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Technical Report No. 2003-04

DEPARTMENT
OF
MATHEMATICAL SCIENCES

University of Delaware
Newark, Delaware

PACS numbers: 03.65.Ge, 02.70.-c, 46.70,05.45.-a

Isospectral Shapes with Neumann and Alternating Boundary Conditions

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The isospectrality of a well-known pair of shapes constructed from two arrangements of seven congruent right isosceles triangles with the Neumann boundary condition is verified numerically to high precision. Equally strong numerical evidence for isospectrality is presented for the eigenvalues of the pair in new boundary configurations with alternating Dirichlet and Neumann boundary conditions along successive edges. Good agreement with theory is obtained for the corresponding spectral staircase functions. Some possible confirmatory experiments involving fluids are suggested.

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Introduction. — As the most accessible realization of the negative answer to Kac's [1] question "Can one hear the shape of a drum?", the pair of isospectral shapes discovered by Gordon et al. [2] (termed respectively "Bilby" and "Hawk" in Figure 1), have subsequently been investigated from a variety of mathematical, numerical, and experimental viewpoints. Chapman [3] showed how domain eigenfunctions can be mapped from the constituent triangles of one shape to the second to prove isospectrality by transplantation, and described a proof by paper-folding. More recently, Okada and Shudo [4] have investigated isospectrality through a technique of successive unfolding of fundamental building block shapes and transplantation of eigenfunctions. Wu et al. [5] achieved a proof by an explicit mode matching method.

The numerical problem concerning the eigenvalues corresponds to solving an eigenvalue problem for the two-dimensional Helmholtz equation subject to the Dirichlet boundary condition (Dbc). Wu et al. [5] verified isospectrality numerically by an extrapolated mode-matching method, apparently to about 8 significant figures, tabulated for the first 25 modes. The "analytical" 9th and 21st modes there, corresponding to known simple modes of the underlying triangles, were not computed but were taken at their exact values. Subsequently, Driscoll [6], using a much more accurate modified domain-decomposition method, verified isospectrality numerically to 12 significant figures for the first 25 modes, including the two "analytical" modes for which the computation was more or less exact. This work showed that the computed results in Ref. [5] were actually accurate to about 4-5 significant figures.

On the experimental side, Sridhar and Kudrolli [7] performed measurements on thin microwave cavities of the appropriate shapes, utilizing the correspondence with a 2-dimensional Helmholtz equation in the electromagnetic formulation. Then

Even and Pieranski [8] constructed actual shaped small "drums" - membranes made from liquid crystal smectic films - and measured their vibrations.

In this paper, we investigate theoretical and numerical aspects of the isospectrality of the two standard Bilby and Hawk shapes when Neumann boundary conditions (Nbc) are present, and make suggestions for possible experimental verification.

Neumann boundary conditions. — The most commonly encountered boundary condition is the Dirichlet bc $\psi=0$ on the boundary. This corresponds to the standard "drum" condition for a vibrating membrane with fixed edges, as well as to the boundary condition for quantum billiards [9]. The Neumann bc $\partial\psi/\partial n = 0$ also has important manifestations [9], especially in acoustics where the pressure satisfies the Helmholtz equation with Nbc at a rigid boundary and for water surface waves. In electromagnetism also, the magnetic field of the transverse electric (TE) mode in a cavity has Nbc. The Nbc corresponds to the vibrational modes of a drum with stress-free edges, as discussed by Hobiki et al. [10], who numerically investigated such a situation for fractal boundary shapes. Russ et al. [11] also considered fractal resonators with Nbc numerically, remarking that this situation could represent transverse acoustical phonons of a 2D irregular crystallite. In the field of quantum billiards, Gremaud and Jain [12] considered rational and irrational rhombus billiards with Nbc. Kohler and Blumel [13] considered ray-splitting billiards including Nbc. Wiersig [14] has used the fact that, for barrier billiards, the Nbc pertains on part of the boundary of a symmetry reduced polygon for the even states of the full shape. Neumann boundary conditions are therefore important, and the problem of isospectral shapes with these conditions will now be addressed.

The spectra under consideration are the eigenvalues obtained from the two-dimensional Helmholtz equation $(\nabla^2 + E)\psi = 0$ together with the Nbc on the edges. In an earlier proof of the existence of isospectral domains in dimensions greater than or equal to 4, Urakawa [15] already showed isospectrality for Dirichlet and for Neumann boundary conditions. In their announcement of Dbc isospectral two-dimensional domains, Gordon et al [16] also stated that the same shapes with Nbc were isospectral; Okada and Shudo [4] noted that their procedures, suitably modified, prove isospectrality for Nbc. Chapman [3] proved isospectrality for the bilby/hawk and other pairs having the Nbc.

Gottlieb and McManus [17] produced explicit eigenfunctions and corresponding exact eigenvalues for some Nbc modes of the two isospectral shapes. From the nodal patterns of these analytical expressions and numerical solutions for low modes, they were able to identify the first five analytical Neumann modes as the 5th, 9th, 15th, 20th and 29th non-zero sequential modes. The finite element computations mentioned there, whilst sufficient for identifying patterns, only verified isospectrality to about three significant figures. The exact modes mentioned above, however, serve as benchmarks in any investigation of Nbc isospectrality for these shapes.

A numerical verification of isospectrality of the two standard shapes of Figure 1 for Nbc, such as has been done previously for Dbc [5], [6] does not seem to have been carried out before. For this paper, the earlier work of Driscoll [6] has been adapted to the Nbc case. In [6] a candidate eigenfunction near a corner with interior angle π/α is expanded in local polar coordinates as $\sum_{n=1}^{M} c_n J_{n\alpha}\left(r\sqrt{E}\right)\sin\left(n\alpha\theta\right)$. Then one finds an eigenvalue E by matching different expansions along the interfaces of a

domain decomposition; numerically, this becomes minimization of the result of a matrix eigenvalue problem. For Nbc we replace the sine by a cosine and start the summation at $n = 0$.

We have verified isospectrality in the Nbc configuration for the first 30 (non-zero) modes to 12 significant figures. The results for both shapes are given in Table I. With fundamental length unit 2, the analytical modes described above have eigenvalues given by $E_{m,n} = (\pi^2/4)(m^2+n^2)$; $m \leq n = 0,1,2,...$ The cases (m,n) = (0,1), (1,1), (0,2), (1,2), (2,2) corresponding to the five analytical modes described above, together with their readily identifiable nodal patterns [17], are essentially recovered exactly.

It may be noted that many of the investigations concerned with quantum chaotic spectral statistics [9] deal with very large numbers of very high levels. The accuracy on these typically was 10^{-2} of the mean level spacing for earlier works, and more recently of the order of 10^{-4}. By contrast, we are here concerned with the first few dozen eigenvalues, computed to very high accuracy, of the order of 10^{-12} of the mean level spacing or better.

The spectral staircase (number-counting) function for these systems is $N(E) = \Sigma_{i=1..\infty}\Theta(E-E_i)$, where Θ is the Heaviside unit step function. This is related to the spectral or trace function $\Phi(t) = \Sigma_{n=1..\infty}\exp(-E_n t)$ via a Laplace transform. Based on the work of several authors [18]-[21], with earlier work discussed extensively in Baltes and Hilf [22], the (smoothed) spectral staircase function (corresponding to the trace function appearing in [17]) for polygons is given by

$$N(E) \sim [A/(4\pi)]\, E + [(L_N - L_D)/(4\pi)]\, E^{(1/2)} + (\Sigma_{DD,NN} - \Sigma_{DN,ND})\, [(\pi^2 - \theta^2)/(24\pi\theta)] \quad (1)$$

where A is the area, L_D (L_N) is the length of that part of the perimeter having Dirichlet (Neumann) boundary condition, and the sums are over corner angles θ subtended by

pairs of sides with boundary conditions as indicated. For basic unit length 2 for the two isospectral shapes of Figure 1 (which have the same area, perimeter lengths and corner angles), this reads, for the Nbc case, $N_N(E) = 1.1141E + 1.6302\sqrt{E} + 0.4167$. We plotted N(E) for the first 31 modes for Nbc (including the zero mode). The agreement with this graph was good, demonstrating the need for inclusion of the zero mode and the plus sign for the second term in the case of Nbc compared with the minus sign for the Dbc case as was plotted in [6].

To our knowledge, no experiments involving isospectral shapes with Neumann boundary conditions have been performed, in contrast to reported experiments for the Dbc case [7], [8]. Some such Nbc experiments could be envisaged, however, based on acoustics and wave propagation in liquids [9, sect. 2.1], where Nbcs are involved. For instance, Blumel et al [23] reported on the nodal patterns of surface waves formed by agitating a tank with circular or stadium-shaped cylindrical walls. Chinnery et al. [24], [25] used a schlieren technique to visualize resonances in insonified water cavities with stadium and circular cross-sections. Hebert et al [26] made an experimental study of resonances of a fractal acoustic cavity.

It seems likely that these experimental techniques could be applied to cross-sectional shapes as in Figure 1 to investigate their isopectrality. Independent checks on the accuracy of such experiments would be available through the nodal patterns of the analytical modes pictured in [17] together with their sequential mode numbers and eigenvalue ratios listed in [17].

Alternating boundary conditions. — There has been some work done on 2D systems with a mixture of Dirichlet and Neumann bcs on different parts of the boundary. Baltes and Hilf [22, p.47] show the appearance of a minus sign in the third

(constant, corner-angle) term of the spectral number counting function for a rectangle whose sides successively alternate Dbc and Nbc (c.f. equation (1) above). In quantum billiards, there has been recent work where parts of a rectangular boundary have Dbc and parts have Nbc, for ray-splitting [13] and barrier [14] billiards. Thus it is important and timely to consider systems with both types of boundary conditions in detail.

Having verified above the accuracy of our modified domain-decomposition method for computationally handling Dirichlet or Neumann bcs in the case of the two standard shapes whose isospectrality has been proved elsewhere mathematically as described above, we turn to the case of the standard shapes with "alternating boundary conditions" (abcs) in which each side is successively Dbc or Nbc as one moves around the perimeter.

The isospectrality in this configuration has not been proved mathematically so far, and does not seem immediately amenable to the standard forms of proof. For instance, the transplantation method for unfolded domains described by Okada and Shudo [4] does not work here because a Dbc edge, upon folding, would yield a Dbc rather than an Nbc external edge as desired. Our aim here is to present strong numerical evidence for isopectrality in this new abc configuration. It can be seen that A, L_D, L_N and the θs in equation (1) are the same for both shapes, so the coefficients of the three terms in (1) are equal for both, a necessary condition for isospectrality. In fact, there are two distinct such isospectral pair configurations. We denote by aDNbc (aNDbc) the situation for which the longest side in the alternating boundary condition configuration in each shape is chosen to have Dbc (Nbc).

Our numerical method now uses $\sin[(n+1/2)\alpha\theta]$ or $\cos[(n+1/2)\alpha\theta]$ in the Fourier-Bessel corner expansions, whichever conforms to the local bc. The computed

eigenvalues for the first 30 modes of the two shapes in the two abc configurations are given in Tables IIa and IIb. In either case, the results for both shapes in the same configuration agree to at least 12 significant figures. Unlike the cases of pure Dbc [5], [17] or pure Nbc [17], we have been unable to construct any exact "analytical" modes or to identify particularly simple nodal patterns in the computed eigenfunction plots. Thus independent checks as for the pure Dbc and Nbc cases do not seem to be available here.

As was noted in [6] for Dbc, the computational abc eigenvalues for the regions are identical when certain decompositions are used. The matrices that play a central role (see (2.6) in [6]) are apparently similar for the two regions. We speculate that if this could be shown formally, a proof of abc isospectrality might follow. (For Tables IIa and IIb these special decompositions were avoided.)

The spectral staircase functions are plotted and compared with the graphs of equation (1) in Figure 2. For basic side length h, $A = (7/2)h^2$, $|L_N - L_D| = \sqrt{2}h$, and the corner angle term has value -5/12. Thus, for h=2, (1) becomes $N_{aDN}(E) = 1.1141E - 0.2251\sqrt{E} - 0.4167$, with a plus sign for the second term in $N_{aND}(E)$. The plots show good agreement and confirm the minus sign for the third term in these cases of alternating boundary conditions.

Conclusion. — The known isospectrality of the two standard shapes (Fig. 1) with Neumann boundary condition has been confirmed numerically to a high degree of accuracy, and good agreement with theory for the spectral staircase function has been obtained.

We have presented numerical evidence that is the first and indeed strong indication of the isospectrality of these two standard shapes in the new boundary

condition configurations with alternating Dirichlet and Neumann conditions on successive sides.

It is suggested that some experimental work involving fluids may illustrate the Nbc case, and that electromagnetic cavities might be relevant for the case of alternating boundary conditions.

One of us (HPWG) would like to thank Professor Y. Okada for illuminating e-mail correspondence.

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TABLE I. Eigenvalues of the first 30 non-zero modes, to 12 significant figures, for the two standard isospectral shapes of Fig. 1 (basic side 2 units), with Neumann boundary condition. (Mode number M = M'+1.)

M'	E	M'	E
1	0.211150088843	16	9.87841180931
2	0.809165742343	17	10.3253541128
3	1.05781354844	18	11.5754753532
4	1.86039671198	19	11.6665580661
5	2.46740110027	20	12.3370055014
6	2.72788980382	21	12.9096372668
7	3.39124050892	22	13.7742157078
8	4.40632478595	23	15.4610825657
9	4.93480220054	24	15.9519127351
10	5.66570714255	25	16.2660218054
11	5.81846754996	26	16.9514271289
12	6.98274727028	27	19.0892095926
13	7.64059043694	28	19.1962817413
14	9.04934361203	29	19.7392088022
15	9.86960440109	30	20.5304462804

TABLE II. Eigenvalues of the first 30 modes, to 12 significant figures, for the two standard isospectral shapes of Fig. 1 (basic side 2 units), with

(a) alternating Dirichlet (longest side) / Neumann boundary conditions;

(b) alternating Neumann (longest side) / Dirichlet boundary conditions.

(a)	aDNbc	(b)	aNDbc
M	E	M	E
1	1.65141342709	1	0.715984505106
2	1.90967315909	2	1.97361411622
3	3.43870897236	3	2.77451509187
4	4.17587478260	4	2.97462791606
5	4.57209973453	5	3.88908349386
6	5.62166889695	6	4.63290823432
7	6.89161511324	7	6.02493079100
8	7.74834106067	8	6.93817184169
9	9.33008358225	9	7.77235813608
10	9.80192274409	10	8.44522884854
11	10.6260535933	11	9.39178897887
12	11.5038125374	12	10.4460416956
13	12.2664713847	13	10.9709402924
14	12.8896196853	14	11.5756538487
15	14.2405332523	15	12.8545214783
16	15.5313799266	16	13.5117617705
17	16.3655578572	17	15.0069728862
18	17.3956907586	18	15.2660433735
19	17.8925543463	19	15.9460361465
20	19.2138859210	20	17.0144500638
21	19.7290650856	21	17.8080832556
22	20.3158331403	22	19.6769099441
23	21.6332697571	23	20.2581964864
24	22.2827868664	24	20.5724674001
25	23.7501148857	25	21.8462979588
26	24.2385971539	26	22.6047777014
27	25.6971017156	27	23.5733311802
28	26.2466652675	28	24.1261381785
29	27.5323151203	29	24.6538070142
30	28.0086952587	30	25.7410633370

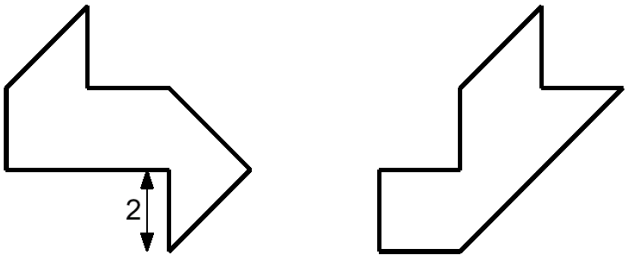

FIG. 1. The two standard isospectral shapes, termed respectively "Bilby" and "Hawk".

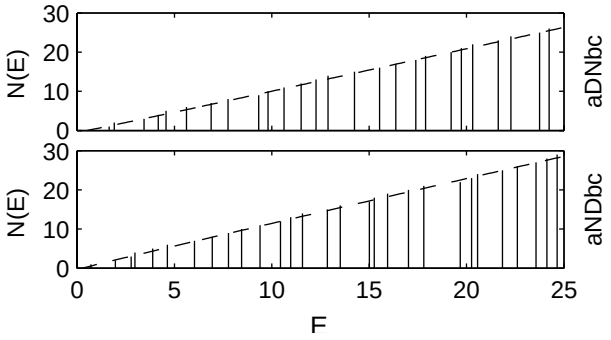

FIG. 2. Spectral staircases (vertical lines) and smooth approximations (dashed curves) for both types of alternating boundary conditions.