehat{\Delta} / \Delta.$$

Further, from the above computations of Δ and $\widehat{\Delta}$

$$\frac{\widehat{\Delta}}{\Delta} = \epsilon_m^{p_0} \frac{\prod_{j=0}^{m-1} (\epsilon_m^{m+1} - \epsilon_m^j)}{\prod_{j=0}^{m-1} (\epsilon_m^m - \epsilon_m^j)} = \epsilon_m^{p_0} \frac{(\epsilon_m^{m+1} - 1)\epsilon_m^{m-1}}{\epsilon_m^m - \epsilon_m^{m-1}} = \frac{2\epsilon_m^{p_0}}{1 - \epsilon_m}.$$

By using the fact that $p_0 = -m/2 - (1 - (-1)^m)/4$, one can easily obtain

$$\lim_{\theta \rightarrow \infty} \widehat{D}(2^{-1}(i\theta/m!)^{1/(m+1)}) / D(2^{-1}(i\theta/m!)^{1/(m+1)}) = -(m!)^{-m/(m+1)} i^{-1/(m+1)} \left(\sin \frac{\pi}{2(m+1)} \right)^{-1}.$$

Finally, (3.2) follows from Theorem 2.1 and the proof is finished after establishing the following fact.

Lemma 3.2 *Given $m \geq 1$, let I and ϵ_m be defined by (3.3) and (3.4), respectively, and let p_0, \dots, p_m be all integers in I with increasing order. Then,*

$$C \equiv \det \left((1 + (-1)^{m+p_q+l+1} i^{(1+(-1)^m)/2}) \epsilon_m^{-p_q l} \right)_{0 \leq q, l \leq m} \neq 0.$$

Proof. Note that $(-1)^{p_0} i^{(1+(-1)^m)/2} = (-1)^m i^{-m+1}$. Hence

$$C = \det \left((1 + (-1)^{q+l+1} i^{-m+1}) \epsilon_m^{-ql} \right)_{0 \leq q, l \leq m} \cdot \prod_{l=0}^m \epsilon_m^{-lp_0}.$$

We first consider the case when $m = 2k$ is even. To simplify the notation, we use " $a \simeq b$ " for the relation between two numbers a and b such that $a = cb$ for some $c \neq 0$. We have

$$C \simeq \det \left((1 + (-1)^{q+l} i) \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k}.$$

To manipulate the determinant on the right hand side, we introduce the notation $[l]_2$ for any non-negative integer l given by $[l]_2 = 0$ if l is even, and $[l]_2 = 1$ if l is odd. Dividing each row by $1 + i$ and using $1 - i = -i(1 + i)$, we have

$$C \simeq \det \left((-i)^{[l+q]_2} \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k} = \det \left((-1)^{l+q} (-i)^{[l+q]_2} \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k} \simeq \det \left(i^{[l+q]_2} \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k}.$$

Then multiply $i^{[q]_2}$ to the q th row for $q = 0, 1, \dots, m$, i.e. multiply i to the odd rows, we obtain

$$C \simeq \det \left(i^{[l+q]_2 + [q]_2} \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k} \simeq \det \left((-1)^{ql} \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k} \neq 0$$

where the second step follows from the fact that

$$i^{[l+q]_2 + [q]_2} = (-1)^{(l+1)(q+1)} \cdot i^{2-[l]_2}$$

by taking out common factors from each row and column, and the last step is due to Vandermonde's formula.

Next consider the case when $m = 2k - 1$ is odd. We need only to prove

$$\det \left((1 + (-1)^{q+l}) \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k-1} \neq 0 \quad (3.5)$$

and

$$\det \left((1 + (-1)^{q+l+1}) \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k-1} \neq 0. \quad (3.6)$$

We only prove (3.5) since the proof of (3.6) is similar. First, we move the original rows with $q = 0, 2, \dots, 2k - 2$ of the determinant in (3.5) to the first k rows without changing their relative order (so the rest of the rows will occupy the next k rows with their relative order being kept); second, we do the same column interchanges. So we obtain

$$\det \left((1 + (-1)^{q+l}) \epsilon_m^{-ql} \right)_{0 \leq q, l \leq 2k-1} \simeq \det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{2k \times 2k} = \det(A) \cdot \det(B) \neq 0$$

where A and B are two $k \times k$ matrices given by

$$A = \det \left(\epsilon_m^{-4rs} \right)_{0 \leq r, s \leq k-1} \quad \text{and} \quad B = \det \left(\epsilon_m^{-(2r+1)(2s+1)} \right)_{0 \leq r, s \leq k-1}$$

and the fact that $\det(A) \cdot \det(B) \neq 0$ can easily be seen from the formula for Vandermonde's determinant.

4 Proof of Theorem 1.2 and 1.3

The proof of Theorem 1.2 is simple and based on both directions of the following well known shift inequalities for symmetric convex sets (see, for example, Dudley, Hoffmann–Jørgensen and Shepp (1979) and de Acosta (1983)).

Lemma 4.1 *Let μ be a centered Gaussian measure in a separable Banach space E with norm $\|\cdot\|$. Then for any $f \in H_\mu$, the reproducing kernel Hilbert space of μ , and $r > 0$,*

$$\exp\{-|f|_\mu^2/2\}\mu(x : \|x\| \leq r) \leq \mu(x : \|x - f\| \leq r) \leq \mu(x : \|x\| \leq r). \quad (4.1)$$

Furthermore,

$$\mu(x : \|x - f\| \leq \varepsilon) \sim \exp\left\{-|f|_\mu^2/2\right\} \cdot \mu(x : \|x\| \leq \varepsilon) \quad \text{as } \varepsilon \rightarrow 0. \quad (4.2)$$

The upper bound follows from Anderson’s inequality $\mu(A + x) \leq \mu(A)$ for every convex symmetric subset A of E and every $x \in E$, see Anderson (1955). The lower bound follows from the Cameron–Martin formula

$$\mu(A - f) = \int_A \exp\left\{-\frac{1}{2}|f|_\mu^2 - \langle x, f \rangle_\mu\right\} d\mu(x) \quad (4.3)$$

for Borel subsets A of E , $f \in H_\mu$, together with Hölder’s inequality and the symmetry of $\langle x, f \rangle_\mu$ on $A = \{x : \|x\| \leq r\}$. Note that $\langle x, f \rangle_\mu$ can be define as the stochastic inner product for μ almost all x in E . A particularly nice proof of (4.3) is contained in Proposition 2.1 of de Acosta (1983). Refinements of (4.1) are given in Kuelbs, Li and Linde (1994) and Kuelbs, Li and Talagrand (1994) which play important roles in the studies of the functional form of Chung’s law of the iterated logarithm.

Proof of Theorem 1.2. Without loss of generality, assume X and Y are independent. Then by applying (4.1), we obtain

$$\mathbb{P}(\|Y\| \leq \varepsilon) \geq \mathbb{P}(\|X - \lambda Y\| \leq \lambda\varepsilon) \geq \mathbb{P}(\|X\| \leq \lambda\varepsilon) \cdot \mathbb{E} \exp\{-2^{-1}\lambda^2 |Y|_\mu^2\}$$

which finishes the proof.

Proof of Theorem 1.3. The existence of the limits in (1.11) can be found in Li and Linde (1998) based on subadditivity. The lower bound for κ_m in (1.12) follows easily from

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} |X_m(t)| \leq \varepsilon\right) \leq \mathbb{P}\left(\int_0^1 |X_m(t)|^2 dt \leq \varepsilon^2\right)$$

and the L_2 -estimate given in Theorem 1.1.

The upper bound for κ_m in (1.12) follows from Theorem 1.2, the L_2 -estimate given in (1.7) and the well known estimate

$$\log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |W(t)| \leq \varepsilon \right) \sim -(\pi^2/8)\varepsilon^{-2}.$$

To be more precise, take $Y = X_m(t)$ and $X = W(t)$ in Theorem 1.2. Then for any norm $\|\cdot\|$ on $C[0, 1]$ and any $\lambda = \lambda_\varepsilon > 0$

$$\mathbb{P}(\|X_m\| \leq \varepsilon) \geq \mathbb{P}(\|W(t)\| \leq \lambda\varepsilon) \cdot \mathbb{E} \exp \left\{ -\frac{\lambda^2}{2} \int_0^1 X_{m-1}^2(s) ds \right\} \quad (4.4)$$

since $|f|_\mu^2 = \int_0^1 (f'(s))^2 ds$ for Wiener measure $\mu = \mathcal{L}(W)$. Taking $\|\cdot\|$ to be the sup-norm on $C[0, 1]$ and $\lambda = \lambda_\varepsilon = \alpha\varepsilon^{-2m/(2m+1)}$ in (4.4) with $\alpha > 0$, it follows from the existence of the constants that

$$\begin{aligned} -\kappa_m &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |X_m(t)| \leq \varepsilon \right) \\ &\geq \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |W(t)| \leq \alpha\varepsilon^{1/(2m+1)} \right) \\ &\quad + \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{E} \exp \left\{ -\frac{\alpha^2}{2} \varepsilon^{-4m/(2m+1)} \int_0^1 X_{m-1}^2(s) ds \right\} \\ &= -(\pi^2/8)\alpha^{-2} - \frac{1}{2}\alpha^{1/m} \left(\sin \frac{\pi}{2m} \right)^{-1}. \end{aligned}$$

Now pick the best $\alpha > 0$, we obtain

$$\begin{aligned} \kappa_m &\leq \min_{\alpha > 0} \left((\pi^2/8)\alpha^{-2} + \frac{1}{2}\alpha^{1/m} \left(\sin \frac{\pi}{2m} \right)^{-1} \right) \\ &= \frac{2m+1}{2} \left(\frac{\pi}{2} \right)^{2/(2m+1)} \left(2m \sin \frac{\pi}{2m} \right)^{-2m/(2m+1)} \end{aligned}$$

which is the upper bound for κ_m in (1.12).

It is of interest to note that both bounds rely on easier L_2 -estimates and the constant bounds for κ_m are the sharpest known. Furthermore, we can also consider the L_p -norm on $C[0, 1]$ with $1 \leq p < \infty$ for $X_m(t)$. It is shown in Li (2001) that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{P} \left(\left(\int_0^1 |X_m(t)|^p dt \right)^{1/p} \leq \varepsilon \right) = -C_{m,p} \quad (4.5)$$

Thus bounds on $C_{m,p}$ can be given and it is similar to what we did for the above sup-norm case. We omit details here.

Finally, we prove (1.13) based on Theorem 1.2 similar to the upper bound argument above. This time we take $Y = X_m(t)$ and $X = X_{m-1}(t)$ in Theorem 1.2. Then for any $\lambda = \lambda_\varepsilon > 0$

$$\mathbb{P} \left(\sup_{0 \leq t \leq 1} |X_m(t)| \leq \varepsilon \right) \geq \mathbb{P} \left(\sup_{0 \leq t \leq 1} |X_{m-1}(t)| \leq \lambda\varepsilon \right) \cdot \mathbb{E} \exp \left\{ -\frac{\lambda^2}{2} \int_0^1 W^2(s) ds \right\}$$

since $|f|_\mu^2 = \int_0^1 (f^{(m)}(s))^2 ds$ for the Gaussian measure $\mu = \mathcal{L}(X_{m-1})$. Taking $\lambda = \lambda_\varepsilon = \beta\varepsilon^{-2/(2m+1)}$ with $\beta > 0$, it follows from the existence of the constants and the L_2 -estimate given in (1.7) that

$$\begin{aligned} -\kappa_m &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |X_m(t)| \leq \varepsilon \right) \\ &\geq \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{P} \left(\sup_{0 \leq t \leq 1} |X_{m-1}(t)| \leq \beta \varepsilon^{(2m-1)/(2m+1)} \right) \\ &\quad + \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/(2m+1)} \log \mathbb{E} \exp \left\{ -\frac{\beta^2}{2} \varepsilon^{-4/(2m+1)} \int_0^1 W^2(s) ds \right\} \\ &= -\beta^{-2/(2m-1)} \kappa_{m-1} - \beta/2. \end{aligned}$$

Now pick the best $\beta > 0$, we obtain for $m \geq 1$

$$\begin{aligned} \kappa_m &\leq \min_{\beta > 0} \left(\beta^{-2/(2m-1)} \kappa_{m-1} + \beta/2 \right) \\ &= ((2m-1)/4)^{2/(2m+1)} \cdot (2m+1)(2m-1)^{-1} \cdot \kappa_{m-1}^{(2m-1)/(2m+1)} \end{aligned}$$

which can be rewritten as (1.13). In the case $m = 2$, we have

$$\kappa_2 \leq (5/3) \cdot (3/4)^{2/5} \cdot \kappa_1^{3/5}.$$

Note that, based on the estimate $\kappa_1 \leq (3/8) \cdot (2\pi)^{2/3}$, the estimate given in (1.12), namely $\kappa_2 \leq (5/8) \cdot (2\pi)^{2/5}$, is better than the foregoing one.

5 Remarks on related problems and approaches

Consider the $(m+1)$ -dimensional diffusion process $X(t) = (X_0(t), \dots, X_m(t))$ with generator

$$\mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial x_0^2} + \sum_{k=1}^m x_{k-1} \frac{\partial}{\partial x_k}. \quad (5.1)$$

The hypoellipticity of this operator can be checked with the help of the Hörmander condition, see Chaleyat-Maurel and Elie (1981) for details, and could possibly yield some informations about the unicity of the solution for certain boundary value problems related to \mathcal{L} .

Consider the first exit time of the unbounded domain $D = \{(x_0, \dots, x_m) : |x_m| < 1\}$,

$$\tau = \inf\{t \geq 0 : |X_m(t)| \geq 1\} = \inf\{t \geq 0 : X(t) \notin D\}.$$

Our main results are then equivalent to

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ -\int_0^t X_m^2(s) ds \right\} = -2^{-(2m+1)/(2m+2)} \left(\sin \frac{\pi}{2m+2} \right)^{-1} \quad (5.2)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\tau \geq t\} = -\kappa_m \quad (5.3)$$

with bounds on κ_m given in (1.12) and (1.13).

Define the linear operator

$$\tilde{\mathcal{L}}u = \frac{1}{2} \frac{\partial^2 u}{\partial x_0^2} + \sum_{k=1}^m x_{k-1} \frac{\partial u}{\partial x_k} - x_m^2 u, \quad u \in C_b^2(\mathbb{R}^{m+1})$$

and let $\sigma(\tilde{\mathcal{L}})$ be the spectrum set of $\tilde{\mathcal{L}}$, where $C_b^2(\mathbb{R}^{m+1})$ is the space of bounded, twice differentiable functions on \mathbb{R}^{m+1} . Further, let $C_b(\mathbb{R}^{m+1})$ be the Banach space of bounded functions on \mathbb{R}^{m+1} with the norm $\|f\| = \sup_x |f(x)|$ and define a semigroup of linear bounded operator T_t ($t \geq 0$) on $C_b(\mathbb{R}^{m+1})$ by

$$T_t f(x) = \mathbb{E}_x \left(\exp \left\{ - \int_0^t X_m^2(s) ds \right\} f(X_0(t), \dots, X_m(t)) \right), \quad x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}.$$

Then $\tilde{\mathcal{L}}$ is the infinitesimal generator of T_t . By Gelfand Theorem, see, for example, Proposition 3.8, p.197 in Conway (1990),

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t\| = \sup \{ \log |\sigma(T_1)| \}. \quad (5.4)$$

On the other hand, it is not hard to see

$$\|T_t\| = \sup_x \mathbb{E}_x \exp \left\{ - \int_0^t X_m^2(s) ds \right\} = \mathbb{E} \exp \left\{ - \int_0^t X_m^2(s) ds \right\} \quad (5.5)$$

where the second equality follows from Anderson's inequality for centered Gaussian measures and the simple fact that $\mathbb{E} Y = \int_0^\infty \mathbb{P}(Y > y) dy$ for $Y \geq 0$. Thus we have from (5.4) and (5.5)

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ - \int_0^t X_m^2(s) ds \right\} = \sup \{ \log |\sigma(T_1)| \}$$

and as a consequence of Theorem 1.1 or (5.2),

$$\sup \{ \log |\sigma(T_1)| \} = -2^{-(2m+1)/(2m+2)} \left(\sin \frac{\pi}{2m+2} \right)^{-1}.$$

It is tempting to conclude directly via the spectrum map theorem that $\sup \{ \log |\sigma(T_1)| \} = \sup \Re \{ \sigma(\tilde{\mathcal{L}}) \}$. But there is no general results for unbounded $\tilde{\mathcal{L}}$ see Davies (1980, Theorem 2.16, 1995), Goldstein (1985) and Ethier and Kurtz (1986).

Similarly, if we view \mathcal{L} as the linear operator defined on the space

$$C_{0,b}^2(\bar{D}) = \{ f \in C_b^2(\bar{D}) : f \equiv 0 \text{ on } \partial D \}$$

then we can also relate the constant κ_m to the corresponding spectral radius.

As a remark, it is interesting to note that the technique of spectrum theory can be applied in broader situation, especially on the existence of the constant. For example, we can apply the above arguments to show easily the existence of the small ball constant under the L_p -norm for X_m by considering

$$T_{t,p}f(x) = \mathbb{E}_x \left(\exp \left\{ - \int_0^t |X_m(s)|^p ds \right\} f(X_0(t), \dots, X_m(t)) \right), \quad x = (x_0, \dots, x_m).$$

In particular, this justifies Theorem 4.1 in Khoshnevisan and Shi (1998) for the case $m = 1$, where the suggested arguments for the proof do not work. A different and direct method for proving existence of small ball constants under various norms, based mainly on scaling rather than subadditivity, is given in Li (2000) for Gaussian processes including X_m .

Our result can also be related to, respectively, the following eigenvalue problems:

$$\tilde{\mathcal{L}}u = -\lambda u \tag{5.6}$$

and

$$\begin{cases} \mathcal{L}u = -\lambda u & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{5.7}$$

In the uniform elliptic operator case, Donsker and Varadhan (1976), see also Pinsky (1996), shows by Feymann-Kac formula that the solutions for above eigenvalue problems exist and their principal eigenvalues are the constants we are looking for. However, this approach does not seem to work in our case mainly because we do not know if the solutions (principal eigenvalues) exist or not. If they exist, they should be the constants we want, in the light of (5.2) and (5.3). Recall that $-\lambda_0$ is the principal eigenvalue of (5.6) if it satisfies (5.6) for some $u_0 > 0$ in the domain of $\tilde{\mathcal{L}}$. Then

$$T_t u_0(x) = e^{-t\lambda_0} u_0(x)$$

by solving the differential equation

$$\frac{d}{dt} T_t u_0(x) = -\lambda T_t u_0(x) \quad \text{with } T_0 u_0(x) = u_0(x)$$

and it is hopeful that as $t \rightarrow \infty$, $\mathbb{E} \exp \left\{ - \int_0^t X_m^2(s) ds \right\}$ behaves like, up to some controlable factors,

$$\mathbb{E} \left(\exp \left\{ - \int_0^t X_m^2(s) ds \right\} u_0(X_0(t), \dots, X_m(t)) \right) = e^{-t\lambda_0} u_0(0) = T_t u_0(0).$$

Note that we encounter some difficulties here since we have an unbounded domain. from above, but not from below. In the literature about eigenvalue problems, it seems that little is known in degenerate case. We hope the special case considered here can attract some interests. Maybe the recent work of Feng and Kurtz (2001) can be applied here.

The third connection we know is the following. Write

$$u(x, t) = \mathbb{E}_x \exp \left\{ - \int_0^t X_m^2(s) ds \right\}, \quad x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}$$

and

$$v(x, t) = P_x \{ \tau \geq t \}, \quad x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}.$$

Then u and v solve, respectively,

$$\frac{\partial u}{\partial t} = \tilde{\mathcal{L}}u, \quad u(x, 0) = 1 \tag{5.8}$$

and

$$\frac{\partial v}{\partial t} = \mathcal{L}v \text{ in } D, \quad v(x, 0) = 1 \quad x \in D, \quad v(x, t) = 0 \quad x \in \partial D \tag{5.9}$$

So (5.2) and (5.3) can be written as

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log u(0, t) = -2^{-(2m+1)/(2m+2)} \left(\sin \frac{\pi}{2m+2} \right)^{-1} \tag{5.10}$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log v(0, t) = -\kappa_m. \tag{5.11}$$

The trouble is, however, we don't know whether or not the bounded solutions of (5.8) and (5.9) are unique. Is it true that all solutions satisfy (5.10) and (5.11) if they are not unique?

A close related and useful technique in studying certain asymptotic problems is the logarithmic transformation $V = -\log v(x, t)$ which changes (5.11) into a nonlinear evolution equation for V . This can then be viewed as a stochastic control problem, see, for example, Fleming and Soner (1993).

Finally, we point out that we do not have any variational representation for the constants κ_m other than the existence with bounds given in (1.12) and (1.13). It would be very nice to have an analytical expression for κ_m . Unfortunately, we don't quite have the technical machinery to carry out any of the approaches mentioned in this section.

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