ROULETTES AND THEIR SOLIDS AND SURFACES OF REVOLUTION

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INTRODUCTION

Definition of a Roulette and Formulation of the Problem of This Paper

The most general roulette R is defined to be the curve generated by a point P fixed in the plane of a curve C as the curve C rolls without sliding on any other curve B.

The problem to be considered here is special in that while C is any continuous curve, B, the curve on which it rolls, is a straight line. Formulas for the areas of the surfaces of revolution and the volumes of the solids of revolution generated by the roulette R when rotated about the line B will be found.

It is then proposed to translate the general results into one of the special cases, namely the case where C is a conic. The areas of the surfaces of revolution and the volumes of the solids of revolution generated as described above by their special roulettes are expressible as elliptic integrals, and the resulting elliptic integrals are reduced to standard forms.

The results of the solution of this problem find an application in mechanics, for example the "soap bubble" problem, and in particular, Celestial Mechanics.

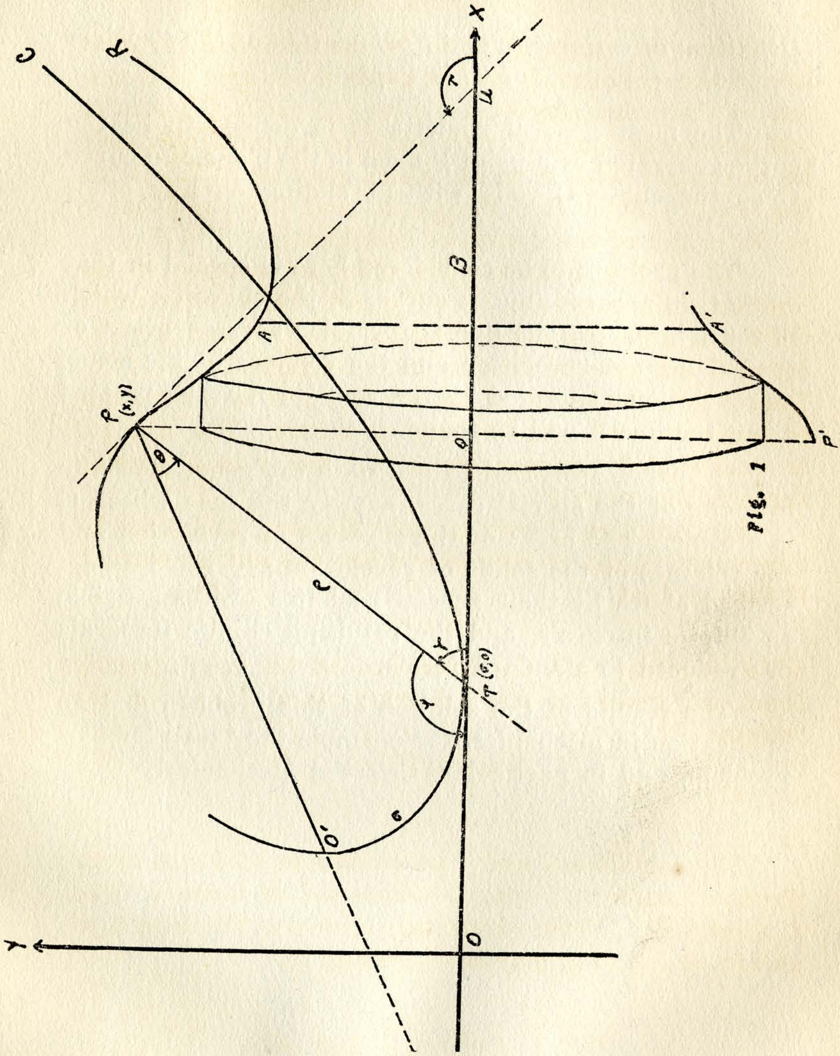

Fig. 1

I. The Parametric Representation of a Roulette in General: Expressions for Areas and Volumes of Solids of Revolution Generated by It

Since the curve B is to be considered a straight line in this problem, it can be made the x-axis of a system of Cartesian coordinates. As the curve C begns to roll some point on C, say O, (in Fig. 1) must obviously be in contact with a point, which may be called O, on the line B. Since B is the x-axis, and O is on B, let O be the origin of the system of coordinates.

Call T the instantaneous point of contact of the line B and the curve C. (See Fig. 1.) Let θ be the angle O′PT and ρ the line connecting the points P and T. The curve C can now be written in polar form as $\rho = f(\theta)$. Denote by σ the length of the arc O′T and by γ the angle of inclination of ρ to the base or x-axis. The line PQ is constructed prerpendicular to the x-axis passing through the point P.

By definition C rolls without slipping, and since by agreement O and O′ coincided when the motion started, it follows that

$$\sigma = OT$$

which together with the definitions of the trigonometric functions, gives for any arbitrary point (x, y) on the roulette

$$(1) \qquad \begin{cases} x = \rho \cos\theta + \sigma \\ y = \rho \sin\theta \ . \end{cases}$$

The variables σ and γ are functions of θ and hence the coordinates x, y may be expressed in terms of θ as a parameter. From elementary Calculus the length of an arc of $\rho = f(\theta)$ may be expressed as

$$\sigma = \int_{O}^{\theta} \left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2 \right]^{1/2} d\theta$$

where O is the initial value of θ when the rolling of the curve begins, and the upper limit of the integral is an arbitrary θ.

In terms of the functions f and its derivative f′, the formula just given for the length of the arc may be expressed by

$$(2) \qquad \sigma = \int_0^\theta \sqrt{f^2+f'^2}\, d\theta \,.$$

From the definition and the conditions imposed, the x-axis is always tangent to the curve at the point T. To determine the angle ψ between the radius vector to the point T on the curve and the tangent at that point, from elementary Calculus

$$\tan\psi = \frac{\rho}{\frac{d\rho}{d\theta}} = \frac{f}{f'}$$

But

$$(180^\circ - \gamma) = \psi$$

whence

$$\tan \gamma = -\tan \psi$$

and

$$\tan \gamma = -\frac{f}{f'}$$

and also

$$\sin \gamma = \frac{f'}{\sqrt{f^2+f'^2}}$$

$$\cos \gamma = \frac{-f'}{\sqrt{f^2+f'^2}}$$

Substituting these expressions and the values $\rho = f$, $\rho' = f'$ in (1), a pair of parametric equations for the roulette in terms of θ results

$$(3) \quad \begin{cases} x = -\dfrac{ff'}{\sqrt{f^2+f'^2}} + \displaystyle\int_0^\theta \sqrt{f^2+f'^2}\, d\theta \\ y = \dfrac{f^2}{\sqrt{f^2+f'^2}} \end{cases}$$

From these expressions the derivatives $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ which determine the derivative of the tangent to the roulette are determined. From

$$x = -\frac{ff'}{\sqrt{f^2+f'^2}} + \int_0^\theta \sqrt{f^2+f'^2}\, d\theta$$

it is found by differentiating with respect to θ that

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(-\frac{ff'}{\sqrt{f^2+f'^2}}\right) + \sqrt{f^2+f'^2}$$

since the derivative of the integral with respect to its upper limit is

$$\frac{d}{d\theta}\int_0^\theta \sqrt{f^2+f'^2}\, d\theta = \sqrt{f^2+f'^2}.$$

After reducing to a common denominator

$$\frac{dx}{d\theta} = \frac{f^2+f'^2(-ff''-f'^2)+f^2f'^2+ff'f''+f^2+f'^2}{(f^2+f'^2)^{3/2}}$$

or after collecting terms and simplifying

$$\frac{dx}{d\theta} = \frac{f^2(f^2-ff''+2f'^2)}{(f^2+f'^2)^{3/2}}\ .$$

In a like manner, starting with the second equation (3), differentiating with respect to θ and simplifying, it follows that

$$\frac{dy}{d\theta} = \frac{ff'(f^2-ff''+2f'^2)}{(f^2+f'^2)^{3/2}}\ .$$

The derivatives of x and y with respect to θ are therefore

$$(4)\qquad \begin{cases} \dfrac{dx}{d\theta} = \dfrac{f^2(f^2-ff''+2f'^2)}{(f^2+f'^2)^{3/2}} \\[2ex] \dfrac{dy}{d\theta} = \dfrac{ff'(f^2-ff''+2f'^2)}{(f^2+f'^2)^{3/2}} \end{cases}$$

From equations (4) and (1) we see at once that the slope of the roulette is

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'}{f} = -\frac{1}{\tan\gamma}$$

from which it follows that the tangent to the roulette is always perpendicular to the line TP.

The area of the surface of revolution generated by revolving an area of the roulette about its x-axis is given by the expression

$$S_x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

or written with respect to θ,

$$S_x = 2\pi \int_{\theta_a}^{\theta_b} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta.$$

Squaring equations (4) and substituting the results together with the second equation (3) in the above integral, gives

$$S_x = 2\pi \int_{\theta_a}^{\theta_b} \frac{f^2}{\sqrt{f_2+f'^2}} \left[\frac{f^2f'^2(f^2-ff''+2f'^2) + f^4(f^2-ff''+2f'^2)}{(f^2+f'^2)^3}\right]^{1/2} d\theta$$

where the arc rotated has for its initial point the point corresponding to $\theta_a = 0$ and its end point corresponding to the arbitrary value $\theta_b = \theta$. After factoring and removing from under the radical such terms as are convenient, this expression becomes

$$(5) \qquad S_x = 2\pi \int_0^{\theta} \frac{f^2(f^2-ff''+2f'^2)}{(f^2+f'^2)^{3/2}} d\theta.$$

If any curve be rotated about its x-axis, the volume of the solid it generates by its revolution is

$$V_x = \pi \int_a^b y^2 \, dx$$

which may be written with respect to θ as

$$V_x = \pi \int_{\theta_a}^{\theta_b} y^2 \frac{dx}{d\theta} d\theta.$$

Squaring the second equation (3) and substituting the result together with the first equation of (4) in the above expression, we find

$$V_x = \pi \int_{\theta_a}^{\theta_b} \frac{f^4}{f^2+f'^2} \cdot \frac{f^2(f^2—ff''+2f'^2)}{(f^2+f'^2)^{3/2}} d\theta$$

where, as in the case of surfaces of revolution of the roulette, new limits can be taken as $\theta_a = \theta$ and $\theta_b = \theta$. Therefore

$$(6) \qquad V_x = \pi \int_0^{\theta} \frac{f^6(f^2—ff''+2f'^2)}{(f^2+f'^2)^{3/2}} d\theta.$$

II. Roulettes Generated by Conics

Roulettes generated by any curve C have been discussed in Section I. An interesting condition on C is the case when C is limited to conics. Expressions which are true of conics will come out of the general case by a mere substitution of the equation of conics in polar form for the general equation $\rho = f(\theta)$.

The integral (2) is a general expression for the length of an arc of the curve C or $\rho = f(\theta)$. Since it includes conics, the polar representation of conics,

$$(7) \qquad f = \frac{p}{1—e \cos\theta}$$

$$(8) \qquad f' = -\frac{pe \sin\theta}{(1-e\cos\theta)^2}$$

when squared and substituted in (2) gives

$$\sigma = \int_0^\theta \left[\frac{p^2}{(1-e\cos\theta)^2} + \frac{p^2e^2\sin^2\theta}{(1-e\cos\theta)^4} \right]^{1/2} d\theta$$

or after simplifying

$$(9) \qquad \sigma = \int_0^\theta \frac{\sqrt{1-2e\cos\theta+e^2}}{(1-e\cos\theta)^2}\, d\theta$$

The parametric equations of the conics come very easily from equations (3) by substituting in (3) the special values (7), (8), and (9). They are

$$(10) \qquad \begin{cases} x = \dfrac{p^2e\sin\theta}{(1-e\cos\theta)\sqrt{1-2e\cos\theta+e^2}} + \displaystyle\int_0^\theta \frac{\sqrt{1-2e\cos\theta+e^2}}{(1-e\cos\theta)^2}\, d\theta \\ y = \dfrac{p^2}{\sqrt{1-2e\cos\theta+e^2}}\ . \end{cases}$$

From the general expression for the area of the surface of revolution of a roulette generated by a curve C, given in (5), the special case for conics may likewise be obtained by substitution.

The differentiation of (8) gives

$$(11) \qquad f'' = \frac{pe\,[2e - \cos\theta\,(1+e\cos\theta)]}{(1-e\cos\theta)^2}$$

which with (7) and (8), changes (5) into

$$(12) \qquad S_x = 2\pi\, p^2 \int_0^\theta \frac{(1-e\cos\theta)\, d\theta}{(1-2e\cos\theta+e^2)^{3/2}}\, d\theta.$$

Knowing the volume of the solid of revolution of a roulette of any curve to be (6), the expression for the case of conics is arrived at by inserting the special values

(7), (8), and (11) for the corresponding general ones in (6), which gives

(13)
$$V_x = \pi \int_0^\theta \frac{(1-e\cos\theta)\,d\theta}{(1-2e\cos\theta+e^2)^{3/2}}\,d\theta.$$

III. Reduction of the Elliptic Integral to Normal Form

The elliptic integrals to be reduced to normal form are the length of the arc, the surface of revolution of the roulette, and the volume of the solid of revolution of the conics. Since the eccentricity of the conic may be greater, equal to, or less than unity, three particular cases will have to be considered in the reduction of each of the above mentioned elliptic integrals.

Reducing the expression for the length of an arc of a curve to normal form is not only a simple operation, but the substitutions employed are carried over directly to the reduction of the remaining integrals.

Consider first the length of the arc of an ellipse $(e<1)$. Recalling (9)

$$\sigma = p\int_0^\theta \frac{\sqrt{1-2e\cos\theta+e^2}}{(1-e\cos\theta)^2}\,d\theta$$

apply to it the substitution

(14)
$$\begin{cases} u = \cos\theta \\ d\theta = \dfrac{du}{-\sqrt{1-u^2}} \end{cases}$$

and it becomes

(15)
$$\sigma = \frac{p}{2e}\int_{u_1}^{1} \frac{(1-2eu+e^2)\,du}{(1-eu)^2\sqrt{(u-\frac{1+e^2}{2e})(u^2-1)}}.$$

To keep the terms in the field of real numbers, they are reduced to the form $(1-t^2)(1-k^2t^2)$, where k and the limits of the integral, t_1 and t_2, are each $\leqq 1$. This is accomplished by setting

$$(16)\qquad \begin{cases} u = \dfrac{1+e^2t}{e(1+t)} \\ du = \dfrac{(e^2-1)\,dt}{(1+t^2)^2}. \end{cases}$$

This substitution may be obtained analytically by using the general linear substitution

$$x = \frac{p+qt}{1+t}$$

in the radical, which occurs in our original expression, setting the coefficients of terms of the first degree equal to zero, and solving for p and q. The values of p and q are found to be e and $\frac{1}{e}$ respectively. After using substitution (16) and simplifying, (15) becomes

$$\sigma = \frac{p}{(1-e^2)} \int_{1/e}^{t_1} \frac{(t^2-1)\,dt}{t^2\sqrt{T}}$$

where

$$(17)\qquad \begin{cases} T = (1-t^2)(1-e^2t^2) \\ t_1 = \dfrac{1-eu_1}{e(u_1-e)}. \end{cases}$$

According to our original assumption, e is less than one, therefore k is less than one, and since $u = \cos\theta$ cannot exceed unity, the limits cannot exceed unity, and a positive radicand is assured.

After breaking into parts, the above expression for the length of an arc becomes

$$(18)\qquad \sigma = \frac{p}{1-e^2}\left[\int_{1/e}^{t_1} \frac{dt}{\sqrt{T}} - \int_{1/e}^{t_1} \frac{(t^2)^{-1}dt}{\sqrt{T}}\right]$$

or in Legendre's notation

$$\sigma = \frac{p}{1-e^2}(Y_0 - Y_{-1}).$$

Apply the standard reduction formula for Y_{-1} to the above expression and it becomes

$$\sigma = \frac{p}{1-e^2}\left[Y_0 - Y_1 - \frac{1}{e^2}\left\{\frac{\sqrt{T}}{t}\right\}_{1/e}^{t_1}\right]$$

or when Y_1 is put in the form $(Y_0 - e^2 Y_1)$

$$\sigma = \frac{p}{1-e^2}\left[\frac{e^2+1}{e^2}Y_0 - \frac{1}{e^2}(Y_0 - e^2 Y_1) - \left\{\frac{\sqrt{T}}{e^2 t}\right\}_{1/e}^{t_1}\right]$$

and therefore, in standard form, the length of an arc of an ellipse may be written

(19)

$$\sigma = \frac{p}{e^2(1-e^2)}\left[(1+e^2)F(k,t) - E(k,t) - \left\{\frac{\sqrt{T}}{t}\right\}_{1/e}^{t_1}\right]$$

where

(20) $$\begin{cases} k = e < 1 \\ t = t_1 = \dfrac{1-eu_1}{e(u_1-e)} \text{ and } t = \dfrac{1}{e}. \end{cases}$$

If, however, the eccentricity is greater than unity, the results of the previous case cannot be used for $k = e > 1$, and the radicand cannot be called positive.

Beginning with (18)

$$\sigma = \frac{p}{1-e^2}\left[\int_{1/e}^{t_1}\frac{dt}{\sqrt{(1-t^2)(1-e^2t^2)}} - \int_{1/e}^{t_1}\frac{dt}{t^2\sqrt{(1-t^2)(1-e^2t^2)}}\right]$$

apply the substitution

(21) $$t = \frac{z}{e}$$

and the above expression becomes

$$\sigma = \frac{pe}{1-e^2}\left[\frac{1}{e^2}\int_1^{z_1}\frac{dz}{\sqrt{Z'}} - \int_1^{z_1}\frac{(z^2)^{-1}dz}{\sqrt{Z'}}\right]$$

when

$$(22)\qquad \begin{cases} Z' = (1-z^2)\left(1-\dfrac{1}{e^2}Z^2\right) \\ z_1 = \dfrac{1-eu_1}{u_1-e}\,. \end{cases}$$

The radical is now in standard form, both requirements being met, namely $k = \dfrac{1}{e} < 1$ and the limits $\leqq 1$.

In Legendre's notation the above expression becomes

$$\sigma = \frac{p}{1-e^2}\,[Y_0 - Y_{-1}]$$

which by the application of the standard reduction formula on Y_{-1} reduces to

$$\sigma = \frac{p}{1-e^2}\,\left[Y_0 + Y_1 - \left\{\frac{e^2\sqrt{Z'}}{z}\right\}_1^{z_1}\right]$$

and therefore written in the standard form the arc of a hyperbola is

(23)

$$\sigma = \frac{p}{e(1-e^2)}\,\left[\,(1+e^2)\;F(k,z) - e^2E(k,z) - \left\{\frac{e^2\sqrt{Z'}}{z}\right\}_1^{z_1}\right]$$

where

$$(24)\qquad \begin{cases} k = \dfrac{1}{e} \\ z = 1, \quad z_1 = \dfrac{1-eu_1}{u_1-e}\,. \end{cases}$$

Finally if e be unity, the integral which expresses the length of an arc of the parabola is not an elliptic integral and can be integrated directly.

Set $e = 1$ in (15), which becomes after simplification

$$\sigma = \sqrt{2p}\int_1^{u_1} \frac{du}{(1-u)^2\sqrt{1+u}}$$

or

$$\sigma = \sqrt{2}\, p\, J$$

where

$$J = \int_1^{u_1} \frac{du}{(1-u)^2 \sqrt{1+u}}\,.$$

Integrating

$$(25) \quad J = \frac{\sqrt{1+u}}{2(1-u)} + \frac{1}{4\sqrt{2}} \log \frac{\sqrt{1+u}+\sqrt{2}}{\sqrt{1+u}-\sqrt{2}} \Bigg\}_1^{u_1}$$

and consequently the length of an arc of a parabola is given by the expression

(26)

$$\sigma = \frac{p}{4} \left\{ \frac{2^{3/2}\sqrt{1+u}}{1-u} + \log \frac{\qquad}{\sqrt{1+u}-\sqrt{2}\,.} \right\}_1^{u_1}.$$

The second elliptic integral to be reduced to normal form is the area of the surface of revolution of a roulette of conics. As before suggested, practically the same substitutions that were used on the elliptic integral expressing the length of an arc of a curve, will suffice here.

Assume first that the eccentricity of conic is less than unity or, in other words, that the conic is reduced to an ellipse. Recalling that the general expression for the area of the surface of revolution (12) of the roulette of any conic is

$$S_x = 2\pi p^2 \int_0^{\theta} \frac{d\theta}{(1-2e\cos\theta+e^2)^{3/2}}$$

proceed as with the arc, using first (14) which gives

(27)

$$S_x = \frac{\sqrt{2}\pi p^2}{e} \int_{u_1}^{1} \frac{du}{(1-2eu+e^2)\sqrt{\left(u-\frac{1+e^2}{2e}\right)(u^2-1)}}$$

and the additional substitution (16) reduces the above to

$$S_x = \frac{2\pi p^2}{(1-e^2)^2} \int_{1/e}^{t_1} \frac{(t+1)^2 dt}{(t-1)^2 \sqrt{T}}$$

where T and t_1 are given by (17). Now multiply both numerator and denominator of the integrand by $(t+1)^2$, in order that it may be divisible into odd and even functions. It becomes

$$S_x = \frac{2\pi p^2}{(1-e^2)^2} \int_{1/e}^{t_1} \frac{(t+1)^4 dt}{(t^2-1)^2 \sqrt{T}}\,.$$

Divide the denominator of the integrand into the numerator and

$$S_x = \frac{2\pi p^2}{(1-e^2)^2} \left[\int_{1/e}^{t_1} \frac{dt}{\sqrt{T}} + 4 \int_{1/e}^{t_1} \frac{(t^3+2t^2+t)\, dt}{(1-t^2)^2 \sqrt{T}} \right]$$

which after separating the odd and even functions becomes

$$S_x = \frac{2\pi p^2}{(1-e^2)^2} \left[\int_{1/e}^{t_1} \frac{dt}{\sqrt{T}} + 8 \int_{1/e}^{t_1} \frac{t^2 dt}{(1-t^2)^2 \sqrt{T}} + 4 \int_{1/e}^{t_1} \frac{t(t^2+1)\, dt}{(1-t^2)^2 \sqrt{T}} \right].$$

Now separate the second integral into its partial fractions and to the third, or the odd function, apply the substitution

(28)

$$y = t^2$$

and the above expression becomes

$$S_x = \frac{2\pi p^2}{(1-e^2)^2} \left[\int_{1/e}^{t_1} \frac{dt}{\sqrt{T}} + 8 \int_{1/e}^{t_1} \frac{dt}{(1-t^2)^2 \sqrt{T}} - \right.$$

$$\left. 8 \int_{1/e}^{t_1} \frac{dt}{(1-t^2)\sqrt{T}} + 2 \int_{1/e}^{t_1} \frac{(y+1)\, dy}{(1-y)^2 \sqrt{(1-y)(1-e^2 y)}} \right].$$

Integrate the non-elliptic integral and call the results C_1 which has the value

$$C_1 = \left\{ \frac{\sqrt{Y}\,[\,12(Y-e^2)-4(1-e^2y)\,]}{3(1-y)^2(1-e^2)^2} \right\}_{1/e^2}^{t_1^2}$$

and the expression for the area of the surface of revolution of the roulette of an ellipse turns out to be, in Legendre's symbols,

$$(29)\quad S_x = \frac{2\pi p^2}{(1-e^2)^2}\,[\,Y_0 + 8Z_2 - 8Z_1 + C_1\,].$$

Using reduction formulas on the Z's, it reduces to

$$S_x = \frac{2\pi p^2}{(1-e^2)^2}\left[\frac{8e\,(1+e^2)}{3(1-e^2)^2}(Y_0 - Y_1) + \frac{3+5e^2}{3(1-e^2)}Y_0 + C\right]$$

where

$$C = \left.\frac{t\sqrt{T}(e^2t^2+t^2-2e^2)}{(1-e^2)(1-t^2)^2}\right\}_{1/e}^{t_1} + \left.\frac{\sqrt{Y}(e^2y+3y-3e^2-1)}{2(1-y^2)^2(1-t^2)}\right\}_{1/e}^{t_1}$$

and the final result can be written in normal form as

$$(30)\quad S_x = \frac{16\pi p^2}{3(1-e^2)^2} - \frac{5+3e^2}{8}F(k,t) + \frac{1+e^2}{1-e^2}E(k,t) + C$$

where k and t are supplied by (20).

Consider e, in the general equation of the conics, greater than unity. The substitutions of the "length of an arc" of a hyperbola apply here as do the remarks concerning the sign of the radicand.

Rewriting (28) which is an expression for the surface of revolution generated by the roulette of any conic,

$$S_x = \frac{2\pi p^2}{(1-e^2)^2}\left[\int_{1/e}^{t_1}\frac{dt}{\sqrt{T}} + 8\int_{1/e}^{t_1}\frac{dt}{(1-t^2)^2\sqrt{T}} - 8\int_{1/e}^{t_1}\frac{dt}{(1-t^2)\sqrt{T}} + C_1\right]$$

and applying the substitution (21) the surface of revolution of the roulette of a hyperbola is

$$S_x = \frac{2\pi p^2}{e(1-e^2)^2}\left[\int_{1}^{z_1}\frac{dz}{\sqrt{Z'}} + 8\int_{1}^{z_1}\frac{dz}{(1-z^2)^2\sqrt{Z'}} - 8\int_{1}^{z_1}\frac{dz}{(1-z^2)\sqrt{Z'}} + eC_1\right]$$

where Z' and z_1 are given by (22).

Writing the above in Legendre's symbols it becomes

$$S_x = \frac{2\pi p^2}{e(1-e^2)^2}\left[Y_0 + 8Z_2 - 8Z_1 + eC_1\right].$$

By applying reduction formulas on Z_1 and Z_2 the area of the surface of revolution of the roulette of the hyperbola is reduced to the normal form

(31)

$$S_x = \frac{16\pi p^2}{3e(1-e^2)^3}\left[\frac{3+5e^2}{8}F(k,z) + \frac{e^2(1+e^2)}{1-e}E(k,z) + C'\right]$$

where

$$C' = \left.\frac{e^2z^2\sqrt{Z'}(z^2+z^2e^2-2e^4)}{(e^2-z^2)^2}\right\}_1^{z_1} \quad \left.\frac{e\sqrt{Y}(e^2y+3y-3e^2-1)}{z(1-e^2)(1-y)^2}\right\}^{t_1^2}.$$

The area of the surface of revolution of the roulette of a parabola, or the case of the conics in which $e=1$, is found without effort since it reduces in the same manner as does the "length of the arc" of a parabola.

From the general result for the surface of revolution of the roulette of conics, the expression true for the parabola is obtained, setting $e=1$ in (27), or

$$S_x = \frac{\pi p^2}{\sqrt{2}}\int_{u_1}^{1}\frac{du}{(1-u)^2\sqrt{1+u}}$$

which in terms of J is

$$S_x = \frac{\pi p^2}{\sqrt{2}}J$$

where J is given by (25).
Whence the area of the surface of revolution of the roulette of a parabola is

(32)

$$S_x = \frac{\pi p^2}{2^{3/2}}\left\{\frac{\sqrt{1+u}}{1-u} + \frac{1}{2^{3/2}}\log\frac{\sqrt{1+u}+\sqrt{2}}{\sqrt{1+u}-\sqrt{2}}\right\}_1^{u_1}.$$

The third elliptic integral to be reduced to normal form is that which expresses the volume of solids of revolution of roulettes of conics. The remarks concerning the method of attack used in the case already reduced apply here.

The general expression for the volume of solids of revolution of all conics has been determined and is given by (13). The case of the ellipse will be considered first, that is, when $e < 1$. Recalling the general expression (13),

$$V_x = \pi p^3 \int_0^\theta \frac{(1 - e\cos\theta)\, d\theta}{1 - 2e\cos\theta + e^2 \sqrt{1 - 2e\cos\theta + e^2}}$$

apply to it substitution (14) and it reduces to

$$(33) \qquad V_x = \frac{\pi p^3}{\sqrt{2e}} \int_{u_1}^{1} \frac{(1 - eu)\, du}{(1 - 2eu + e^2) \sqrt{\left(u - \frac{1+e^2}{2e}\right)(u^2 - 1)}} \,.$$

Employing (16) and multiplying both numerator and denominator by $(t+1)^2$ it becomes

$$V_x = \frac{\pi p^3}{1 - e^2} \int_{1/e}^{t_1} \frac{(t^4 + 3t^3 + 3t^2 + t)\, dt}{(t^2 - 1)^2 \sqrt{T}}$$

where Y and t_1 are given by (17).

Divide the above integral into odd and even functions and after reducing the rational part of the integrand to a proper fraction the integral takes the form

$$V_x = \frac{\pi p^3}{1 - e^2} \left[\int_{1/e}^{t_1} \frac{dt}{\sqrt{T}} + \int_{1/e}^{t_1} \frac{(5t^2 - 1)\, dt}{(t^2 - 1)^2 \sqrt{T}} + \int_{1/e}^{t_1} \frac{t(3t^3 + 1)\, dt}{(t^2 - 1)^2 \sqrt{T}} \right].$$

Breaking the even functioned integral into its partial fractions and using substitution (28) on the odd functioned part of the above expression, gives

$$V_x = \frac{\pi p^3}{1 - e^2} \left[\int_{1/e}^{t_1} \frac{dt}{\sqrt{T}} + 4 \int_{1/e}^{t_1} \frac{dt}{(1 - t^2)^2 \sqrt{T}} - 5 \int_{1/e}^{t_1} \frac{dt}{(1 - t^2) \sqrt{T}} + \tfrac{1}{2} \int_{1/e^2}^{t_1^2} \frac{(3y + 1)\, dy}{(1 - y)^2 \sqrt{Y}} \right]$$

where $Y = (1 - y)(1 - e^2 y)$. Call the non-elliptic integral K_1 and integrate. It is found that

$$(34) \quad K_1 = -\left.\frac{\sqrt{Y}(5+3e^2-9y+e^2y)}{3(1-e^2)(1-y)^2}\right\}_{1/e^2}^{t_1^2}$$

Using Legendre's symbols the expression for the volume can be written as

$$(35) \qquad V_x = \frac{\pi p^3}{1-e^2}[Y_0 + 4Z_2 - 5Z_1 - K_1]$$

and upon reduction becomes

$$(36) \quad V_x = \frac{\pi p^3}{3(1-e^2)^2}\left[-(7+5e^2)F(k,t) + \frac{7+e^2}{1-e^2}E(k,t) - K\right]$$

where

$$K = \left.\frac{t\sqrt{T}(-3-5e^2+7t^2+e^2t^2)}{(1-t^2)^2(1-e^2)}\right\}_{1/e}^{t_1} + \left.\frac{\sqrt{Y}(5+3e^2-9y+e^2y)}{(1-y)^2}\right\}_{1/e^2}^{t_1^2}$$

and k and t are as given by (20).

Now assume the eccentricity greater than unity, which reduces the general case of conics to the hyperbola, when a value greater than one is assigned to e.

In determining the volume of the solid of revolution of the roulette of an ellipse it has been found that

$$V_x = \frac{\pi p^3}{1-e^2}\left[\int_1^{z_1}\frac{dz}{\sqrt{T}} + 4\int_1^{z_1}\frac{dz}{(1-t^2)^2\sqrt{T}} - 5\int_1^{z_1}\frac{dz}{(1-t^2)\sqrt{T}} - K_1\right]$$

which in this case is not in legitimate form since $e > 1$.

To change the radical to the desired form, substitution (21) is employed, which reduces the above expression to

$$V_x = \frac{\pi p^3}{e(1-e^2)}\left[\int_1^{z_1}\frac{dz}{\sqrt{Z'}} + 4\int_1^{z_1}\frac{dz}{\left(1-\frac{1}{e^2}z^2\right)^2\sqrt{Z'}} - 5\int_1^{z_1}\frac{dz}{\left(1-\frac{1}{e^2}z^2\right)\sqrt{Z'}} - eK_1\right]$$

where Z' and z are given by (22), and K by (33). Writing this in Legendre's symbols

$$V_x = \frac{\pi p^3}{3e(1-e^2)} [Y_0 + 4Z_2 - 5Z_1 - eK_1]$$

and after applying the reduction formula on the Z's, the volume of the solid of revolution of the roulette of an hyperbola may be written in normal form as

(37)

$$V_x = \frac{\pi p^3}{3e(1-e^2)} [(3+e^2) F(k,z) + e^2(7+e^2) E(k,z) + K']$$

where

$$K' = \left. \frac{e^2 z\sqrt{Z'}(-3e^2-5e^4+7z^2+e^2z^2)}{(e^2-z^2)^2(1-e^2)} \right\}_{z_1}^{1} - \left. \frac{e\sqrt{Y}(5+3e^2-9y+e^2y)}{(1-y)^2} \right\}_{1/e^2}^{t_1^2}$$

and k and z are given by (24).

The last assumption on the eccentricity e of the case of the volumes of solids of revolution of roulettes of conics is that it shall be equal to unity. The parabola again becomes integrable and easily handled.

Letting $e = 1$ in the general expression for the volume of the solid of revolution of the roulette of conics, the following expression is obtained

$$V_x = \frac{\pi p^3}{2^{5/3}} \int_1^{u_1} \frac{du}{(1-u)^2\sqrt{1+u}}$$

and recalling (25) the above expression is seen to be

$$V_x = \frac{\pi p^3}{2^{5/2}} J.$$

Inserting the value of J which is given by (25) the volume of the solid of revolution of the roulette of the parabola is

(38)

$$V_x = \frac{\pi p^3}{2^5} \left\{ \frac{\sqrt{1+u} + \sqrt{2}}{\sqrt{1+u} - \sqrt{2}} \log \frac{\sqrt{1+u} + \sqrt{2}}{\sqrt{1+u} - \sqrt{2}} \right\}_1^{u_1} .$$

SUMMARY

	Case	*Standard Form*
Area of Surface of Revolution of Roulette of Conics	*Ellipse* *(30)*	$\frac{16\pi p^2}{3(1-e^2)^2}\left[-\frac{5+3e^2}{8}F(k,t)+\frac{1+e^2}{1-e^2}E(k,t)+C\right]$
	Hyperbola *(31)*	$\frac{16\pi p^2}{3e(1-e^2)^3}\left[\frac{3+5e^2}{8}F(k,z)+\frac{e^2(1+e^2)}{1-e^2}E(k,z)+C'\right]$
	Parabola *(32)*	$\frac{\pi p^2}{\sqrt{2}}J$
Volume of Solids of Revolution of Conics	*Ellipse* *(36)*	$\frac{\pi p^3}{3(1-e^2)^2}\left[-(7+5e^2)F(k,t)+\frac{e^2+7}{1-e^2}E(k,t)+K\right]$
	Hyperbola *(37)*	$\frac{\pi p^3}{3e(1-e^2)^2}\left[(3+e^2)F(k,z)+\frac{e(7+e^2)}{1-e^2}E(k,z)+K'\right]$
	Parabola *(38)*	$\frac{\pi p^3}{2^{5/2}}J$

FOR CONICS

Case	*Explanation of Construction*
Ellipse (30)	$$C = \overset{k=e}{\left.\frac{t\sqrt{T}(e^2t^2+t^2-2e^2)}{(1-e^2)(1-t^2)^2}\right\}_{1/e}^{t_1}} + \overset{t=t_1>1/e}{\left.\frac{\sqrt{Y}(e^2y+3y-3e^2-1)}{2(1-e^2)(1-y^2)}\right\}_{1/e^2}^{t_1^2}}$$
Hyperbola (31)	$$C' = \overset{k=1/e}{\left.\frac{e^2z\sqrt{Z'}(e^2z^2+z^2-2e^4)}{(e^2-z^2)^2(1-e^2)}\right\}_{1}^{z_1}} + \overset{z=et_1>1}{\left.\frac{e\sqrt{Y}(e^2y+3y-3e^2-1)}{z(1-e^2)(1-y)^2}\right\}_{1/e^2}^{t_1^2}}$$
Parabola (32)	$$J = \left.\frac{\sqrt{1+u}}{2(1-u)} + \frac{1}{2^{3/2}}\log\frac{\sqrt{1+u}+\sqrt{2}}{\sqrt{1+u}-\sqrt{2}}\right\}_{1}^{u_1}$$
Ellipse (36)	$$K = \left.\frac{t\sqrt{T}(-3-5e^2+7t^2+e^2t^2)}{(1-t^2)^2(1-e^2)}\right\}_{1/e}^{t_1} - \left.\frac{\sqrt{Y}(5+3e^2-9y+e^2y)}{(1-y)^2}\right\}_{1/e^2}^{t_1^2}$$
Hyperbola (37)	$$K' = \left.\frac{e^2z\sqrt{Z'}(-3e^2-5e^4+7z^2-e^2z^2)}{(e^2-z^2)^2(1-e^2)}\right\}_{1}^{z_1} - \left.\frac{e\sqrt{Y}(5+3e^2-9y+e^2y)}{(1-y)^2}\right\}_{1/e^2}^{t_1^2}$$
Parabola (38)	J as above

IV. Historical Remarks on Roulettes

A roulette ("little wheel") is a very common curve and in our ordinary experience is frequently seen. A nail in the rim of a wheel, in fact at any fixed point of the wheel, describes a roulette as the wheel rolls. More generally the roulette is the curve R, described by the path of a point, which is rigidly attached in the plane of a curve C, as the curve C rolls, without sliding, on any other curve B. If restrictions be placed on the general roulette such that the curve C is to be a circle, B a straight line, and P a point on the circumference of the circle, then a very important particular roulette, called the cycloid, is generated by the point P. The nail in the rim of the wheel mentioned above describes a cycloid as the wheel rolls.

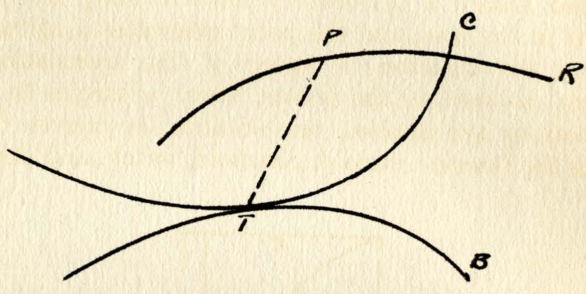

Fig. 2

A moment's reflection on the particular roulette, or the cycloid, suggests that its early history might, very likely, have been connected with an experimental attempt to measure the circumference of a circle by the use of a wheel. It is possible that wheels were the practical circles of early geometers. As far back as Rameses II, chariot wheels (1)* have always had six spokes, seldom four, which has an interesting geometric association. In the writings of the ancients, however, there seems to be but one passage which indicates that the roulette was

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* The numerals in parentheses in this section refer to the numbered list of references at the end of this paper.

known. Iamblicus (2), in speaking of the quadrature of the circle, says, "Carpos (has accomplished it) by means of a line which he calls simply 'of *double motion.*'" It is probable that these three words describe a roulette, and to justify this statement let us consider for a moment some of the curves known to the ancient Greeks.

If the curves of the Greeks are analyzed, they are found to be generated by certain selected points as motions of translation and rotation with given speeds and directions are applied separately to points, lines, or circles. For example, rotate one line E about its end point, translate another line F in the direction of the arrows in Figure 3, and if the speeds be such that the lines arrive at PB simultaneously, the point of intersection of E and F generates a quadratrix. Or rotate a line and assign a translatory motion along the line to a point in the line, and the point generates a spiral. By varying the directon or speed of the translation and rotation, as well as the points, lines, or circles to which the motions are applied, one obtains the curves famous among the Greeks; cissoid, conchoid, witch, and others.

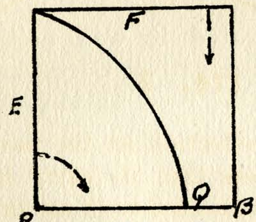

Fig. 3

Now if the rotation and translation be both applied to the same line, the resulting motion of the line might be described, in the words of Carpos, as a "double motion," or better as a rolling motion. In the same manner in which the quadratrix and the spiral are obtained, that is, by applying the motions of translation and rotation separately, a roulette may be obtained by combining these motions and applying them together. Tannery believes Carpos' curve to be a cycloid. (3)

Though the cycloid was known to the Greeks, its properties were very likely uninvestigated, and we pass to the second period in the history of the roulette, which may be characterized as one of independent discovery and frequent disputes.

Three names were given to the curve in this period, each at base meaning "wheel." The first name recorded was "cycloid" which was applied to the curve in 1590 by Galileo. "Roulette" was added in 1615 by Pater Mersenne, and Roverval chose to call it "trochoid" (1634). Very aptly, however, this curve has been called the "Helen of Geometers" because of its beautiful properties* and the many controversies over it. (4)

In his history of roulettes, Pascal (5) said that Mersenne had the gift of framing beautiful questions and, having remarked this curve while watching the rolling of wheels, had first proposed it to the world as a curve whose properties should be investigated. From Italy came Torricelli's (6) declaration that his most illustrious teacher, Galileo, had known the cycloid in 1590 and with material figures had tried to determine its properties experimentally. John Wallis (7) pointed out that early manuscripts indicated it was neither of the abovementioned men, but rather Cardinal Cusa and Charles de Bouvelles who first recognized roulettes.

About 1450 Cardinal Nicolas of Cusa used a curve generated by a point p, on the circumference of a circle, as the circle rolled from a, on a straight line, until p again came into contact with the line at the point b. (See Fig. 4) Careful investigation of Cusa's works have satisfied historians (8) that to Cusa this curve was but an auxilliary line in his solution of the quadrature of the circle, and as a distinct curve it was unknown to him.

The discovery of the roulette is therefore credited to Charles de Bouvelles, who observed the cycloid in

* In addition to its properties, we might mention that Galileo considered its form a most beautiful arch for use in architecture.

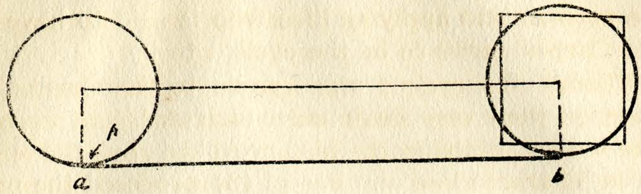

Fig. 4
(Cusa's Construction)

1503. Bouvelles was a man of keen mind, and his intellectual activities held him high in the regard of his contemporaries. As he tells of his discovery one can almost place the atmosphere about it. "I was sitting, one time, on a little bridge of Paris watching the wheel of a coach turning on the flat pavement; there came to me a clear and easy method of realizing my intention (the quadrature of the circle). It is known that when a wheel has made a complete turn it is equal to the circumference of the said wheel. Whence there remained only to find exact incidences of the points of the quadrant, and of the half, and of the whole wheel upon the pavement, so that by this means we can find a straight line equal to the parts and also to the whole of the circumference. I returned to my lodgings and on a bronze table, with the aid of a ruler and compass, I found what I sought." In this solution of the quadrature he constructed and recognized the cycloid. (9)

The quadrature of the cycloid was the subject of many controversies. Descartes and Fermat found themselves at odds on this score, for each had an independent solution. In a very gentlemanly way, Torricelli conceded Roberval priority on the same point when the latter had harshly accused him of publishing work not his own. Torricelli, however, stoutly maintained he had not stolen, but had effected his own solution. Again when Descartes suggested that Roberval's quadrature was not a brilliant one, there was another quarrel and a frequent exchange of letters. Probably the first quadrature was an experimental one by Galileo who is said to have obtained the quadrature of the cycloid to be nearly three times that of the generating circle by comparing the weights of their respective paper figures.

Another discussion was provoked by the prize offered by Blaise Pascal. Pascal (3) proposed the problem of finding the area CZY (see Fig. 5) and its center of gravity, the volume of the solid of revolution of CZY around ZY or CY, and finally the center of gravity of halves of the areas and solids. Forty pistoles of Spanish gold was the first prize, and twenty pistoles the second. Pascal agreed to publish his work if no solutions were offered before the end of the contest, October 1, 1658. The only competitors were Wallis and LaLoubere, but their work scarcely merited a prize and consequently no prize was awarded. Pascal published his own findings, according to his promise, and they were accepted immediately as both final and elegant.

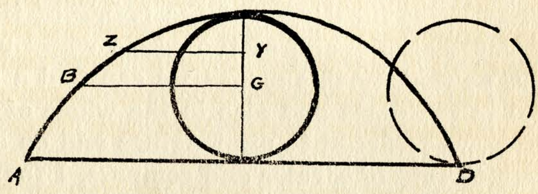

Fig. 5

Other outstanding achievements in this period were the construction of the normal to the cycloid by Descartes in 1637 and its tangent in the following year by Fermat. The cycloid was rectified in 1658 by Christopher Wren, the architect of St. Paul's Cathedral in London, and in 1659 Wallis found its solid of revolution.

The discovery of the Calculus marks the beginning of a period of expansion. More powerful methods served to clarify and to generalize the work already done while a volume of new theorems were added to its properties. The cycloid now became a particular case of the roulette, and attention was directed toward the general form, that is to say, when any curve was rolling.

The important work in this period was done by Huygens, Leibnitz and the Bernoulli brothers. The work of Huygens deserves particular mention. In a book (10) dedicated to King Louis XIV. of France, dated March 25, 1673, many new theorems appear, among which, theorems 15 and 25 are noteworthy. If any curve C rolls on a straight line, according to theorem 15, and if the point on the line which it continually touches be called P and the point on the surface of the curve C that generates the roulette R, be called P′, then the line PP′ is always normal to the roulette R.* Theorem 25 points out the tautochrone property of the cycloid. Huygens also showed that the cycloid was generated by its own evolute.

Roulettes, and in particular cycloids, have been carefully investigated and the properties uncovered found so beautiful, that they have been drawn upon constantly when a nice illustration or an example was needed. When the followers of Leibnitz and Newton were challenging each other with new problems involving Calculus, many roulette problems were hurled back and forth by the contenders.

There have been many papers written on roulettes during the nineteenth and twentieth centuries, including a treatise on *Roulettes and Glissettes* by Besant in 1890, but its rôle has now become a secondary one, and its field scarcely yields with the same fertility as in its earlier days.

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* This theorem arises and is developed quite elegantly in Section I (page 6) of this paper.

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