FOSTERING MATHEMATICAL MODELING THROUGH LESSON STUDY

by

Jenifer Hummer

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Education

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I dedicate this dissertation to my grandmother, Mary Phillis Hummer, who was a dedicated teacher. She always believed in me and inspired me to become who I am today.

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ABSTRACT

The Common Core State Standards for Mathematics (CCSSM) suggested that mathematical modeling, an important mathematical process, be incorporated into school mathematics. However, due to sparse resources for teaching mathematical modeling in the United States (Gould, 2013), it is expected that teachers need support to teach mathematical modeling. Turner et al. (2014) suggested that lesson study can support the implementation of mathematical modeling. Moreover, lesson study has been found to influence improvements in teachers' content knowledge, pedagogical knowledge, beliefs, and attention to student thinking (Lewis, Perry, & Hurd, 2009).

To address this issue, qualitative methods were employed to investigate how lesson study on mathematical modeling that made use of the "Five Practices" for orchestrating mathematical discussions (Stein, Engle, Smith, & Hughes, 2008) supported three secondary teachers' efforts to teach mathematical modeling. Additionally, the teacher participants were interviewed before and after the lesson study about their conceptions of teaching mathematical modeling.

The findings reveal how the teachers: anticipated valid and emerging student responses for a modeling task, used the five practices to ask questions and provide guidance to advance student thinking, and focused on student thinking to refine lesson plans. Also, the interview data provided evidence of ways in which the teachers' conceptions of teaching mathematical modeling evolved with respect to their views on mathematical modeling tasks and the benefits and challenges of teaching mathematical modeling. These findings indicate that lesson study can support the teaching of mathematical modeling.

Chapter 1

INTRODUCTION

Mathematical modeling is an important mathematical process that supports the understanding of real-world phenomena (Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), 2016). The implementation of modeling in K – 12 classrooms has the potential to provide opportunities for students that extend beyond conventional classroom practices. For example, when students engage in mathematical modeling, they are afforded opportunities to gain problem-solving skills that are transferrable to other contexts (GAIMME, 2016). Yet, because it is expected that teachers will experience challenges with teaching mathematical modeling (Gould, 2013), leaders in the U.S. mathematical modeling movement (e.g., Turner et al., 2014), have recommended that teachers receive support for teaching mathematical modeling through professional development (PD). Lesson study, a common form of professional development in Japan, is one strategy recommended for addressing these challenges. Consequently, the research discussed in this dissertation investigated how to foster the teaching of mathematical modeling through lesson study. The study described in the following chapters was guided by two research questions:

1. In what ways does teachers' participation in lesson study focused on mathematical modeling support their engagement with student thinking?

2. What are teachers' conceptions of teaching mathematical modeling before and after participating in lesson study on mathematical modeling?

This opening chapter will introduce the study and provide information about the organization of the dissertation. Then, in Chapters Two and Three, two separate papers are presented. Data related to the first research question is presented in Chapter Two, and data regarding research question two is presented in Chapter Three. Each of these chapters effectively functions as a separate manuscript and includes its own literature review, methods, findings, discussion, and conclusions sections. Finally, a fourth, concluding chapter is included to share overall reflections on the study, additional observations from the data, and suggestions for future research on lesson study and mathematical modeling.

Different Methods - Same Dissertation

Although this dissertation, as a whole, is about one study, different methodologies were employed to answer each of the two research questions. While both studies employed qualitative methodologies, each methods section has a different focus. Paper 1 contains specific details about the lesson study on mathematical modeling and describes the author's role as a participant observer (see Hatch, 2002) in the data collection and subsequent analysis. Paper 2 features interviews about teaching mathematical modeling that were conducted before and after the lesson study.

Organization of the Findings

The findings for each paper also influenced the organization of the dissertation. In Paper 1, the nature of the data influenced a cross-data analysis to investigate how the participating teachers engaged with student thinking as a lesson study team. This analysis was the most appropriate as the collaborative nature of lesson study influenced the outcomes. Even though each teacher enacted the lessons individually, they were implementing lesson plans that had been collaboratively developed. Thus, the findings are presented through themes that emerged from the analyses. In the next chapter, for Paper 2, interviews were the primary data source. Through constant comparative analyses of the interview transcripts, three cases emerged to show how the teachers' conceptions of teaching mathematical modeling evolved. In Paper 2, the findings are presented as case studies. Hence, given the differences in methodology and the outcomes for each study, I chose to present the research as two empirical studies rather than a single study.

Making Conscious Choices as a Researcher of Lesson Study

In the planning of the methodology, many factors were considered for this study. Given that lesson study is relatively new in the United States, I did not want the teachers to perceive lesson study as conventional PD or believe that the goal was to only develop lesson plans. Thus, I was careful to impress upon the teachers that the main idea was not to produce perfect lesson plans. I emphasized that lesson study, as it is practiced in Japan, is focused on teacher and student learning (Takahashi & McDougal, 2016; Stigler & Hiebert, 1999). Also, because Stigler and Hiebert (2016) suggested that re-teaching of the lesson may be more important in the context of the United States, I included multiple enactments of the lesson in the study design. I also carefully considered particular structural aspects of the U.S. school system that have influenced modifications to Japanese lesson study in previous U.S. studies (e.g., Lewis et al., 2009). As an example of a possible adaptation, based on logistical challenges, researchers used video-recordings of lessons rather than observing live lessons (e.g., Lewis et al., 2009; Inoue, 2011). Other research teams utilized summer sessions in the United States to hold lesson study meetings (e.g., Suh & Seshaiyer, 2014). More specifically, as indicated in Chapter Two, I made use of video-recorded lessons, as needed, and teachers' flexible schedules during the summer as a necessary adaptation due to the U.S. school structure.

In addition to attending to the school structure, I wanted to foster a collegial environment where all voices of the participants were respected and validated. In doing so, one goal of this study that went beyond the research questions was providing PD that would support teachers to engage in professionalism as described by Berry and Berry (2017). This type of PD involves a collective responsibility for growth and improvement by collaborating with and learning from other educators. Through providing this type of PD, I hoped to see that lesson study would: encourage the visibility of teacher knowledge through written lesson plans, promote teacher ownership, and support building norms for improvement among colleagues, as seen by previous researchers (e.g., Lewis et al., 2009).

What is in a Name?

In the spirit of maintaining and considering the professionalism of the teachers in the study, purposeful decisions were made about the pseudonyms for the teachers. Three secondary teachers participated in the study. In Paper 1, the teachers' work is often presented through transcript excerpts of their lesson enactments. Thus, in Paper 1, the teachers are referred to by last names as they would be by their students (e.g., Ms. Dain). However, in Paper 2, I chose to consider the teachers as my colleagues while we were discussing the teaching of mathematical modeling through interviews. Hence, the teachers are referred to by first names. The full pseudonyms are Loren Dain, Anne Maronis, and Karen Denvers. Additional details about the participants are included in subsequent chapters.

Optimizing the Presentation of Reviewed Literature

In setting up the next two chapters, it is important to note that, in contrast to a traditional dissertation with a single literature review, this dissertation has two distinct literature reviews which support the two related studies. Paper 1, presented in Chapter Two, attends to lesson study, the “Five Practices” for orchestrating mathematical discussions (Stein, Smith, Engle, & Hughes, 2008), and teaching mathematical modeling. In both papers, ideas about lesson study and mathematical modeling are important. However, in Paper 1 the literature on lesson study, which includes attending to student thinking, is foregrounded, while the literature on mathematical modeling is backgrounded. The reverse is true for Paper 2.

Chapter 2

TEACHERS' ENGAGEMENT WITH STUDENT THINKING WHILE PARTICIPATING IN LESSON STUDY ON MATHEMATICAL MODELING

Researchers have found that increased awareness of student thinking can improve teaching (e.g., Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hiebert & Stigler, 2004). When teachers attend to student thinking they can be better prepared to facilitate rich mathematical discussions that lead to conceptual understanding (Stein, Engle, Smith & Hughes, 2008). However, mathematics teachers have struggled with attending to student thinking (Pang, 2017; Smith & Stein, 2011, 2018). To further complicate the situation, teachers have struggled with attending to student thinking while implementing mathematical modeling (e.g., Pereira de Olivera & Barbosa, 2013; Thomas & Hart, 2013). This is unfortunate because teaching mathematical modeling can provide many opportunities for students.

One reason to teach mathematical modeling is that engaging students in mathematical modeling provides them with opportunities to experience an increasingly important mathematical process that is becoming more prominent in many science, technology, engineering, and mathematics (STEM) fields (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). Despite a need for the implementation of mathematical modeling, classroom teachers have reported challenges with teaching mathematical modeling due to its complexity (e.g., Ang, 2013; Kuntz, Siller, & Vogl, 2013). These challenges are possibly accentuated by a lack of training and curricula for teaching this important process (e.g., Meyer, 2015; Newton, Madea, Alexander, & Senk, 2014). Thus, researchers such as Ang (2013) and Kuntz et al. (2013) have recommended that preservice and inservice teachers receive formal training and professional development (PD) on implementing mathematical modeling. One type of

PD, lesson study, has specifically been recommended as potential PD for modeling (Turner et al., 2014). Lesson study consists of four essential activities: Curriculum Study, Lesson Planning, Teaching and Observing, and Debriefing (see Figure 2.1[1]).

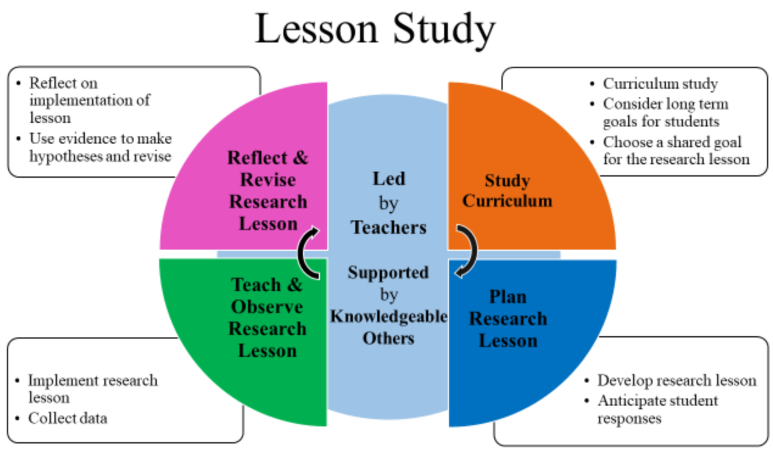

Figure 2.1 The Lesson Study Cycle

Researchers have suggested that, given adequate conditions, lesson study, a Japanese continuous improvement approach, can be an effective form of professional development (Takahashi & McDougal, 2016). While research is still emerging on how to best achieve those conditions in the United States (Lewis, C., 2016), some researchers have found that the activities and goals of lesson study naturally support engagement with student thinking, teacher learning, and the improvement of teaching (e.g., Lewis et al., 2009). For example, researchers observed how participating in collaborative planning sessions supported teachers in gaining content knowledge by revising or creating lessons plans and engaging in discussions about mathematics (see, e.g., Cajkler et al., 2015; Meyer & Wilkerson, 2011). The learning outcomes were influenced by a focus on student thinking while teachers participated in lesson study

[1] Figure 2.1 was inspired by the common components of lesson study as found in C. Lewis (2016), Suh and Seshaiyer (2014), and Takahashi and McDougal (2016).

activities (Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012; Stigler & Hiebert, 1999). This is important because this focus on student thinking could support teachers to engage with student thinking in ways that could improve the teaching of mathematical modeling.

The aim of this study was to investigate how teachers engaged with student thinking as they worked to improve their teaching of mathematical modeling through lesson study on mathematical modeling. The lesson study team consisted of one university researcher, who is the author of this paper, and three secondary teachers who completed two cycles of lesson study over the course of several weeks.

Background and Theoretical Framework

In this review of literature, the following ideas are discussed: mathematical modeling, lesson study, and a framework for engaging with student thinking. Within the discussion of mathematical modeling, opportunities and challenges of teaching mathematical modeling are presented. Then, relevant literature about lesson study is reviewed. Next, the “Five Practices” for orchestrating mathematical discussions (Stein, Engle, Smith, & Hughes, 2008) are described as a framework that supports teachers to engage with student thinking. Finally, these discussions are followed with suggestions for how integrating mathematical modeling, lesson study, and the Five Practices could support teachers’ engagement with student thinking, thereby improving their teaching of mathematical modeling.

What is Mathematical Modeling?

Mathematical modeling is an important mathematical process that is used in a variety of disciplines within fields of mathematics, science, engineering, and technology. Mathematical modeling is employed to better understand and control real-world phenomena and make predictions (GAIMME, 2016; Cirillo et al., 2016). Because there is not one standard definition of mathematical modeling, for the purpose of this study, a working definition of *mathematical modeling* is inspired by Cirillo et al.'s (2016), description of the features of mathematical modeling:

> Mathematical modeling is an iterative process that authentically connects to real-world situations; it requires creativity and making choices, assumptions, and decisions; and there can be multiple approaches or answers to developing and understanding a mathematical model.

Mathematical modeling is used in diverse contexts and is necessary for rapidly evolving societies. Thus, mathematicians and mathematics educators alike have recommended that mathematical modeling be taught across grade levels (e.g., GAIMME, 2016; Turner et al., 2014).

When students explore mathematical modeling, they have opportunities to engage with various aspects of the mathematical modeling cycle. According to GAIMME (2016), one possible description of the mathematical modeling cycle includes the following activities: "identify the problem, make assumptions and identify variables, do the math, analyze and assess the solution, iterate, and implement the model" (p. 12). While mathematical modeling has been recommended by policy documents (e.g., NCTM, 2000, 1989; National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA & CCSSO), 2010) and it may be necessary for modern careers, mathematical modeling was chosen as a topic for this study because teaching mathematical modeling has the potential to provide exciting new opportunities for students that go beyond conventional mandated mathematics content.

Why is Teaching Mathematical Modeling Important?

When teachers implement ill-structured mathematical modeling activities, students are provided opportunities to engage with rigorous mathematics. As an example of mathematical modeling's positive effects on learning, Ang (2013) worked with secondary students who reported that, through the process of engaging in mathematical modeling, they gained new knowledge of trigonometry. While learning mathematical content can be achieved through engaging in typical textbook tasks, the participating students reported finding it refreshing to apply their mathematical knowledge to a real-world context rather than within conventional textbook tasks that offer contrived contexts with simple solutions. Similarly, Stillman, Brown, and Galbraith (2013) observed that high school students' interest in mathematics increased when they were given the opportunity to choose relevant and important modeling activities related to real-world issues.

In addition to the potential positive outcome of increasing students' interest in mathematics, implementing mathematical modeling can also provide students with opportunities to address issues of equity and to learn important problem-solving skills. Equity can be addressed in a variety of contexts, such as providing students opportunities to explore the location and pricing of unhealthy food options (Cirillo, Bartell, & Wager, 2016), the locations of alternative financing institutions (e.g., check cashing shops; Rubel, Lim, Hall-Wieckert, & Katz, 2016), and environmental crises such as the water scandal in Flint, Michigan (Aguirre, Anhalt, Cortez, Turner, & Simic-Miller, 2019). Other researchers (e.g., Galbraith & Clatworthy, 1990; Gann, Avineri, Graves, Hernandez, & Teague, 2016) have observed that engaging in the mathematical modeling process can provide students with opportunities to: persevere through the challenges of real-world problem solving, collaborate with others other students, and gain problem-solving skills that are transferable to other contexts.

The various studies mentioned above indicated that teaching mathematical modeling can support students in gaining content knowledge, increasing interest, and exploring equity. Yet, despite policy recommendations and the purported benefits of teaching and learning mathematical modeling, teachers face various challenges with the implementation of mathematical modeling.

What are Some Challenges of Teaching Mathematical Modeling?

Challenges of teaching mathematical modeling are complicated by scarce research on teaching mathematical modeling (Kaiser, 2017). These challenges could be further influenced by a lack of teacher preparation. For instance, Newton, Madea, Alexander, and Senk (2014) found that only 15% of secondary programs surveyed (n = 72) required a mathematical modeling course as part of their teacher preparation program. Also, practitioners might be challenged in finding mathematical modeling activities in textbooks. Tasks labeled as "mathematical modeling" in textbooks may only provide limited opportunities for students to engage in mathematical modeling. An example of the need for better modeling tasks was illustrated by Meyer (2015) who discovered that items labeled "mathematical modeling" tasks in an Algebra 1 and a Geometry textbook primarily provided students only with opportunities to practice the modeling skills of "performing operations or interpreting results" (p. 581). The tasks did not provide many opportunities to engage with other important facets of modeling such as identifying variables or validating results in the analyzed tasks. Thus, more research and formal support for teachers to implement mathematical modeling may begin to address these challenges.

While a lack of preparation and resources is likely to influence the implementation of mathematical modeling, gaps in pedagogical knowledge could also hinder the teaching of mathematical modeling. Although the open-ended nature of mathematical modeling activities has the potential to provide many opportunities for students, this open-endedness can also lead to challenges for teachers. Some of these challenges are centered on the need to focus on student thinking while teaching mathematical modeling. For example, preservice teachers who completed modeling eliciting activities as part of their coursework reported a lack of self-efficacy for teaching modeling. The preservice teachers indicated that they may not be prepared to respond to multiple approaches to a modeling activity in the moment and may find it challenging to facilitate discussions about various student responses (e.g., Thomas & Hart, 2013). Some of these concerns about teaching mathematical modeling became evident in a secondary mathematics classroom observed by Pereira de Oliveira and Barbosa (2013). The participating teacher had completed two courses on modeling for teachers at a university in Brazil. However, when he implemented open-ended mathematical modeling activities in his own classroom, he struggled with facilitating discussion and responding to student thinking, especially when attending to unanticipated student responses. In this case, because he had completed two courses on modeling, the teacher likely had the content knowledge for teaching mathematical modeling, but his coursework had not adequately prepared him for engaging with student thinking with respect to teaching mathematical modeling.

In the study discussed above, the teacher struggled with facilitating discussion about a mathematical modeling activity. However, Warner, Schorr, Arias and Sanchez (2013) found that teachers *can* be supported to engage with student thinking and facilitate discussion about open-ended tasks (e.g., mathematical modeling tasks) through sustained PD. For example, during a four-year PD program, Warner and colleagues observed two middle school teachers in the United States improve their instruction by learning how to change their directive questions to guiding questions that built on student thinking. As a result, the participants' students began to develop their own mathematical ideas without teacher guidance. Although teachers may have concerns about teaching mathematical modeling, and there is evidence that it can be difficult to teach, sustained professional development, such as lesson study, that supports teachers' engagement with student thinking could support teachers in overcoming these challenges.

Supporting Teaching Through Lesson Study

In Japan, lesson study is often part of a large systematic school-wide structure that focuses on improving a specific problem or achieving specific goals (Takahashi & McDougal, 2016). To achieve these goals, C. Lewis (2016), indicated in her recent theoretical model of lesson study that Japanese lesson study consistently maintains five activities. In particular, teachers: study content and curriculum materials and consider long-term goals for students; collaboratively plan a lesson aligned to agreed-upon learning goals; observe the research lesson and record student responses; discuss and analyze the data collected during instruction and interact with other educators to discuss learning and content. Lesson study has been viewed as a means of PD for learning more about the process of teaching (Stigler & Hiebert, 2016) and could be a reliable way to store teachers' knowledge through the production of instructional materials such as lesson plans (Hiebert & Morris, 2012). These aforementioned attributes of lesson study could be a useful approach to support teachers in understanding the process of teaching mathematical modeling. Lesson study also has potential to provide teachers with opportunities to address another challenge of teaching modeling by developing shared knowledge through instructional materials.

Tools and Resources. The above-mentioned studies indicate that lesson study is a promising form of professional development. However, since it is a relatively new practice outside of Japan, for the benefits of lesson study to be realized, school culture and structure must be considered in order to provide necessary tools and resources for the success of lesson study. To better understand which tools and resources teachers may need, lesson study practitioners must attend to system features such as teacher schedules, school funding, teacher needs, and student needs. Additionally, in order for lesson study to be successful, teachers should be provided with high-quality curriculum materials and opportunities to focus on student thinking (Lewis, C., 2016). As a new content area within the United States, mathematical modeling tasks within curriculum materials are often scarce or inauthentic. Thus, engaging in lesson study can provide an opportunity for teachers to study existing curriculum materials and develop new materials such as lesson plans.

One important tool used in Japanese lesson study which may also be valuable for use outside of Japan is a lesson plan format consisting of the following components: information about students, lesson goals, anticipated student responses, planned instructor responses, evaluation of student learning, and hypotheses for future teachings (e.g., Gorman, Mark, & Nikula, 2010; Lewis & Hurd, 2011). This lesson plan format is often divided into three or four columns where the first column includes student activities and anticipated responses; the second column includes planned instructor responses, and the third and fourth columns include evaluation of student learning and hypotheses for future lessons. This lesson plan format could be especially useful for supporting the teaching of mathematical modeling activities for which students typically produce a wide variety of approaches and responses. Furthermore, to support lesson study in the United States, researchers have also leveraged particular tools and resources such as debrief protocols and data collection tools (Murata et al., 2012). These resources and tools have the potential to assist teachers in focusing on student thinking during lesson study.

To further assist teachers with the lesson study process, knowledgeable others, such as education researchers, school administrators, or veteran teachers, have been valuable resources for lesson study in Japan and in the United States (e.g., Lewis et al., 2009; Fernandez, 2005; Murata et al., 2012). During the lesson study process, knowledgeable others provide feedback, observe lesson enactments, facilitate meetings, and provide final comments on lessons (Takahashi & McDougal, 2016). As an example, an education researcher may serve as a facilitator and a knowledgeable other by organizing meeting structure, providing scholarly articles, and supporting growth in content knowledge and curriculum materials. As mentioned before, teachers may need outside support to teach the relatively new school content of mathematical modeling. Hence, the participation of knowledgeable others in lesson study can provide support that teachers may not receive in teacher preparation programs or other inservice teacher PD.

The role of a knowledgeable other in the United States might be quite complex as teachers in the United States have not historically participated in lesson study. In the case of mathematical modeling, the content may also be new for teachers. Thus, in order for lesson study to go well, it is recommended that several factors be in place. In particular, facilitators need to be familiar with current school trends. They also need to be prepared to solve mathematics tasks with teachers and demonstrate relevant mathematics to support teachers' content knowledge (Lewis, J. 2016; Murata et al., 2012). J. Lewis (2016) also found that facilitators must be prepared to address delicate situations such as teacher resistance or lack of confidence due to low content knowledge. For instance, facilitators my improve teacher collaboration and address resistance and discomfort by purposefully limiting their own participation in PD meetings. A facilitation move such as this could support participants to increase collaboration and participation in the group discussions (Lewis, J., 2016). When facilitators use the aforementioned tools and resources, they can support teachers in collaborating and focusing on student thinking to improve teaching and learning. These ideas suggest that facilitators of lesson study in the United States can provide critical opportunities for teachers to engage with student thinking in ways that could support the implementation of mathematical modeling.

Focus on Student Thinking. As mentioned earlier, there is evidence to suggest that the activities of lesson study tend to support teacher learning. One of the features of lesson study that guides each activity is a focus on student thinking. This focus on student thinking may not be included in traditional PD models (Murata et al., 2012). Attention to student thinking in teacher PD is important because when teachers understand student thinking, they are better prepared to provide student-centered instruction, facilitate rich mathematical discussions, and make connections that support student learning (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Franke & Kazemi 2011). Thus, prioritizing student thinking can encourage teachers to engage with student ideas and build on those ideas while leaving room for students to explore rigorous mathematics (Jacobs et. al, 2011). This focus on student thinking could be especially important for the teaching of mathematical modeling as teachers need to be prepared to respond to a variety of solution pathways and ideas from students. Teachers can also support the validation of mathematical models through the facilitation of whole-class discussions. Moreover, mathematical modeling can be a rigorous, cognitively demanding process, so a focus on student thinking could prepare teachers to provide opportunities for their students to explore rigorous mathematics.

Researchers have observed several opportunities for teachers to focus on student thinking within the lesson study process. As an example of engagement with student thinking, secondary teachers, observed by Inoue (2011) and Suh and Seshaiyer, (2014) used their range of content knowledge to serve as resources for one another while collaborating to plan lessons and anticipate student responses. Then, while teaching, those anticipated student responses supported the teachers in using student thinking to facilitate rich mathematical discussions. Collaborating to anticipate multiple student responses is likely to address challenges of responding to student thinking in the moment. Additionally, the anticipation of student responses could support the facilitation of whole-class discussions about multiple student solutions during mathematical modeling lessons.

In addition to anticipating student responses, observing lesson enactments and participating in debrief sessions can provide other opportunities for engaging with student thinking. For instance, participants observed by Cajkler et al. (2015) reported that it was especially beneficial to focus on student thinking while observing lesson enactments. Through focusing on student thinking while observing lesson enactments teachers can make improvements to lesson plans. In particular, pre-service teachers observed by Ricks (2011) participated in two cycles of lesson study which supported the generation of additional anticipated student responses. Another benefit for observing lesson enactments during lesson study was observed by Lewis et al. (2009). In that study, teachers collected evidence of student thinking during lesson enactments and decided their chosen task for the lesson may not have elicited rich student responses. As a result, during the debrief session, the teachers eliminated a worksheet and created a more open-ended task for students to complete. This revision positively influenced student learning during the reteaching of the lesson. Given the results from these empirical studies, one can hypothesize that participating in various activities of lesson study can influence the improvement of mathematics teaching through a focus on student thinking.

These various opportunities to engage with student thinking during the lesson study activities are likely to improve the teaching of mathematical modeling. For example, adding student responses that were observed during lesson enactments could address challenges of responding to partial solutions or unanticipated solutions during modeling lessons. Also, collecting evidence of student thinking during multiple lesson enactments and collaboratively debriefing with other teachers could influence the improvement of mathematical modeling activities. Additionally, developing authentic and rigorous mathematical modeling activities through lesson study could support teachers in building a catalog of mathematical modeling activities and student responses.

It is important to note that teachers may not only be influenced by the structured activities within the lesson study process. Additionally, as teachers learn about the process of teaching and continuous improvement, teacher practice may be influenced beyond the formal process of lesson study. For example, when Bruce and Ladky (2011) worked with 12 teachers who participated in two lesson study cycles, the teachers applied their learning from lesson study informally. More specifically, in between lesson activities, the participants in the study spent time, on their own, researching curriculum materials and brainstorming ideas. Also, after planning the research lesson, teachers used key ideas from the research lesson to support the planning of other lessons in their curriculum. Because these teachers learned systematic approaches that they could apply outside of formal lesson study, it is possible that lesson study can be a mechanism for supporting teachers in their overall improvement of practice. The transferability of knowledge gained during lesson study is especially important for an area like mathematical modeling where outside support for teachers may be limited.

Engaging with Student Thinking through The Five Practices

As mentioned above, a focus on student thinking is likely to support the teaching of mathematics and may be especially useful in the teaching of mathematical modeling. Thus, to examine how teachers engage with student thinking while participating in lesson study, this study employs a framework that could be well-suited for lesson study. Stein et al. (2008) introduced five practices (5Ps), which were later written about extensively in two editions of *5 Practices for Orchestrating Productive Mathematics Discussions* (Smith & Stein, 2011, 2018), a book that is considered to be an important resource for teaching open-ended tasks and hence could support the teaching of mathematical modeling activities. The 5Ps are as follows:

1. *anticipating* likely student responses to challenging mathematical tasks and questions to ask students who produce them;

2. *monitoring* students' actual responses to the tasks (while students work on the tasks in pairs or small groups);

3. *selecting* particular students to present their mathematical work during the whole-class discussion;

4. *sequencing* the student responses that will be displayed in a specific order; and

5. *connecting* different students' responses and connecting the responses to key mathematical ideas (Smith & Stein, 2018, pp. 9 – 10).

The 5Ps seem especially well-suited to complement the activities of the lesson study process. The alignment of the 5Ps and lesson study is exemplified through the connection of lesson study's focus on student thinking. When executing the 5Ps teachers must engage with student thinking in ways that will advance student thinking and learning. An example of this connection is in the first of the 5Ps, anticipating, because as part of planning activities in lesson study, teachers anticipate student responses. Additionally, the 5Ps are likely to be a valuable framework in the teaching of mathematical modeling activities which have complex solutions and require skilled orchestration of mathematics discussions.

Stein et al. (2008) theorized the 5Ps by drawing on the work of other mathematics education researchers such as Schoenfeld (1998), Smith (1996)[2], and Wood, Cobb, and Yackel (1991). The remainder of this section includes descriptions of the 5Ps as developed by Stein et al. (2008) as well as additional related literature that influenced the data analysis for this study.

[2] To be clear, because the works of two well-known, yet different authors, with the last name Smith are discussed in this section, to avoid confusion, Smith (1996) refers to J., Smith (1996).

Anticipating. As described in *A Mathematics Leader's Guide to Lesson Study in Practice*, anticipating student responses includes anticipating "things students might do, say, think, or feel as they tackle the lesson activities and mathematics" (Gorman et al., 2010, p. 85). The first step in anticipating student responses to a task is for teachers to solve the task themselves. Stein et al. (2008) recommended that teachers "put themselves in the position of their students" while solving the task (p. 323), and teachers should attempt to solve the task in multiple ways. In addition to anticipating correct solutions, teachers should also anticipate partially correct strategies or possible misconceptions that students may have. Engaging in the tasks as learners during lesson study could be especially useful for teaching mathematical modeling as teachers could learn more about the process of modeling, a process in which they have likely had little experience. Furthermore, multiple teacher solutions within a lesson study team could support the anticipation of multiple student responses.

Solving the task and anticipating student responses supports teachers for a variety of reasons. One such reason, as suggested by Smith (1996), is that when teachers anticipate possible student responses and language they intend to use while solving tasks, the teachers are better prepared to facilitate mathematical discourse. Researchers have also suggested that anticipating student responses supports teachers in planning how to engage with student thinking in the moment and allows teachers to plan in advance about how to organize the sharing of student responses, further contributing to the main point or learning goals of the lesson (e.g., Gorman et al., 2010; Inoue, 2011). Anticipating multiple student responses, including partial or incomplete responses, is especially important for mathematical modeling activities which can have multiple solution approaches and results.

Monitoring. The anticipated student responses can then inform monitoring as the teacher circulates around the room and attends to student thinking (Stein et al., 2008). While monitoring, the teacher attends to anticipated student responses as well as unanticipated student responses. Yet, as some researchers have pointed out, monitoring consists of more than simply walking around the room and choosing responses to share with the class. One way to think about monitoring is through what Japanese teachers refer to as *kikan shido* described as “between desks instruction” (Clarke et al., 2007, p. 287) or “instruction at students’ desks” (Shimizu, 2007, p. 65). During monitoring, the teacher observes student progress by listening to group discussions and reviewing written work. The teacher uses discretion as to when to interact with students (Clarke et al., 2007). Knowing when to interject requires that teachers pay close attention to the mathematical ideas that students say or write while they solve a task. During this time, the teacher assesses students’ progress, guides students who are struggling, and selects student responses to share with the class (Clark et al., 2007; Shimizu, 2007; Stein et al., 2008). The teacher’s monitoring is purposeful and guided by student thinking, planned instructor responses, and planned selection and sequencing of responses. While students are exploring modeling activities, they may need support to make decisions, define variables, perform computations, and validate results. Thus, monitoring while teaching modeling could be quite complex for teachers. Planning ahead to monitor student progress during modeling could enhance students’ learning opportunities.

Assessing and Advancing. While monitoring, teachers ask questions to assess and guide student thinking. This includes asking questions that can be planned ahead of time. For example, *assessing questions* can act as monitoring tools for teachers to ask in order to understand how students are thinking about a problem-solving process (Smith, Bill, & Hughes, 2008; Smith & Stein, 2018). Instances of the use of assessing questions and their importance have been observed by various researchers. For example, assessing questions were critical, in a case study by Wood, Cobb, and Yackel (1991), where an elementary teacher discovered how to support students' deeper understandings of mathematics by asking questions that encouraged students to explain their thinking in detail. Schoenfeld (1998) observed a high school physics teacher using clarifying questions to elicit students' thinking. Through asking clarifying questions, the teacher provided students with opportunities to provide most of the "intellectual content" during the lesson (p. 54). Once a student's thinking has been made visible, the teacher may also ask *advancing questions* that are intended to support the student in making further progress towards the goals of the lesson (Smith et al., 2008; Smith & Stein, 2018). Asking assessing and advancing questions supports student agency, which is important for mathematical modeling, in the sense that doing so positions students as people who are capable of making sense of challenging mathematics. To support teachers in making in-the-moment decisions, many of the assessing and advancing questions may be planned ahead of time based on the anticipated student responses. Doing so could be especially useful for teaching mathematical modeling because teachers must be prepared to assess students' progress due to the iterative nature of modeling activity. Collaboratively planning assessing and advancing questions for modeling activities during lesson study could address challenges of responding to student thinking in the moment.

Judicious Telling. Another monitoring tool to support the advancement of student thinking is judicious telling (Smith, 1996). When teachers anticipate student responses, they can also plan how they will judiciously tell to advance student thinking. Judicious telling can support student discourse and student thinking in intentional ways (Smith, 1996). When teachers judiciously tell, they provide purposeful additions such as specific terminology, mathematical representations, or counterexamples. This type of telling does not include a simple demonstration of steps and procedures that negatively influences the cognitive demand of mathematics tasks. Teachers may also choose to revoice students' ideas or remind them of strategies that were used in earlier lessons (Lobato, Clarke, & Ellis, 2005). Judicious telling can allow students to explore mathematics and maintain their agency while deciding on and verifying a valid mathematical approach (Lobato et al., 2005; Hiebert et al., 2007). Teachers can plan ahead to be intentional and practice judicious telling when deciding which mathematical ideas they will share to advance student thinking. Judicious telling may be critical in supporting students' creativity through various aspects of the modeling process such as making decisions and assumptions, defining variables, and developing a formula.

Selecting. Another goal of monitoring student thinking is to observe student responses which may be selected to share with the class (Stein et al., 2008). The teacher may choose specific student responses to be shared with the class in order to advance student thinking during the problem-solving process, or the teacher may save the selected responses for a whole-class discussion to bring closure to a task and achieve a learning goal of the lesson (Stein et al., 2008). When the teacher selects student responses for sharing out solutions, the responses can include both correct and incorrect answers (e.g., Clarke et al., 2007). Being prepared to select incorrect or partially correct solutions aligns well with mathematical modeling since part of the process requires validating and verifying solution pathways. Selecting the student responses relies heavily on the anticipated student responses but may also include unanticipated responses. This is important as unanticipated student responses can be prevalent for modeling activities. Teachers' skillful work with unanticipated student approaches to modeling activities may support students in making connections and building understanding for mathematical modeling.

Sequencing. Once the teacher has selected which student responses will be presented, the teacher must decide how to order the responses for sharing (Stein et al., 2008). This decision is also guided by the anticipated student responses, and teachers can plan ahead for how they will share the responses. The sequencing of student responses is purposeful and should be done in such a way that supports students in building understanding and making connections across student responses and other mathematics concepts (Schoenfeld, 1998; Smith & Stein, 2018). The teacher may choose to share unique solutions or solutions that were common among students' groups. The teacher may also choose to share incorrect or incomplete solutions first in order to lead a whole-class discussion about how to learn from students' errors or provide students an opportunity to guide each other towards a solution. Sequencing in a way such that simple strategies are shown first may support students in understanding more complex solutions (Stein et al., 2008). Sequencing partial solutions or different approaches is likely to be a useful strategy for supporting students in exploring mathematical modeling activities that can have a variety of approaches that need to be verified by the class. When student responses are purposefully selected and sequenced, those student responses can support students in making connections to the mathematical points of the lesson.

Connecting. One purpose of selecting and sequencing student responses is to support students in connecting one another's responses to gain deeper understanding of mathematical content (Stein et al., 2008). When guiding whole-class discussions focused on student responses, one of the teacher's goals should be to facilitate students in making connections. This may involve asking questions to guide students in making connections across the different student responses, or teachers may choose to remind students how various strategies relate to previous mathematics content. For example, Wood, Cobb, and Yackel (1991) described how an elementary school teacher used a variety of student responses to facilitate whole-class discussions allowing the students to make connections, build understanding, and explore other students' thinking about mathematics. The teacher said things like "Explain how you got your answer," and "Did somebody get it a different way?" along with encouraging several groups of students to share their responses. In asking connecting questions, such as those observed by Wood, Cobb, and Yackel (1991), students were encouraged to compare their different responses, interpret the mathematics, and verify valid solutions, as is well-aligned with the mathematical modeling process.

Integrating Mathematical Modeling, Lesson Study, and the Five Practices

The Society for Industrial and Applied Mathematics (SIAM) **–** National Science Foundation (NSF) Workshop, *Modeling Across the Curriculum II* report, (Turner et al., 2014) included suggestions that explicitly recommended teachers' use of lesson study and the 5Ps when implementing mathematical modeling. At the same time, previous studies have indicated that the combination of lesson study and use of the five practices can support teachers in engaging with student thinking (Lim, Kor, & Chia 2016; Pang, 2016). Thus, it is likely that integrating mathematical modeling, lesson study, and the 5Ps will have positive outcomes on teachers' practice. For example, Pang observed how teachers' participation in lesson study improved their use of the 5Ps over time. At first, the teachers struggled with anticipating student responses and responding to student thinking in the moment. However, by the third cycle of lesson study, the teachers were more likely to use anticipated student responses to purposefully and strategically make connections, select, and sequence student responses while conducting meaningful whole-class discussions. In another case, Lim and colleagues (2016) observed that over three lesson study cycles, teachers improved their engagement with student thinking with respect to the 5Ps. For instance, the teachers refined and revised their catalog of anticipated student responses. These evolving catalogs of student responses influenced how the teachers selected and sequenced student solutions to share with the class. The findings from both of these studies show how the use of the 5Ps can be improved through participation in lesson study. Thus, one can hypothesize that an integration of the 5Ps and lesson study will support teachers' engagement with student thinking while teachers work to improve their teaching of mathematical modeling.

Integrating mathematical modeling, lesson study, and the 5Ps may address multiple challenges of teaching modeling. More specifically, a focus on student thinking which makes use of the 5Ps could alleviate challenges of engaging with student thinking in the moment while teaching mathematical modeling (see e.g., Pereira de Oliveira & Barbosa, 2013). Teachers' participation in modeling-focused lesson study also addresses a lack of teacher preparation for a relatively new topic to school mathematics. Additionally, because modeling tasks in mainstream textbooks are currently limited, it is reasonable to hypothesize that lesson study can support teachers with selecting and developing authentic mathematical modeling activities through curriculum study and collaboration with other teachers. Hence, this study seeks to explore the possibilities and outcomes of lesson study, which makes use of the 5Ps to support teachers to engage with student thinking while working to improve their teaching of mathematical modeling.

In particular, this study will investigate how the teachers use the 5Ps to engage with student thinking while planning, enacting, and debriefing lessons on mathematical modeling. The following question guided the research:

> In what ways does teachers' participation in lesson study focused on mathematical modeling support their engagement with student thinking?

This study will make progress on understanding the challenges of using student thinking productively in the context of teaching mathematical modeling and ways to overcome those challenges. The findings from this study also inform future research and implementation of lesson study in the United States. In the subsequent sections, details are included about how this study employed qualitative methodology to reveal how the teachers engaged with student thinking while participating in lesson study.

Research Methods

To investigate ways in which secondary teachers engaged with student thinking during lesson study on mathematical modeling, qualitative research methods were employed. In the methods section, the participants, lesson study activities, data, and analytic methods are described in detail.

Participants

Three secondary teachers teaching in a vocational high school with a diverse student population in the mid-Atlantic region of the United States participated in this study (see Table 2.1; all names are pseudonyms). The participants were recruited based on their interest in improving their teaching of mathematical modeling. The teachers committed to participation while funding for the project was pending[3], and they were willing to participate on a voluntary basis. However, by the start of the project, funding had been obtained, and the teachers were promised a stipend as well as curriculum materials on mathematical modeling.

Table 2.1 Credentials of Lesson Study Participants

Teacher	**Education**	**Years Teaching**	**Experience Teaching Modeling**
Ms. Dain	• Bachelor of Arts in Secondary Mathematics Education	1	None
Ms. Maronis	• Bachelor of Science in Electrical Engineering • Master of Science in Curriculum and Instruction	6	None
Ms. Denvers	• Bachelor of Science in Computer Information Systems • Master of Arts in Elementary Education • Specialization Credits in Secondary Mathematics	21	Some experience

At the time of the study, Ms. Dain, the most novice participating teacher, was in her second year as a high school mathematics teacher. She recently earned her bachelor's degree in secondary mathematics education. As part of her degree program, Ms. Dain completed one course on mathematical modeling for secondary teachers. However, she had not implemented mathematical modeling in her own classroom. The

[3] The author applied for a dissertation support award that was granted through the University of Delaware.

second teacher, Ms. Maronis, had six years of teaching experience. After working as an engineer for several years, she changed her career to education. She had a master’s degree in curriculum and instruction. Ms. Maronis also had no experience implementing mathematical modeling. The most experienced teacher, Ms. Denvers, had 21 years of experience teaching. She had a bachelor’s degree in computer information systems and a master’s degree in elementary education. While teaching elementary school, Ms. Denvers completed coursework for specialization in secondary mathematics. She had previously participated in professional development on mathematical modeling, and she had been implementing mathematical modeling activities in her classroom.

Description of the Lesson Study Activities

This section describes, in detail, the lesson study activities in which teachers engaged with student thinking. The lesson study activities (see Figure 2.2) were facilitated by the researcher. The researcher served as a “knowledgeable other” by providing information on lesson study, curriculum materials on mathematical modeling, relevant articles from practitioner journals on teaching mathematics, and guiding discussions during meetings. After the teachers were introduced to lesson study, and they explored curriculum materials, they planned a two-day lesson on mathematical modeling. Then Ms. Dain enacted the lesson while the other teachers observed. After her enactment, the team participated in a debrief session. Next, Ms. Denvers and Ms. Maronis enacted the lessons. Finally, a second debrief session was held to discuss Ms. Denvers’ and Ms. Maronis’ lesson enactments. The details for these activities are described below.

Figure 2.2 Lesson Study Sequence

Introducing the Lesson Study. Because the teachers had never participated in lesson study, the researcher presented the main ideas of lesson study to the teachers (see Appendix A: Meeting Agendas). First, as part of an overview of lesson study, the participants watched a three-minute video clip introducing lesson study. This clip included student and teacher testimonies about the benefits of lesson study (Fischman, Aikin, & Wasserman, 2018). Next, the lesson study team watched four video clips, totaling twelve minutes of video, of teachers participating in each activity of lesson study (see Lewis & Hurd, 2011). After each video clip, the researcher facilitated a discussion about what the participants noticed pertaining to the video teachers' engagement with student thinking.

During the introduction to lesson study, the researcher emphasized how lesson study maintained an important focus on student thinking. First, the 5Ps were introduced as a tool for engaging with student thinking during the lesson study. Then the researcher introduced the lesson plan template that would be used for the lesson planning process (see Appendix B). The template was used so that the lesson plan could be annotated with anticipated student responses and instructor support. Last, as a transition to the curriculum study, the teachers anticipated student responses to a mathematical modeling task[4] and discussed how anticipating responses might be useful for implementing mathematical modeling lessons.

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[4] This task was different from the task (see Appendix A) implemented in the actual lesson study.

Studying Curriculum. After learning about lesson study, the teachers engaged in a curriculum study where they explored documents such as the Common Core State Standards for Mathematics (CCSSM; NGA & CCSSO, 2010) and other resources that could support the teaching of mathematical modeling. In this study, the teachers were introduced to the working definition of mathematical modeling that was introduced earlier in this paper. The teachers also had the opportunity to compare different modeling cycles (see, e.g., Giordano, Weir, & Fox, 1997). The teachers were given a binder containing resources such as: observation forms for lesson study, a chapter introducing the 5Ps, standards related to mathematical modeling, and printed PDFs of books containing mathematical modeling tasks (see Table A.1 in Appendix A for the list and description of the binder materials). Teachers were also provided with electronic copies of all the materials. To support the curriculum study, the researcher facilitated discussions about the materials in the binder. Then the teachers were given one hour to individually explore the different modeling tasks within the distributed materials. As part of their exploration of modeling tasks, the teachers were asked to consider which tasks, if any, they would like to use in the lesson plan.

Planning the Lessons. The lesson planning took place over two meetings during the summer. The lesson study team met for four hours on the first day and three hours on the second day. The second planning meeting occurred a little over a week later than the first planning meeting. The teachers had four days in between the curriculum study meeting and the first lesson planning meeting to review mathematical modeling tasks in their materials and choose their top two tasks for the lesson plan. They were also encouraged to bring a modeling task if they had implemented any in their classroom as one of their two choices. In the beginning of the first planning meeting, each teacher shared their two picks for a mathematical modeling task. Two of the teachers suggested the “State Apportionment Task” (Sanfratello, 2012, pp. 133 – 140) from the *Mathematical Modeling Handbook* (Gould, Murray, & Sanfratello, 2012). Thus, after some discussion, all three teachers decided to use this task (see Appendix C for Version 2 of the lesson plans and task). The task provides opportunities for students to apportion state representatives, as is done for the U.S. Congress, using a variety of methods, for a fictional country. After choosing the task, the lesson study team proceeded to develop lesson plans.

Due to the length of the mathematical modeling task, the teachers decided that the lesson would span two days of classroom instruction (i.e., two 90-minute periods). Thus, for each day of instruction, a separate lesson plan was written. The task was divided into several sub-tasks. From here on out, these sub-tasks are referred to as “tasks.” Both modeling lesson plans contained their own launch activity and a set of four tasks. The teachers anticipated student responses based on their own approaches to the activity. This is important because mathematical modeling activities usually do not have one single approach. Thus, students could give responses that are partially correct or have partial answers. The teachers decided to plan for “valid responses” and “emerging responses” rather than “correct” and “incorrect” responses. Valid responses would include any logical approach that could be validated through the modeling process. Whereas, emerging responses were incomplete, partial, or otherwise inconclusive approaches to a task. While planning the possible student responses, the teachers also planned strategies for monitoring such as types of questions to ask or what to say in order to “judiciously tell” (Lobato et al., 2005). Also, the teachers planned which student responses they would select to report out.

While the teachers anticipated student responses and planned teacher actions, the researcher facilitated the meeting and filled in the lesson plan template using the ideas provided by the teachers. Although the researcher only introduced the 5Ps, the teachers prompted each other to incorporate the 5Ps into the lesson plan. To make efficient use of their time together, the researcher typed the teachers' anticipated student responses and their planned monitoring, selecting, and sequencing moves into the template as the teachers collaboratively planned the lessons. The template was displayed on a screen for the teachers' viewing so that they could verify accuracy. The researcher asked clarifying questions and asked the teachers to review the lesson plan periodically to ensure that the plan accurately captured the teachers' thoughts and ideas rather than the researcher's interpretations. Table 2.2 includes a detailed timeline of events for the lesson study activities.

Table 2.2 Lesson Study Timeline

Activities	**Timeline**
Introduction to Lesson Study (2 hours) Curriculum Study (2 hours)	Thursday, July 26, 2018
Lesson Planning Meeting 1 (4 hours) Lesson Planning Meeting 2 (3 hours)	Tuesday, July 31, 2018 Friday, August 10, 2018
Ms. Dain Enacted the Lesson Ms. Denvers Enacted the Lesson Ms. Maronis Enacted the Lesson	Tuesday, September 25, 2018 Wednesday, September 26, 2018 Monday, October 15, 2018 Tuesday, October 16, 2018 Thursday, November 8, 2018 Friday, November 9, 2018
Debrief Ms. Dain's Lesson Enactment Debrief Ms. Denvers' and Ms. Maronis' Lesson Enactments	Wednesday, September 26, 2018 Friday, November 9, 2018

Enacting and Debriefing the Lessons. About one month after the final lesson planning meeting, Ms. Dain was the first to enact the lesson. On the first day of the lesson enactment, both Ms. Maronis and Ms. Denvers observed the teaching of Ms. Dain, and the researcher video-recorded the lesson enactment. Ms. Maronis and Ms.

Denvers used the observation form (see Appendix D) to keep track of student thinking during the first lesson enactment. On the second day, Ms. Maronis was not able to observe the teaching due to a family emergency. Ms. Denvers observed Day 2 of Ms. Dain's lesson, and the lesson study team met after school on the second day of Ms. Dain's teaching. Ms. Maronis joined the meeting via telephone. The researcher executed the lesson study debrief protocol (see Appendix E) to facilitate the discussion. First, Ms. Dain shared her reflections on the lesson enactment. Then the other group members responded to Ms. Dain's reflections and shared their initial observations. Next, the team reflected on whether or not the learning goals were met and revised the learning goals to better align with the teachers' intentions for student outcomes. Then, in following the debrief protocol, each activity in the lesson was revised using evidence of student thinking that had been collected during the observations.

For the second lesson enactment, about three weeks after Ms. Dain's enactment, Ms. Denvers taught the lesson. For this enactment, Ms. Dain observed live, and Ms. Maronis watched the video-recordings of Ms. Denvers' lessons. Ms. Maronis taught the final enactment about three weeks later. There was no debrief meeting after Ms. Denvers' enactment due to time constraints. However, between the second and third teaching cycle, Ms. Denvers shared her Desmos (n.d.) activities with Ms. Maronis in between the second and third enactments. This informal interaction between Ms. Denvers and Ms. Maronis influenced minor changes to the third lesson enactment. For the third enactment, Ms. Dain observed Ms. Maronis' lesson live, but because Ms. Denvers was not able to have her classes covered, she watched the videos of the lesson enactments prior to the debrief meeting. The final debrief session was held on the second day of Ms. Maronis' teaching of the lesson. The researcher executed the debrief protocol in the same way that the first debrief session had been facilitated. Final changes were made to the lesson plans so that the teachers could use the improved lesson plans in the future.

Data

All meetings, lesson enactments, and debrief sessions were video and audio-recorded. Then video and audio segments were selected for transcription. The decisions for selecting the video and audio segments are described in the next section. The data analyzed for this study included transcripts from the planning meetings, lesson enactments, and debrief sessions. Additionally, the written lesson plans were used as further evidence to support findings related to how the teachers engaged with student thinking. Table 2.3 shows the data sources that were used for analysis. In the next section, details are presented on the analytic methods as well as the data reduction process that resulted in the final selection of units of analysis.

Table 2.3 Data Collected

Data	Number
Planning Meeting Transcripts	2
Written Lesson Plans	6 (2 lesson plans x 3 versions)
Lesson Enactment Transcripts	6 (2 lesson plans x 3 lesson enactments)
Debrief Meeting Transcripts	2

Data Analysis

Using ideas from the literature described earlier as well as emerging themes and trends from the transcripts (see, e.g., Hatch, 2002; Strauss, 1987), the transcripts were analyzed using deductive and inductive analytic techniques. First, for data reduction, particular tasks from the lesson plans influenced the selection of units of analysis. Then a coding dictionary, developed from the 5Ps, was applied. Through multiple iterations of coding, the coding dictionary was revised. The initial findings and themes were organized into analytic memos consisting of charts from Dedoose (2016), main themes, transcript excerpts, and diagrams such as flow-charts. Following is a discussion of the data analysis, including the process for data reduction, the units of analysis, and the coding categories.

In an effort to reduce the data, specific units of analysis were selected. The data reduction was guided by a decision to analyze transcripts centered around specific tasks in the mathematical modeling lesson. These tasks were chosen because they met at least one of the following criteria:

A. The teachers revised the task during the first debrief meeting.
B. The task was open-ended and provided students with opportunities to develop their own methods of apportionment.
C. The task had a wide variety of valid approaches or responses.

These features were chosen because the author hypothesized that tasks meeting these criteria would provide more elaborate evidence for how teachers attended to student thinking; whereas, some of the tasks required simple solutions, so there were limited opportunities for teachers to engage with student thinking.

Table 2.4, below, includes the tasks selected for analysis as well as the selection criteria that were met. All but one of the tasks, Day 2 Launch, met criteria C. The Day 2 Launch, met criteria A, and was selected for analysis because Ms. Dain added this task to the lesson in between the lesson planning meeting and the first lesson enactment, as indicated by the greyed-out cell. The remaining tasks in the lesson required simple calculations, one valid approach, or were not centered around mathematical approaches. Each “X” indicates a unit of analysis (n = 39). Transcripts about each task within the lesson study activities were selected for analysis.

Table 2.4 Units of Analysis: Transcripts of Discussions and Enactments of Selected Tasks Analysis

Tasks Selected for Analysis	**Criteria for Selection**	**Lesson Study Activities**					
		Lesson Planning Meetings	**Lesson Enactment 1 (Loren)**	**Debrief Session 1 (Loren)**	**Lesson Enactment 2 (Karen)**	**Lesson Enactment 3 (Anne)**	**Debrief Session 2 (Karen & Anne)**
Day 1 Launch	A, B, C	X	X	X	X	X	X
Day 1 Task 1	B, C	X	X	X	X	X	X
Day 1 Task 2	C	X	X	X	X	X	X
Day 2 Launch	A		X	X	X	X	X
Day 2 Task 1	C	X	X	X	X	X	X

Table 2.5, below, includes examples of tasks that were selected for the data analysis. For contrast of the different types of tasks, one task is included in the table that was not selected for data analysis. In particular, Day 1 Task 3 provided students with an opportunity to explore Hamilton’s method of apportionment, but it was not selected for data analysis because the solution approach required simple computations, and there was only one valid response to this task. In contrast, Day 1 Task 1 provided opportunities for students to develop their own state apportionment method and could yield a wide variety of approaches. Day 2 Task 1 provided students with an opportunity to explore Thomas Jefferson’s method of state apportionment and develop various approaches. These examples illustrate how the tasks selected for analysis were more likely to provide teachers with opportunities to engage with student thinking and use the 5Ps.

Table 2.5 Example Tasks

<table>
<tr><td colspan="2">Selected for Data Analysis</td></tr>
<tr>
<td>
Day 1 Task 1
For simplicity, imagine that a newly formed country wishes to copy the U.S. House of Representatives. This new country has just 100,000 people split up into only four different states, listed in the table below.
<table>
<tr><th>State</th><th>Population</th></tr>
<tr><td>A</td><td>15,000</td></tr>
<tr><td>B</td><td>17,000</td></tr>
<tr><td>C</td><td>28,000</td></tr>
<tr><td>D</td><td>40,000</td></tr>
</table>
A. If the new country plans on having 25 representatives in its House of Representatives, how many should each state receive?
B. What if they plan to have only 17 representatives? How did you calculate how many representatives each state should receive? Did you use the same method for both 25 and 17 representatives?
</td>
<td>
Day 2 Task 1
Watch the video on the Jefferson method, apply the Jefferson method for 25 representatives.
Link: https://tinyurl.com/SGJefferson

Use the tables to apply the Jefferson method.

Jefferson's Apportionment for 25 Seats
<table>
<tr><th>State</th><th>Population</th><th>Calculation</th><th>Number of Seats</th></tr>
<tr><td>A</td><td>15,000</td><td></td><td></td></tr>
<tr><td>B</td><td>17,000</td><td></td><td></td></tr>
<tr><td>C</td><td>28,000</td><td></td><td></td></tr>
<tr><td>D</td><td>40,000</td><td></td><td></td></tr>
</table>
Why did Jefferson use this method?
What are the differences and similarities between the Jefferson Method and the Hamilton Method?
</td>
</tr>
<tr><td colspan="2">Not Selected for Data Analysis</td></tr>
<tr><td colspan="2">
Day 1 Task 3
The Hamilton Method was devised by Alexander Hamilton as a technique for fair apportionment. Investigate what the Hamilton Method was and if you agree or disagree with its fairness. Do either of your methods share any similarities with the Hamilton Method?
Watch the video on the Hamilton Method: https://tinyurl.com/SGHamilton
</td></tr>
</table>

Table 2.6, below, includes the coding categories as they were applied to the units of analysis for each lesson study activity. Once all of the data were analyzed, final memos were developed in which the data were organized by lesson study activity and themes. Those memos informed the findings for the study which are described in the next section.

Table 2.6 Five Practices Coding Categories

5Ps Category	Lesson Study Activities: Planning the Lesson	Enacting the Lessons	Debriefing the Enactments
Anticipating	• Anticipated Emerging Response • Anticipated Valid Response • Developed Teacher Approach to the Tasks • Referred Back to Teachers' Approaches to the Tasks	• Used Anticipated Emerging Response • Used Anticipated Valid Response • Used Unanticipated Emerging Response	• Refined & Added Emerging Responses • Refined & Added Valid Responses
Monitoring	• Planned Advancing Questions (e.g., Smith et al., 2008) • Planned Judicious Telling (Lobato et al., 2005)	• Asked Advancing Questions • Asked Assessing Questions (e.g., Smith et al., 2008) • Use Judicious Telling	• Revised planned monitoring o To support student responses o To support student actions
Selecting	• Planned Selecting	• Selected Emerging Response o Anticipated o Unanticipated • Selected Valid Response o Anticipated o Unanticipated	• Revised Selecting • Hypothesized Outcomes for Changes to Selecting
Sequencing	• Planned Sequencing • Planned Building and Connecting (e.g., Simple to Complex)	• Sequenced Emerging Response o Anticipated o Unanticipated • Sequenced Valid Response o Anticipated o Unanticipated	• Revised planned sequencing
Connecting	• Planned Connections • Planned Student-Led Connections	• Asked Connecting Questions • Connected Student Responses • Connected to the Real World • Used Revoicing to Connect Student Responses	• Modified Tasks

Findings

The findings are organized according to how the participating teachers engaged with student thinking, as they worked to improve their teaching of mathematical modeling, during each of the lesson study activities: Planning the Lesson, Enacting the Lesson and Debriefing the Lesson. Evidence within the findings consists of samples from the analyses of the lesson plans and transcripts. Since the analysis was conducted across the data, the findings within each sub-section are organized by themes about how the teachers (a) anticipated student responses, (b) monitored, and (c) selected, sequenced, and connected student thinking.

Lesson Study Activity: Planning the Lesson

While the teachers were Planning the Lesson, they were provided with time to collaboratively hypothesize student thinking. The findings within this section are organized by themes focused on teachers' *developing solutions to* anticipate *multiple student responses*, planning to *advance student thinking* while monitoring, and planning to connect, select and sequence *simple to complex* student responses. Through engaging in student thinking, the teachers worked to improve their implementation of mathematical modeling.

Developing Approaches and Anticipating Multiple Student Responses. For the first theme, as part of anticipating student responses, the teachers decided to *develop their own responses* for the modeling tasks. In the transcript below, Ms. Maronis suggested that she and her colleagues complete the tasks to more accurately anticipate student responses:

Ms. Maronis	I'm wondering if it's worth doing the task. I guess it won't take too long, or at least kind of exploring it a little bit so that we

know what [students] need to know. ‘Cause we can sit here and look at it, but we don't really know what they need to know.

Ms. Denvers Yeah cause [the answer key] says, research this (points to teacher answer key) and I don't know what that means.

Ms. Maronis I've given them stuff, and I hadn't [solved them] and I was like, "Oh gosh, that's a lot harder than I thought." I know we don't have time to do the whole thing, but if we could just kinda work through. ‘Cause I don't know that we know.

Ms. Dain It has some of the answers, but a lot of them vary.

As Ms. Maronis mentioned, teachers might not always choose to solve a task prior to implementing it during a lesson, but during the lesson study, the teachers took the time to do so. This is important because, as recommended by Stein et al. (2008), solving the task is the first step in anticipating student responses. It is also important to note that in order to understand how lesson study naturally supported these teachers' engagement with student thinking, the researcher did not explicitly tell the teachers to solve the task or share that aspect of the 5Ps with the teachers during the introductory meetings. Despite this purposeful omission, prior to anticipating student responses, the teachers decided to brainstorm their own approaches to the tasks. In doing so, teachers were given an opportunity to experience the process of mathematical modeling. The teachers' exploration of the mathematical modeling activity seemingly prepared the teachers to anticipate how students might engage with the task.

The teachers' approaches to the mathematical modeling tasks became useful during the planning process when the teachers were anticipating student responses. Because they had solved the task, they each had different solutions to offer to the group, and the teachers served as resources to each other by collaborating and sharing their developed solutions. Many times, the teachers drew upon their own responses to the tasks. For example, when the teachers anticipated student responses for Day 2 Task 1, Ms. Denvers said, “Yeah so, the only thing that’s hard with [the Jefferson Method] is it took us a lot of guessing and checking to come up with that number. Didn't it take us a few tries?” and Ms. Maronis agreed, “It did.” Also, in realizing that, even as teachers, they struggled with arriving at a valid approach for Day 2 Task 1, they were able to anticipate how students might struggle with the task and plan for supporting emerging responses. Through anticipating the multiple responses and engaging in the iterative process for completing the modeling task, the teachers experienced various aspects of mathematical modeling such as making decisions and verifying choices. Thus, the teachers could be better prepared to support students to engage in those aspects of mathematical modeling.

Because the teachers themselves had considered multiple approaches to each task, they were able to anticipate *multiple student responses*. Mathematical modeling activities typically do not have a single correct answer or solution pathway, so it was important that the teachers recognized that the tasks in their modeling lessons would not have only one right solution. The following transcript presents a discussion that teachers had about Day 1 Task 1.

Ms. Maronis	Valid responses would be that… (overlapping talk)
Ms. Denvers	Students appropriately use percentages.
Ms. Maronis	That students, do the ratio and percentage times the number of seats being allocated right?
Researcher	And then students round up or down
Ms. Denvers	Appropriately
Ms. Maronis	But it isn't really appropriate because…some of them are

> wrong. There's not really a right or wrong answer, I would like to see what they're thinking.

In this example, Ms. Maronis rightly pointed out that students could have multiple valid responses to the task not just a single correct answer. Also, in alignment with the focus on student thinking in lesson study, Ms. Maronis indicated that she wanted to “see what they’re thinking.”

Students’ Learning Activities, Teacher’s Questions and Anticipated Student Responses	**Teacher’s Support**
Launch/Warm Up (10 minutes) How might you arrange a system so that each state is represented fairly? What obstacles do you think might be present? <u>Valid Responses</u> Students will tie the population to the number of representatives. I will calculate the percent population of each state. I will find the total number of congressional seats. I will use the average. I would make sure the states with a higher population have higher seats. Smaller states have fewer, larger states have more (with regards to population). <u>Emerging Responses</u> I will divide by the total number of states. States with a larger area should have the highest number of representatives.	Can you provide an example from the map? How do you know how many seats should be allocated? Why do you think that method is fair? For example, should Delaware have the same number of representatives as California? Why or why not?

Figure 2.3 Anticipated Responses for Day 1 Launch

Furthermore, focusing on student thinking is aligned with preparing to implement complex mathematical modeling activities. Lesson study provides opportunities during the planning activity for teachers to collaborate to anticipate student responses. The teachers in this study not only anticipated *multiple* solutions, they were also careful to consider both valid and emerging responses. This is important for modeling because, in addition to having a variety of valid responses, students may also have emerging solutions that could include partial solutions or solutions that cannot be validated as opposed to completely incorrect answers. For instance, Figure 2.3, an excerpt from the lesson plan, shows the anticipated student responses for the Day 1 Launch. For this launch, the students were asked to describe their own process for state apportionment. As shown in the lesson plan, the teachers recognized that students' valid responses would make use of proportional reasoning, based on the population of the country, to devise a system of apportionment. However, the teachers also recognized that students' emerging responses could include simpler ideas that were not as accurate. As mentioned above by Ms. Maronis, solutions for this task were not exactly "right or wrong" as students could make a variety of assumptions and choices for how they would choose to divide up state representatives. Through collaborative planning, lesson study provided an opportunity for teachers to serve as resources for each other in anticipating multiple student responses in preparation for implementing modeling tasks. Hence, they were able to anticipate how students might use their creativity to engage in various aspects of mathematical modeling such as making decisions and defining variables.

Planning to Advance Student Thinking. The prominent theme that emerged while teachers planned how to monitor student groups based on the anticipated student responses was *an intention to advance student thinking*. Through collaboration, the teachers determined how they could use judicious telling (Lobato et al., 2005) to support students' emerging responses. For instance, Ms. Maronis and Ms. Denvers discussed how to support students who might struggle with Day 1 Task 1. They did not want to "tell" students how to solve the task, as demonstrated in the following exchange with the researcher:

Researcher	What could you say? To push them or get them thinking? You're walking around, and you see just one group sitting there, they don't have anything on their papers, what do you say?

Ms. Maronis I guess they need to be steered towards the percentage because maybe there's some that wouldn't even do that initially, right?

Ms. Denvers Right, so how do we steer them there without like taking the steering wheel.

When planning how to monitor the class and support students who might struggle with this task, the teachers wanted to *guide* their students rather than *tell* them answers. In doing so, the teachers could promote student agency while implementing modeling. Moreover, students would be afforded more opportunities to engage in the modeling process through making their own choices and developing their own approaches.

In their efforts to avoid "telling," teachers also planned advancing questions to support students with emerging responses, or to extend the thinking of students who had valid responses. In one example, the teachers discussed what to say to students who did not calculate the correct number of representatives for Day 2 Task 1. Ms. Denvers suggested that they ask questions such as:

- What's the purpose?
- What do you have to accomplish by doing this?
- It's a guess and check. So how do you know if you need to do a different guess?
- How do you know when to stop guessing?

By asking these types of questions, the teachers hypothesized that their students might be prompted to reflect on their own work or think about the next part of a solution path. Because teachers were provided with opportunities to plan to ask advancing questions rather than telling students answers or guiding them to one right answer, the teachers could provide opportunities for students to engage in the iterative process of modeling as well as support students in making choices, interpreting results and validating their own approaches. As described earlier, these activities are important features of mathematical modeling.

Planning to Connect Student Responses. While teachers were Planning for Selecting and Sequencing Student Responses, the teachers' intentions to *make connections across student responses*, was a prominent theme. The teachers began by discussing how they could support students to build on each other's work prior to reporting their results. For instance, the teachers suggested that they would encourage students to consult with one another or ask another student group for support when they were struggling with a concept. Ms. Dain was concerned that students would struggle with Day 1 Task 1, so she said,

> This is the part in the lesson where I feel like if I saw one group starting to do something in the right direction and all the other groups were completely stuck, I would take the time to say, 'Hey why don't you just explain what you're doing real quick?

Ms. Dain was discussing how teachers could support students to build on each other's thinking prior to sharing out complete approaches to the modeling tasks. In doing so, Ms. Dain was planning to support students to collaborate when reporting results and validating different solution pathways. Ms. Denvers and Ms. Maronis agreed with Ms. Dain's strategy, but they also wanted to make sure that students had a chance to struggle with the task, and they did not want students to tell each other answers.

In addition to discussing how to connect student thinking while students completed the tasks, lesson study provided opportunities for teachers to collaboratively make decisions about how to sequence student responses for the reporting out of results. First, the teachers planned to show more *simple strategies before complex* solution strategies to support students in understanding different approaches. This is important because sharing multiple student responses supports students in validating and interpreting multiple approaches to modeling activities. Also, in the lesson plan, the teachers decided to add connecting questions for the whole-class discussion such as "What do you notice about each method?" and "What are the pros and cons of each method?" By adding connecting questions to ask during the whole-class discussion, the teachers planned to have students make connections and build on one another's thinking. In planning to ask connecting questions, teachers planned to support the validation and interpretation of various modeling approaches for State Apportionment. Because the teachers planned ahead to support these important modeling competencies, the teachers were more likely to implement modeling in ways that would provide positive opportunities for students to learn mathematics. For example, Figure 2.4 shows the planned sequencing of student responses for Day 1 Task 1 in the first column. The second column shows the plans for selecting the responses and the questions they would ask to support students in making connections across the different responses.

Students' Learning Activities, Teacher's Questions and Anticipated Student Responses	**Teacher's Support**
Share Out of Exploration Part 1 (10 minutes) Student Response Sequencing First: Guess and check weighting method Second: Percentage method & rounded to get too many or too few representatives Final: Rounded and ended up with the right number of representatives (compare two groups that rounded differently)	 Monitor students and ask students to share out their responses using the document camera. Questions to ask during share-out: What do you notice about each method? What are the pros and cons of each method? What are your revisions for your initial responses?

Figure 2.4 Teachers' Planned Sequencing for Day 1 Task 1

Enacting the Lesson

Through the iterative process of lesson study, because all three of the teachers enacted the lesson, each teacher had two chances to observe the lesson and collect evidence of student thinking. As will be demonstrated through the data, when the teachers enacted the lesson, they likely advanced student thinking to support students' creativity and problem-solving while students explored modeling tasks. The teachers also selected and sequenced student responses in ways that provided students with opportunities to make connections across student responses and validate various approaches to State Apportionment models while students reported their results to the class.

Advancing Student Thinking. The main theme that emerged for the teachers' monitoring was how they used student responses to *advance student thinking*. Although the teachers did not explicitly plan assessing questions, the teachers had planned to use judicious telling (Lobato et al., 2005). Thus, asking assessing questions was a natural first step. Then based on the student responses and the planned monitoring, the teachers chose their next move. During the second lesson enactment, in a discussion between Ms. Denvers and her student, Sam, about Day 2 Task 1, Ms. Denvers started the discussion by asking the following assessing questions: "I would like to see is yours exactly the same? Tell me exactly what you did." Then as Sam guided Ms. Denvers through his written work, the discussion proceeded:

Ms. Denvers And it worked out perfectly for 25 people?

Sam Yes, but you have to round 'cause this one is 3.85, this one is 4.25 and you can't have part of a person, so you have to round.

Ms. Denvers So, you didn't round them all?

Sam We only rounded the first two. So, if we do the same thing, for 17, we'll get the answer.

By asking assessing questions, Ms. Denvers was able to observe the student's valid approach. After doing so, she encouraged the group to consult with each other to agree on an approach, and she moved on to the next group to assess their work. This example from Ms. Denvers' enactment is representative of how the teachers used assessing questions to engage with student thinking in the moment. Once Ms. Denvers observed that the students had developed one of the anticipated valid responses, rather than evaluate their work, she encouraged the students to use one another's thinking to verify their work and prepare for the whole-class discussion. Then she moved on to the next group. Ms. Denvers promoted student agency by allowing the students to make decisions and validate their modeling approaches. Lesson study had provided opportunities for Ms. Denvers to work with her colleagues to focus on student thinking through the following activities: anticipating multiple student responses, observing Ms. Dain teach the lesson, and collaborating to revise and improve the lesson plan during a debrief session. These activities and the focus on student thinking were likely to support Ms. Denvers in providing opportunities for her students to engage in various aspects of the mathematical modeling process.

In addition to asking assessing questions to understand more about student thinking, the teachers employed advancing questions and judicious telling to support students' emerging responses. In the lesson plan, for Day 2 Task 1, the teacher support suggested that if students were struggling with the task, teachers could encourage different groups or students within a group to choose different denominators and allocate one to each group member. Ms. Dain's suggestion could provide teachers with support to facilitate student collaboration if students were struggling to do so on their own. During the first lesson enactment, Ms. Dain encountered a group that was struggling with the task. The following discussion shows how she used judicious telling and advancing questions, as planned, to advance students to the next step in the Day 2 Task 1:

Ms. Dain	Use your lower quota. See if they add to 25.
Salina	It's only 24.
Ms. Dain	It's only 24, so if I were Hamilton, we would say we're going to look for the highest decimal remainder, but this is Jefferson, so I need to adjust the divisor. What number was the divisor again?
Salina	4,000
Ms. Dain	4,000 so you need a new divisor. Think about if you want to use a check method for the Jefferson method. To save some time, what could you do if you have a group?
Jamie	We each try a different number.

Although she was the first to enact the lesson, Ms. Dain implemented a collaboratively developed, annotated lesson plan that allowed her to use anticipated student responses and the planned monitoring to support a student group with an emerging response. She first reminded the group about the process for using the Jefferson method. Then, once they understood how to proceed, she asked an advancing question to prompt students to collaborate to complete the task. This is important because collaboration is often a necessary part of solving modeling tasks. Additionally, in this example, Ms. Dain elicited the ideas from the students rather than telling them which denominator to use or exactly how to work together as a group to complete the task. In doing so, Ms. Dain provided important modeling opportunities by allowing students to use their own creativity, make choices, and develop problem-solving skills.

Connecting Student Responses. The main theme that emerged when teachers selected and sequenced student responses to report for a whole-class discussion was that the teachers maintained fidelity to their lesson plans in selecting and sequencing student responses from simple to complex approaches. As an example of this selecting and sequencing, when Ms. Dain taught the lesson, the student responses aligned well with the anticipated student responses, and she sequenced the responses to report as planned. Then she asked connecting questions and encouraged students to engage in discussions with each other. In the following transcript excerpt, Ms. Dain facilitated the discussion in ways that supported students to respond to one another during the whole-class discussion about Day 1 Task 1:

Ms. Dain	Ok so there's an issue there, so why did your group decide to take away the representative from State A when you had 18?
Jamal	Because we added up to four, but you can't have .55 for a person and we didn't know what to do. And we think that's what, we didn't know any other way to do it. If we add all this together it made 17.
Ms. Dain	I think Alisha wants to add to the question I just asked.
Alisha	Well, my group, did the [approach] that you said, but we just looked at the decimal numbers and we chose the lowest one that could be rounded, 2.55. And we just rounded it down to 2 since it was like in the middle.
Ms. Dain	Excellent, I heard other groups saying that too. .5 is so close, it's half of the way. You know, traditionally rounding rules would say bump it up, but we decided [those rounding rules] didn't work.

In this case, Jamal was unsure about his group’s methods, so Alisha jumped in to explain why she thought their method was valid. Ms. Dain then connected Alisha’s response to other groups’ responses. Ms. Dain’s facilitation of the students’ report of their modeling approaches provided students with opportunities to validate each other's methods and explore different types of valid responses.

Another theme that emerged was that the teachers *adapted to unanticipated student responses* so that the responses would still be sequenced from simple to complex approaches. Since the teachers had anticipated specific and general methods for student responses, and they had planned to sequence from simple to complex, they were able to quickly adapt unanticipated student responses into the reporting out of student responses. Ms. Denvers was the second teacher to enact the lesson and the first to encounter an unanticipated and unique approach. She chose to share the response as one of the more complex responses. Ms. Denvers also asked connecting questions and encouraged students from other groups to explain Jake's approach to the modeling task. Below is the transcript of the discussion about Jake's approach to Day 1 Task 1, which was presented after a simpler and more common approach was discussed:

Ms. Denvers	We're trying to understand what Jake's group was thinking. Anybody have any thoughts about what those calculations are able to achieve? I'll be honest, 'cause when they showed it to me, it took me a whole like two minutes to ponder it for me to think like them, and that's fine. Let's share an idea.
Kelsey	He has 100,000/15,000, uh 6.666.
Ms. Denvers	So 6 and 2/3? What would that mean? I take 25 and divide it by 6 and 2/3. What's that 6 and 2/3 represent?
Kelsey	Is it because a certain portion out of the whole will go into each?
Ms. Denvers	And that will give you 25/6 and 2/3, right?
Ms. Denvers	Jake, can you tell us? … Like what were you thinking when you did that calculation?

Jake	So, the uh, the size of the state compared to the size of the overall population.
Ms. Denvers	So that sounds like what Kelsey was saying right? So, the size of the country is 6.666 times bigger than the size of the states…

Lesson study provided time for the teachers to predict that students would determine state apportionment using guess and check methods, percentages, or proportions. In this example, the students used a weighted average approach, so both Ms. Denvers and the other students in the class needed time to verify and interpret this valid result. Although this student response was unanticipated, Ms. Denvers was able to adapt how to sequence the approach because the teachers had considered sequencing from simpler to more complex solutions during the planning phase of the lesson study. Being prepared to productively use unanticipated student approaches is useful when teaching mathematical modeling as modeling activities can have multiple valid responses that may be difficult to predict. Also, because lesson study provided opportunities for teachers to observe each other, Ms. Maronis was then prepared to facilitate a discussion about the same approach when it appeared in her lesson enactment later on in the lesson study.

A final theme that was evident across teachers, including the examples above, was that teachers *asked connecting questions* while students reported their approaches. In the following excerpt, which came from the transcript of the third lesson enactment, Ms. Maronis provided opportunities for multiple students to share their responses. She also used revoicing to make connections across the student responses. Her connecting statements and questions are representative of how all of the teachers encouraged their students to consider multiple approaches to the modeling tasks. In the next transcript excerpt, Ms. Maronis shared student responses for Day 1 Task 2. This task asks students if the states might disagree on their apportionment.

Ms. Maronis	Question two says which states if any, would disagree with the apportion? Sarah, what did you guys get?
Sarah	The states would disagree because it's randomized.
Ms. Maronis	She was saying that they would disagree because it's randomized. They create the same issue and it's hard to make it balanced. Alright. Sasha, what did you guys get?
Sasha	We said we felt like states A and D may have some disagreements. We feel like because D has many more representatives that A, they may not like that.
Ms. Maronis	Is that in both scenarios or just the 17?
Sasha	Both.
Ms. Maronis	So, you think even though it's based on population they still might be upset. What did your group think?
Ben:	We thought State A and State B see a would disagree the most because they have the least amount of seats.
Ms. Maronis:	You said both scenarios create the same problem. There's always going to be a state with a lower number…

Here, Ms. Maronis facilitated a whole-class discussion by asking students from different groups to share their thoughts about the fairness of their state apportionment methods. At times, she used revoicing so that students could keep various student responses in mind. Then, as the whole-class discussion continued, she asked students if they agreed or disagreed with each other. Lesson study provided opportunities for the teachers to include connecting questions in the lesson plans. Doing so allowed the teachers to orchestrate productive mathematics discussions in the moment.

Additionally, in facilitating a discussion about multiple student responses, the students were provided with opportunities to engage in important aspects of mathematical modeling, namely, interpreting and validating each other's responses.

During Ms. Maronis' enactment, her selecting and sequencing did not always follow the lesson plan. For the most part, she chose to share out emerging responses and build on them by following up with valid responses. However, the student responses were entered into Desmos (n.d.), an online platform. Using the technology to share student responses added a challenge to the sequencing of the student approaches because they were entered in random order. Thus, Ms. Maronis did not always follow the planned sequencing, but she did make connections across the student responses.

Debriefing the Lesson

As part of the lesson study process, the teachers observed each other teach the lessons, and they used an observation form to take notes and collect evidence about student thinking. Then while Debriefing the Lesson, the teachers shared their observations and used their collected evidence to revise and improve the lesson plans. During the debrief sessions, the main themes that emerged were *refining and enhancing* anticipated student responses, *revising teacher support*, and *revising the strategy for reporting* and connecting student responses. In collaborating to complete these revisions based on student thinking, the teachers continued to work towards their improvement of teaching mathematical modeling.

Refining and Enhancing Anticipated Student Responses. The lesson study team used data collected during the observation to *refine and enhance* anticipated student responses in the lesson plan. For instance, during the first debrief session, after

Ms. Dain's teaching of the lesson, the group shared various student responses for the Day 1 Launch. They acknowledged that the student responses were not aligned with a learning goal about students developing their own methods for state apportionment. Thus, the teachers decided to modify the Day 1 Launch to support students in achieving the learning goal. As a result, the student responses in the subsequent lesson plans were more specific, and the student responses in the future enactments were better aligned to the learning goal. Figure 2.5 shows Version 1 of the Day 1 Launch and Figure 2.6 shows Version 2 of the Day 1 Launch in the lesson plan. These examples demonstrate how the teachers modified the task, added to the valid and emerging responses, and hypothesized that the revised question would influence students to write more specific responses. The Day 1 Launch was an important part of the modeling process as it provided opportunities for students to make assumptions and consider variables that might be included in State Apportionment. Thus, in revising the task, the teachers planned to better support the students in important aspects of mathematical modeling. Observing other teachers' enactments of the lesson, collecting evidence of student thinking, and spending time sharing that evidence in order to improve the lesson plan enhanced ways that teachers could support aspects of mathematical modeling in the State Apportionment Task.

Students' Learning Activities, Teacher's Questions and Anticipated Student Responses	**Teacher's Support**
Launch/Warm Up (10 minutes) How might you arrange a system so that each state is represented fairly? What obstacles do you think might be present?	

Figure 2.5 Version 1 Day 1 Launch

Students' Learning Activities, Teacher's Questions and Anticipated Student Responses	**Teacher's Support**	**Notes/Reflection Include hypothesis to try out in the future.**
Launch/Warm Up (10 minutes) If you were in charge of determining how many representatives each state in the United States should have, what information would you need. How would you use that information? What obstacles do you think might be present? <u>Valid Responses</u> Students will tie the population to the number of representatives. I will calculate the percent population of each state. I will find the total number of congressional seats. I will use the average. I would make sure the states with a higher population have higher seats. Smaller states have fewer, larger states have more (with regards to population). 1 representative per 100,000 people <u>Emerging Responses</u> I will divide by the total number of states. States with a larger area should have the highest number of representatives. Students think they need to know political parties, or the conversation turns into a discussion about who to elect.	Once students have two or three valid responses, choose those students to share out. Can you provide an example from the map? How do you know how many seats should be allocated? Why do you think that method is fair? For example, should Delaware have the same number of representatives as California? Why or why not?	Students only said they would use the population. They did not give these responses or specific responses. We might need to make the question more specific.

Figure 2.6 Version 2 Day 1 Launch

The teachers also used their observation notes to *refine anticipated student responses*. In the first version of the lesson plan, for some of the emerging responses, the teachers anticipated vague student responses. For instance, for Day 1 Task 1, the original anticipated emerging response said, "Students do not round correctly, or students' allocations may not sum to 25." During the observations, the teachers were able to take notes on student thinking to inform the refinement of this anticipated emerging response. During Ms. Dain's enactment of the lesson, Ms. Denvers observed a student response that she shared during the debrief session: "[The student] rounded

5,882 to 6,000 basically. Technically what she did was she scaled it down to like 40/5.882 and then she changed to it to 40/6." In this example, the student should not have rounded the divisor, so this error resulted in an emerging response; the number of allocated seats did not sum to 25. This example is representative of how lesson study provided opportunities for the teachers to refine anticipated emerging responses with specific instances of student thinking. By adding this possible error to the lesson plan, teachers could be better prepared to support students in validating and interpreting their modeling approaches so that students could revise errors.

In addition to refining anticipated student responses, the teachers also *enhanced* anticipated valid and emerging responses by adding student responses that surfaced during each lesson enactment. As discussed in the previous section, the teachers observed Jake's unique method to apportion representatives for Day 1 Task 1. Focusing on student thinking during the observations provided opportunities for teachers to improve the lesson plans for future enactments. Although, due to time constraints, the team did not debrief, formally, after Ms. Denvers' enactment, Ms. Maronis observed the various student responses. Thus, she was prepared for Jake's unique method when it surfaced during her lesson enactment. The observers were also able to collect detailed notes to share during the final debrief session.

Because Ms. Denvers was fully engaged in teaching the lesson, she was not able to record the exact student response. However, Ms. Dain, an observer, was able to write down specific notes about the student's response. Ms. Dain then showed Ms. Denvers her detailed notes which included the details of Jake's strategy as shared in the transcripts earlier. The detailed notes allowed Ms. Denvers and the other teachers to deeply reflect on that student's response. The lesson study team determined that the student had used weighted averages, a new mathematical concept for the students, to apportion the representatives. Then, since the response had been evident in both Ms. Denvers' and Ms. Maronis' lesson enactments, the teachers decided to add that student response to the anticipated valid responses in the lesson plan. The iterative process of lesson study included multiple enactments and debrief sessions. These multiple cycles of lesson study supported teachers in continuing to refine and enhance anticipated student responses so that teachers would be better prepared for future enactments of these mathematical modeling lessons.

Revising Teacher Support. In addition to adding to the catalog of anticipated student responses, the debrief sessions allowed time for the *revising of teacher support* and adjusting some of the suggested monitoring actions. The teachers revised how they would monitor emerging responses based on some of the unexpected responses that arose as well as revised some monitoring moves that could support student connections and whole-class discussion. For example, Ms. Dain noticed that in the beginning of the task it might be necessary to include monitoring statements or questions that would re-focus students if they veered off topic. She said:

> The reason I was going around to the groups is because I remembered from the first class that they started to think about politics right away, and their opinion. So, I wanted to nip that in the bud and be like, we're not talking about who you would elect. We're just talking about how many.

Thus, the lesson study team added that students might want to discuss political affiliation as a possible emerging response so that teachers would be supported in how to refocus students. Since modeling activities can be ill-structured and messy, teachers need to be prepared to monitor students and support students to stay focused and to determine relevant assumptions and variables during launch activities.

Revising the Strategy for Reporting and Connecting Student Responses. The final theme that emerged and influenced the way teachers sequenced student responses was a *revision to their strategy for reporting out* student responses. As discussed earlier, during the first debrief session, the teachers modified the Day 1 Launch because the students did not respond with specific solution approaches. The teachers hypothesized that a different strategy for reporting responses might influence students' sense of urgency to complete the tasks and support teachers in sharing multiple student responses with the class. Hence, to improve the lesson plan, the teachers added a suggestion to the teacher support that recommended the use of Desmos (n.d.), an online tool that includes an online knowledge sharing platform for mathematics teachers and students, when selecting and sequencing student responses. Thus, Ms. Denvers and Ms. Maronis incorporated Desmos allowing students to simultaneously view the State Apportionment tasks and type in their responses (see Figure 2.7). As a result, the teachers were able to show more than one student response at a time, and the students wrote more detailed responses. The student responses were also saved electronically, so the teachers could quickly review student responses.

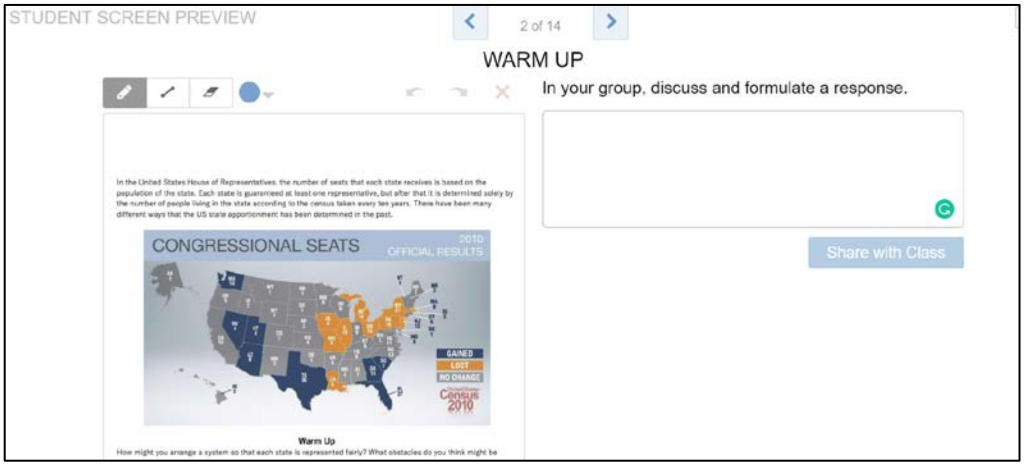

Figure 2.7 Student Preview of Day 1 Launch in Desmos

Being able to observe student thinking through the Desmos application was valued by the teachers. During the second debrief session, the teachers commented on how they thought the use of Desmos (n.d.) improved the lesson. For example, when Ms. Maronis reflected on her lesson enactment, she said:

> I liked the idea of including the Desmos. That seemed to really help kind of with the flow of the class and the kids kind of knew what to do, and I was able to pause it and stop it so that they couldn't go further. And I was able to see their responses while they were typing them and things like that. I knew when they were all done and whatnot.

Ms. Maronis described how using Desmos supported her in monitoring student responses and keeping track of their progress from a single location (i.e., computer stand at the front of the classroom). Then, as she circulated, she could focus more in-depth on student thinking and support students to engage with the modeling process rather than simply checking for progress. Ms. Dain, who was the first to teach the lesson, was impressed with the use of Desmos for the Day 1 Launch in Ms. Denvers' and Ms. Maronis' enactments. She indicated her enthusiasm for Desmos by saying, "The first question. It was so much better on Desmos. They actually answered it. In my class, nobody answered that." She also acknowledged that the prompt had been changed for the final enactments, but Ms. Dain noted that it was "probably a combination of both" the reworded prompt and the inclusion of Desmos that influenced more specific and thoughtful student responses. Ms. Denvers, who used Desmos in her enactment of the lesson, also remarked:

> I think if [students] know they're typing into a computer versus 'it's on my paper and whatever, [the teacher is] probably not going to read this.' I think they feel more pressure. Like more accountable I guess is the word. That's one of the things I like about Desmos is that I feel like [the students] are more accountable.

Ms. Denvers' had observed that students were more thoughtful in their responses when they knew their responses could be on display. The addition of Desmos to the lesson plans was influenced through multiple activities of lesson study. First, the teachers observed Ms. Dain and collaboratively revised a task based on evidence of student thinking. Then, Ms. Denvers piloted the use of Desmos in the second lesson enactment, and, after observing Ms. Denvers, Ms. Maronis further refined the use of Desmos during the third lesson enactment. Finally, the use of Desmos was discussed in the second debrief session so that the lesson plan could include this suggestion for future enactments. Desmos created another opportunity for students to share their responses and further supported teachers in engaging with student thinking. By providing a way for many students to report their results at once, Desmos also served as a valuable tool for mathematical modeling in that students could share, validate, and interpret many different approaches and results.

Discussion

This study was conducted in order to understand more about how lesson study can influence teachers' engagement with student thinking that can support teachers as they work to improve their teaching of mathematical modeling. A feature of lesson study is that it provides opportunities for teachers to engage with student thinking in ways that PD does not typically do. Furthermore, the researcher hypothesized that the use of the 5Ps would enhance teachers' engagement with student thinking while participating in lesson study on mathematical modeling. Thus, the findings of this study provided insight into how specific conditions can influence teachers' engagement with student thinking while planning, enacting, and debriefing mathematical modeling lessons through participation in lesson study.

One of the ways that lesson study supported these teachers was through the anticipation of student responses. Moreover, the teachers in this study anticipated multiple student responses and they focused on both emerging and valid responses. This finding is contrary to previous studies that have shown how teachers struggled with anticipating multiple student responses, leading to further challenges during the implementation of mathematical modeling activities (e.g., Doerr & English, 2006). Also, teachers have tended to focus solely on *correct* responses when anticipating student responses for open-ended mathematics tasks including mathematical modeling tasks and non-modeling tasks (e.g., Doerr & English, 2006; Pang 2016). However, the teachers in this study anticipated both valid responses and emerging responses. This finding is important with respect to modeling because not only is it possible for a modeling activity to have more than one valid solution approach, there could also be several valid models or results. This separates modeling from other types of open-ended mathematics activities, because although, any open-ended task may have multiple valid approaches, the correct end -result or answer of non-modeling tasks may often be the same.

Then to make use of multiple student responses, the teachers' plans for monitoring included judicious telling (see Lobato et al., 2005) and asking questions that could support the advancement of student thinking and encourage them to persevere with the problem-solving process. The teachers' first inclination was to plan to use judicious telling (see Lobato et al., 2005) to support students who had emerging responses, rather than to lower the cognitive demand of the modeling tasks by supplying students with ideas on how to proceed. While judicious telling can support teaching any mathematics topic, planning to use judicious telling was especially useful in teaching the mathematical modeling tasks. In doing so, the teachers could be better prepared to engage with student thinking in ways that provided students with opportunities that are important to the process of mathematical modeling such as being creative, making choices, defining variables, and collaborating to develop their own approaches.

When teachers planned to select and sequence student responses, they focused on emerging and valid responses. In previous cases of teaching open-ended mathematics tasks, researchers have found that mathematics teachers have focused on correct responses (e.g., Schoenfeld, 1998) or did not make use of student errors (Bray, 2011). However, unlike those teachers, the participants of this lesson study planned to use emerging responses to build student knowledge during the whole-class discussion. In this study, the teachers planned to sequence student responses so that simple or emerging strategies would be shown before more complex strategies. This strategy is well-aligned with the suggestion from Stein et al. (2008). This strategy is important for teaching modeling because it might not be possible to anticipate every single student response. Thus, planning to sequence from simple to complex can support teachers in making use of unanticipated student responses for modeling activities. Additionally, using emerging and valid responses for modeling activities can provide opportunities for students to interpret and validate mathematical models by comparing various approaches. The participants in this study further planned to support students to build on each other's responses by planning connecting questions and using strategic sequencing for the reporting out of various approaches to the modeling tasks. By planning how to select, sequence, and make connections, these teachers were prepared to engage with student thinking in ways that could support students to interpret and validate a variety of modeling approaches and results.

After planning how to engage with student thinking, the teachers used their anticipated student responses and their planned monitoring to maintain the cognitive demand of the tasks and facilitate student discussion. In previous studies, teachers have tended to lower the cognitive demand of tasks through their hesitation to allow students opportunities to explore and learn through their mistakes (e.g., Boston & Smith, 2009; Hiebert & Stigler, 2000; Stein & Lane, 1996). One way the teachers in this study maintained the cognitive demand of the tasks was through judicious telling (Lobato et al., 2005). Also, the teachers had planned several advancing questions to ask in order to support students who had emerging responses. Through the use of judicious telling and advancing questions, the teachers advanced student thinking and facilitated mathematical discussions rather than *telling* students the answers, as teachers have done in previous studies (see e.g., Schoenfeld, 1998; Smith, 1996). While using judicious telling and asking advancing questions can be used in a variety of mathematics contexts, these strategies could be especially useful when teaching mathematical modeling. For example, when teaching modeling, students need to be supported to validate and revise results. Advancing student thinking and using emerging responses are important for the iterative process of mathematical modeling as students may need to revise their approaches if a model cannot be validated with real-world data.

While monitoring, the teachers supported students in developing their own approaches to the modeling tasks. The teachers also made decisions about which tasks to select and sequence. As planned, the teachers selected and sequenced student responses that would make connections across student responses. In a few instances, students responded to each other with little interference or need of facilitation from the teacher. This is similar to the interaction that Wood, Cobb, & Yackel (1991) observed where an elementary school teacher used multiple student responses to support students in making connections and exploring other students' thinking about mathematics. Although making connections is important when teaching any type of mathematics tasks, making connections across students' responses could be especially important during modeling lessons. Part of the modeling process includes reporting results and collaborating to validate the results so that the model or models can be revised and implemented in the real world. In making connections across student responses, the teachers in this study likely supported student collaboration during the reporting of results.

The teachers in this study were also able to make connections because these teachers had carefully planned how they would sequence specific student responses. These results were especially notable for the novice teacher, Ms. Dain, who selected, sequenced, and connected student responses with fidelity to the lesson plan whereas, early career teachers in previous studies have struggled with anticipating multiple student responses (e.g., Stein & Smith, 2018). Those teachers' struggles with anticipating student responses influenced a lack of efficacy in selecting and sequencing responses. Moreover, in this study, the teachers planned to sequence approaches from simple to complex. Doing so supported the teachers to make selecting and sequencing decisions in the moment for unexpected emerging and valid responses that contained similar strategies as those anticipated in the lesson plan. This is particularly important when teaching mathematical modeling because it might be difficult to predict or anticipate every single student approach or result for a modeling task. Through participating in lesson study, the teachers in this study were provided with time to strategically plan how they could select, sequence, and connect student responses. Consequently, during the lesson enactments, the teachers provided opportunities for students to make connections across various modeling approaches for State Apportionment and draw conclusions about the fairness of the methods.

In addition to engaging with student thinking within the formal lesson study activities, these teachers reflected on the lesson enactments outside of the formal debrief sessions. As Bruce and Ladky (2011) observed in their study, reflecting on student thinking influenced how the teachers in this study made decisions in between the formal lesson study activities. For example, Ms. Maronis and Ms. Denvers discussed the use of Desmos in between formal lesson study meetings. Doing so supported the teachers in facilitating discussions about multiple approaches and results for the modeling tasks. Given the challenges of teaching mathematical modeling, it is promising to see that lesson study could influence the improvement of teaching outside its formal structure.

Finally, the findings indicated that the debrief sessions influenced the teachers' engagement with student thinking to support their teaching of mathematical modeling. This was particularly evident when, during the debrief sessions, the teachers continued to use student thinking to guide their revisions and improvements to the lesson plans. The lesson study process provided opportunities for the teachers to observe student thinking and take notes using the observation form. Focusing on student thinking during the lesson enactments influenced the improvement of the lesson plans. For example, during the first debrief meeting, the teachers modified a task after observing that students did not make strong connections to the mathematical points of the lesson. The modification of the task supported teachers in implementing a modeling task for which students were likely able to make better assumptions and make choices about potential variables that needed to be considered when developing a State Apportionment method.

It was also the case that observing student thinking during the lesson enactments allowed the participants to share specific student responses in order to build catalogs of anticipated student responses for the modeling tasks within the lesson plans. Building a catalog of possible student responses for modeling tasks could be extremely useful for teachers. Although it might not be possible to anticipate every single student response for a modeling task, it is likely that common student responses will emerge over time. Thus, continuously improving a mathematical modeling lesson plan by adding to and refining anticipated student responses could support teachers. Robust catalogs of student responses could prepare teachers to facilitate discussions about the most common responses and encounter fewer unanticipated responses to make decisions about in the moment. Additionally, in this study, when the teachers improved the lesson plan, they stored their knowledge as suggested by Hiebert and Morris (2012) so that the lesson plan could influence future lesson enactments.

Conclusion

Previous literature (e.g., Lewis et al., 2009; Murata et al., 2012) has indicated that lesson study can support attention to student thinking with respect to the teaching of a variety of mathematics topics. The findings of this study indicate that lesson study can support teachers to engage with student thinking in ways that could be especially beneficial to the implementation of mathematical modeling. The opportunities provided for the teachers in lesson study are not common in all types of PD. For instance, lesson study provided opportunities for teachers to engage with student thinking while collaboratively planning lessons, observing colleagues enact the lessons, and use student thinking to collaboratively revise the lesson plans. It is also the case, that for two of the three teachers in this study, this lesson study was their first PD experience on teaching mathematical modeling. Even the teacher who had attended PD and had prior experience teaching modeling benefited from the of lesson study activities. For instance, as noted in the literature review, anticipating student responses is especially challenging when teaching mathematical modeling. Yet, lesson study provided opportunities for these teachers to serve as resources for one another when anticipating multiple approaches to modeling tasks. The teachers' collaboration allowed them to build a catalog of potential student responses. As a group, they also planned ways to judiciously tell and monitor in ways that would not be a hindrance to students as they engaged in mathematical modeling. Then, planning how to engage with student thinking during the lesson enactments supported the teachers to implement a complex modeling activity. Planning ahead to engage with student thinking likely supported the teachers in providing more opportunities for students to engage in a process of mathematical modeling. The debrief sessions provided unique opportunities for teachers to reflect on their own teaching and make improvements to their lesson plans. Revising the lesson through multiple cycles of lesson study provided teachers with resources they needed to support students to engage in mathematical modeling. Findings from other studies indicated that engaging with student thinking while teaching mathematical modeling has been challenging for teachers. However, in many ways, lesson study influenced the teachers in this study to consider student thinking in ways that could support students in engaging with mathematical modeling.

Special Conditions

Although this study had a small sample size, using qualitative methodology provided detailed examples of how lesson study can support teachers to engage with student thinking while working to improve their teaching of mathematical modeling. To generalize the findings or consider how to conduct a similar study, it is important to note that special conditions likely influenced the outcomes of the study. For example, before beginning the lesson study, the teachers were oriented to the nature of lesson study, and the focus on student thinking within lesson study was made explicit. Because the 5Ps is a useful framework to support a focus on student thinking (Smith & Stein, 2018), in the beginning of the lesson study, the researcher encouraged the teachers to use the 5Ps throughout each phase of lesson study. Additionally, the lesson study team used a lesson plan template adapted from Japanese lesson study which included cells for anticipated student responses and planned instructor support. The orientation meeting and the tools used during the lesson study likely influenced the teachers' engagement with student thinking throughout the phases of the lesson study. The participants in the study also had a variety of backgrounds and experiences, and they were all in favor of teaching mathematical modeling. Each participant had some, albeit limited, experience with mathematical modeling as a student, a teacher, or a worker in industry. They had also agreed to participate in the study, so they had a specific interest in improving their teaching. It is possible that given similar conditions, other teachers may be influenced by lesson study on mathematical modeling in similar ways. Future research should be conducted to determine which conditions might influence other populations of teachers.

Implications and Future Research

The findings in this study provide insight into how lesson study activities can support teachers in engaging with student thinking in ways that could provide opportunities for students to explore mathematical modeling tasks. Although lesson study has been researched more frequently in recent years, theories on the inner workings of the features of lesson study are still emerging. In this study, the features of lesson study that seemed most important were the focus on student thinking, teacher collaboration, and iterating the lesson study cycle. Future studies that include a similar teacher population along with differing teacher populations are likely to provide insight into how these features of lesson study support the teaching of mathematical modeling. Additionally, further research incorporating an emphasis on the 5Ps into lesson study would be a contribution to the field of mathematics education in general. Moreover, additional studies that incorporate lesson study, the 5Ps, and mathematical modeling could contribute to broader knowledge on how to best support teachers in implementing this important mathematical process. Long-term research goals should also include attempts to connect improved teaching of mathematical modeling through lesson study to student learning.

Chapter 3

TEACHERS' CONCEPTIONS OF TEACHING MATHEMATICAL MODELING BEFORE AND AFTER LESSON STUDY

Although national policy documents have called for the incorporation of mathematical modeling into the curriculum (e.g., NCTM 2000, 1989; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) emerging research has indicated that teachers may have challenges with teaching this important mathematical process. The challenges of teaching modeling include limited experience and resources (e.g., Meyer, 2015; Newton, Madea, Alexander, & Senk, 2014). These challenges are unfortunate because mathematical modeling can provide students with opportunities to engage in ill-structured problems about authentic real-world situations that are unlike typical textbook tasks. These opportunities can leverage students' personal knowledge and experiences, which may lead to increased interest in doing mathematics (Cirillo, Bartell, & Wager, 2016). Thus, researchers have recommended that preservice and inservice teachers receive professional development (PD), including lesson study, on implementing mathematical modeling (e.g., Ang. 2013; Turner et al., 2014; Kuntz et al., 2013). Lesson study is an iterative process that includes curriculum study, lesson planning, teaching and observing, and reflecting/debriefing (see Figure 3.1[5]).

[5] Figure 3.1 includes common attributes of lesson study as suggested by Lewis, C. (2016), Suh and Seshaiyer (2014), and Takahashi and McDougal (2016).

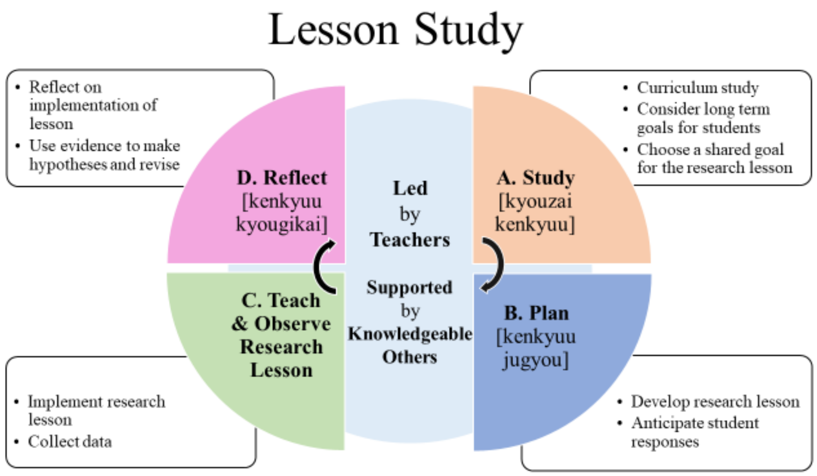

Figure 3.1 The Lesson Study Cycle

Lesson study could support the teaching of mathematical modeling because lesson study employs a variety of tools and resources that influence improvement of teaching. For instance, to support lesson study, the provision of curriculum documents by knowledgeable others have contributed to the positive outcomes of lesson study (e.g., Lewis, C., 2016; Takahashi & McDougal, 2016). Consequently, as a system of continuous improvement, lesson study has been found to provide teachers with opportunities to attend to student thinking as well as improve their content and pedagogical knowledge (Cajkler, Wood, Norton, Pedder, & Xu, 2015; Lewis, Perry, & Hurd, 2009; Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012). This is important because in order to improve the teaching of mathematical modeling, teachers need support to engage with student thinking and gain pedagogical and content knowledge.

In this study, one university researcher, who is the author of this paper, and three secondary teachers engaged in lesson study on mathematical modeling. The teachers were interviewed, about their conceptions of teaching mathematical modeling before and after the lesson study.

Background and Theoretical Framework

First, this literature review begins with a discussion of mathematical modeling as an important mathematical process. Opportunities provided through teaching and learning mathematical modeling are then discussed. Next, challenges that have influenced the current context of teaching mathematical modeling are presented. In the last section, an overview of lesson study is provided.

What is Mathematical Modeling?

Mathematical modeling is an important mathematical process that is used by many professions within the fields of science, technology, engineering, and mathematics. A mathematical model represents a real-world phenomenon and supports the understanding and controlling of real-world situations or making predictions (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016; GAIMME, 2016). There are many descriptions of mathematical modeling, but there is no agreed upon definition. Hence, for the purpose of this study, a working definition of *mathematical modeling* is inspired by Cirillo, Pelesko, Felton-Koestler, and Rubel's (2016) description of the features of mathematical modeling:

> Mathematical modeling is an iterative process that authentically connects to the real world. It is used to explain phenomena in the real world and/or make predictions about the future behavior of a system in the real world. Mathematical modeling requires creativity and making choices, assumptions, and decisions, and can have multiple approaches and solutions.

The iterative process of modeling can be communicated through a mathematical modeling cycle. To represent the cyclical nature of mathematical modeling GAIMME (2016) developed the diagram in Figure 3.2 which illustrates how the process of mathematical modeling is not linear. Thus, the modeler may go back and forth between different phases of the cycle (as indicated by the bilateral arrows). When students engage in the mathematical modeling process, they have opportunities to improve their problem-solving skills and to gain skills that are transferable to other contexts (GAIMME, 2016).

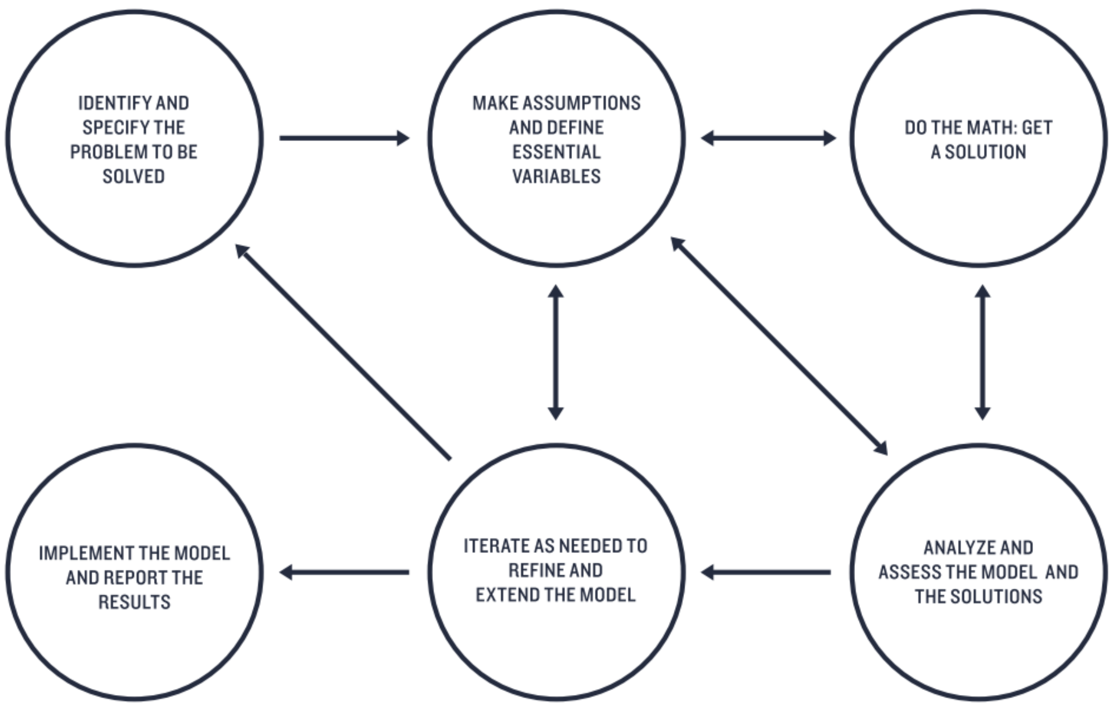

Figure 3.2 The Mathematical Modeling Cycle (GAIMME, 2016, p. 13)

Teaching and Learning Mathematical Modeling

Kaiser (2017) indicated that, globally, research on teaching mathematical modeling is scarce. This was evident through an extensive review of the literature as it was more common to find edited books (e.g., Lesh, Galbraith, Haines, & Hurford, 2013; Stillman, Kaiser, Blum, & Brown, 2013), where the contributing authors come from countries outside of the United States (e.g., Australia, Austria, and Germany), rather than articles in peer-reviewed journals. In other words, there have been few empirical studies on teaching mathematical modeling reported in the literature. A review of the international literature revealed the following themes, which provide additional context for this study:

- Preservice and inservice secondary mathematics teachers reported a lack of self-efficacy with respect to pedagogical strategies for teaching mathematical modeling, specifically because of the ill-structured nature of mathematical modeling activities (e.g., Kuntze, Siller, & Vogl, 2013; Chan, 2013).
- Secondary teachers struggled with anticipating multiple student responses for mathematical modeling activities. The lack of anticipated responses led to pedagogical challenges with respect to classroom management, the handling of multiple student approaches, and the facilitation of whole-class discussions during implementation (e.g., Pereira de Oliveria & Barbosa, 2013; Borromeo Ferri & Blum, 2013).

While the global body of literature on teaching mathematical modeling is still emerging, because mathematical modeling is still a relatively new topic within the U.S. curricular landscape, studies conducted in U.S. classrooms are even sparser. However, because context matters, this literature review primarily focuses on U.S. studies and papers. This decision was made because teachers in the current study are more like the teachers in the U.S. studies in the sense that they had little background for and experience with teaching mathematical modeling and because context is important.

Opportunities for Students. One hypothesized opportunity for students in teaching mathematical modeling is that, given the right conditions, implementing authentic modeling activities can intrinsically provide equitable opportunities for students through equitable teaching practices (NCTM, 2018). For instance, students can develop agency through persevering to understand complex modeling activities and reporting their thinking and ideas. These types of opportunities can provide *all* students with access to rigorous mathematical content. As an empirical example, Rubel, Lim, Hall-Wieckert, and Sullivan (2016), worked with students in an urban high school who had struggled to meet their grade-level requirements in mathematics. The students investigated the fairness of local lotteries and discovered that members of their own community were spending money on lotteries that could have been allocated towards community improvements. According to Rubel and colleagues, leveraging student knowledge about their own communities increased the students' motivation and interest. Furthermore, the students engaged in rigorous mathematics and produced complex mathematical solutions. In addition to providing opportunities for *all* students to engage in rigorous mathematics, mathematical modeling activities can provide many opportunities for students to explore issues of equity. For instance, students might explore fairness in a variety of contexts, such as the location and pricing of unhealthy food options (Cirillo, Bartell, & Wager, 2016), the location of alternative financing institutions such as check cashing shops (Rubel, Lim, Hall-Wieckert, & Katz, 2016), or environmental crises (e.g., Aguirre, Anhalt, Cortez, Turner, & Simic-Miller, 2019).

The above examples indicate that engaging in mathematical modeling can provide opportunities that conventional textbook tasks may lack. It is also important to note that while students are engaged in interesting mathematical modeling activities, they can develop and use mathematical content knowledge as required by curricula and gain other important skills such as communication and collaboration. For example, Alhammouri, Foley, and Dael (2018) examined how high school students gained knowledge and problem-solving skills while engaging in a mathematical modeling activity about maintaining trout populations in a local lake. The students engaged in an iterative process to develop valid models, and they learned from each other through collaborating with classmates and participating in a whole-class discussion about different models. While engaging with mathematical modeling can provide many opportunities for students, ill-structured problems can also present challenges for students.

Challenges for Students. In previous studies, when secondary students have engaged in modeling, they often struggle with making assumptions, defining variables, attending to important contextual details, and using mathematics to justify the validity of their results (Bleiler-Baxter, Barlow, & Stephens, 2016; Doerr & English, 2003). An example of students struggling with a complex modeling activity surfaced in a study involving secondary students (Gould & Wasserman, 2014). While the students were engaged in a modeling activity that involved choosing the best gas station, they oversimplified or overcomplicated their solutions. The student errors were most likely caused by a lack of understanding about which variables and assumptions were appropriate to use in the real-world context. For instance, students assumed that gas tank capacity was only one gallon. Although students have experienced challenges when exploring mathematical modeling, the opportunities for students seem to be compelling enough to make mathematical modeling worth pursuing.

Challenges of Teaching Mathematical Modeling. Teachers need to be prepared to support students in overcoming the challenges of mathematical modeling. Yet, teachers may not be implementing mathematical modeling as recommended, and they may not be prepared to do so. Evidence of a lack of teacher preparation was revealed through a survey of teacher preparation programs (n = 72). Of those who responded, only 15% of teacher preparation programs required a mathematical modeling course for preservice secondary teachers (Newton, Madea, Alexander, & Senk, 2014). This survey finding could indicate that many novice secondary teachers may not implement modeling as recommended by curriculum documents (e.g., CCSSM).

Furthermore, follow-up preparation for inservice teachers seems unlikely or is not always effective. This hypothesis is supported by findings from a national survey of inservice teachers (n = 274) from 35 states (Gould, 2013). More specifically, Gould (2013) found that teachers had a variety of misconceptions about mathematical modeling. Overall, the teachers who responded to the survey correctly understood that mathematical modeling consists of using mathematical representations to explain real-world phenomenon. However, many of the surveyed teachers also indicated that mathematical modeling could include unrealistic situations with only one solution. These teachers also thought that mathematical models included physical manipulatives (e.g., pattern blocks, fraction tiles). In addition to the survey results, Gould presented case studies of five inservice teachers enrolled in a PD course on modeling. Despite efforts to correct misconceptions about mathematical modeling, some of the teachers still held problematic beliefs such as the idea that mathematical models could be represented using physical manipulatives.

While Gould's (2013) results indicate that content knowledge could be problematic for teaching mathematical modeling, teachers may also hold beliefs that hinder their willingness to teach mathematical modeling. An example of teachers' beliefs was reported by Anhalt, Cortez, and Bennett (2018) who found that after completing a mathematical modeling activity, preservice teachers acknowledged that implementing modeling could provide many opportunities for students to engage in mathematical reasoning, rich discussions, and learn multiple solution approaches. At the same time, those teachers were concerned that the mathematics content would be too difficult for most secondary students. These types of beliefs could influence teachers' decisions about whether or not to implement mathematical modeling activities.

Moreover, inservice teachers have reported that a lack of resources, such as time and access to good mathematical modeling activities prevented them from implementing mathematical modeling regularly, if at all (Gould, 2013; Huson, 2016). Teachers' concerns about curriculum resources are further complicated in that activities labeled as mathematical modeling tasks in textbooks may not provide a range of opportunities for students to engage in all aspects of the mathematical modeling process (Meyer, 2015). Modeling tasks analyzed by Meyer (2015) primarily focused on mathematical computations and procedures but did not include many opportunities for students to be creative, define their own variables or validate results.

Lesson Study as a Means for Improving the Teaching of Mathematical Modeling.

Lesson study is likely to support teachers in addressing the challenges of teaching mathematical modeling. Through the process of lesson study, teachers have opportunities to learn from one another while collaboratively planning, observing teaching, and debriefing together to improve lesson plans (Lewis et al., 2009). Previous studies have found that lesson study provides opportunities for teachers to learn content knowledge, pedagogical strategies, and focus on student thinking (e.g., Murata et al., 2012; Lewis et al., 2009). An example of collaborative learning through lesson study was found by Inoue (2011) who indicated that when teachers observed each other teach, and they reflected on their teaching through collegial feedback, the teachers learned important pedagogical strategies. The opportunities provided by lesson study are likely to support the teaching of mathematical modeling by providing time to explore mathematical modeling curriculum materials, spend time on collaboratively developing a lesson, and focus on student thinking to foster students' engagement in rigorous mathematics.

The focus of student thinking during lesson study can have important consequences for teacher learning (e.g., Suh & Seshaiyer, 2014). For instance, the teachers observed by Inoue (2011) anticipated student responses that guided teachers in facilitating discussions and supporting students to engage with the reasoning of others. Reflecting on student thinking can also influence teachers' beliefs about students. For example, Cajkler et al. (2015) observed how teachers' beliefs were changed through their direct observation of students who did not always stand out in class. This finding about debrief sessions is further supported by the work of Widjaja et al. (2017) who, likewise, indicated that reflecting on student responses influenced teachers' beliefs about their students' abilities to solve problems. This kind of attention to student thinking is likely to support teachers in the implementation of open-ended mathematical modeling activities.

This study examined how teachers expressed their conceptions of teaching mathematical modeling before and after participating in lesson study on mathematical modeling. Because lesson study has the potential to improve teachers' content and pedagogical knowledge and influence beliefs, there is reason to believe that, under the right conditions, lesson study could influence teachers' conceptions about teaching mathematical modeling. Hence, the following question guided the research:

> What are teachers' conceptions of teaching mathematical modeling before and after lesson study?

Research Methods

While the teaching of mathematical modeling is not a new idea, the practice of doing so in U.S. classrooms is a relatively new phenomenon. Thus, at this early stage of research on the teaching of mathematical modeling, it is important to understand more about how teachers are perceiving and approaching the teaching of mathematical modeling. In this study, interviews and case study methodology were used to investigate the conceptions of three secondary teachers who participated in lesson study on mathematical modeling. The use of “conceptions” in this study draws on Lloyd’s and Wilson’s (1998) notion of teacher conceptions which they described as a combination of teachers’ “knowledge, beliefs, understandings, preferences, and views” (p. 249). In a similar vein, Knuth (2002) studied teachers’ conceptions of mathematical proof.

This study was conducted over three phases. First, the teachers were interviewed about their conceptions of teaching mathematical modeling. Second, these teachers participated in two cycles of lesson study. Third, the teachers were interviewed to see how their conceptions of teaching mathematical modeling were different after participating in lesson study on mathematical modeling. Figure 3.3 represents the overall sequence of the phases, as well as the data collection method (top bullet), and the resulting data artifact (bottom bullet). See Appendix F for a detailed timeline of the study. This study focuses on data from the interviews. Next, further details about the participants and methodology for the study are provided.

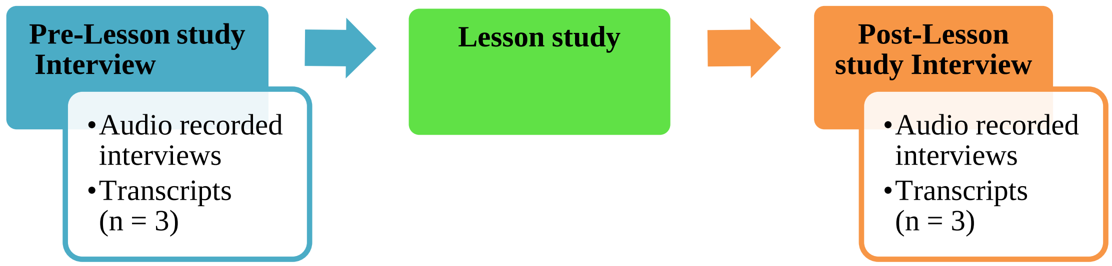

Figure 3.3 Study Sequence and Data

Participants

Three secondary teachers who were teaching in a vocational high school at the time of the study participated. This school has a diverse student population and is located in the mid-Atlantic region of the United States. The participants were recruited based on their interest in improving their teaching of mathematical modeling. The teachers committed to participation while funding for the project was pending, and they were willing to participate on a voluntary basis. However, by the start of the project, funding had been obtained,[6] and the teachers were promised a stipend as well as curriculum materials on mathematical modeling.

The three teachers, Loren, Anne, and Karen varied with respect to career stage and background (see Table 3.1). Loren was in her second year of teaching, and she recently earned her bachelor's degree in secondary mathematics education. As part of her degree program, Loren completed one course on mathematical modeling for

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[6] Funding was provided by the University of Delaware through a dissertation support award.

secondary teachers. Anne, who had previously been an engineer, had six years of teaching experience. Anne had earned a bachelor’s degree in electrical engineering and a master’s degree in curriculum and instruction. Karen had 21 years of experience teaching and had earned a bachelor’s degree in computer information systems and a master’s degree in education. Karen had participated in professional development on mathematical modeling, and she had some experience teaching modeling.

Table 3.1 Teacher Participant Credentials

Teacher	**Education**	**Years Teaching**	**Experience Teaching Modeling**
Loren	• Bachelor of Arts in Secondary Mathematics Education	1	No prior experience
Anne	• Bachelor of Science in Electrical Engineering • Master of Science in Curriculum and Instruction	6	No prior experience
Karen	• Bachelor of Science in Computer Information Systems • Master of Arts in Elementary Education • Specialization Credits in Secondary Mathematics	21	Some experience

Lesson Study on Mathematical Modeling

The interviews for this study were part of a larger study that included lesson study aimed at supporting the teaching of mathematical modeling. To orient first-time lesson study participants, the researcher began the process by introducing the concept of lesson study. In the first lesson study activity, curriculum study, the investigator introduced mathematical modeling, and the teachers explored curriculum materials related to mathematical modeling (see Appendix A for the complete list of materials). For instance, the teachers explored modeling tasks such as those provided in the

Mathematical Modeling Handbook (Gould, Murray, & Sanfratello, 2012). The teachers chose the “State Apportionment Task” (Sanfratello, 2012, pp. 133 - 140) to include in a two-day lesson plan. This task allows the students to develop their own model and explore historical methods for apportioning United States House of Representatives.

The next activity of the lesson study involved teachers collaboratively planning a lesson on mathematical modeling. The researcher facilitated the meeting and guided teachers in completing an annotated lesson plan format, similar to the format that is commonly used in Japanese lesson study (e.g., Gorman et al., 2010; Lewis & Hurd, 2011). The lesson plan template contained spaces for clear learning goals, anticipated student responses, planned instructor actions, rationale for tasks, and hypotheses for future teaching (see Appendix B for a sample lesson plan template). To complete the lesson study cycles, each teacher enacted the lesson and participated in two debrief sessions. Because of time constraints, the team only held two debrief sessions. The team met after the first enactment (i.e., Loren’s) and the third enactment (i.e., Karen’s). During the debrief sessions, the lesson plan was revised using evidence of student thinking collected by the teachers while observing each other.

Teacher Interviews

Prior to the lesson study, interviews were conducted regarding the teachers’ conceptions of teaching mathematical modeling (see Appendix G for the interview protocols). The questions asked provided teachers with opportunities to share their conceptions of teaching mathematical modeling. For example, one of the questions was: “Are you currently teaching mathematical modeling, or have you ever taught mathematical modeling? Describe your teaching approach to mathematical modeling (e.g., frequency, resources for tasks, aspects of the modeling cycle addressed).” Two different sets of prompts followed this question. If the teacher was currently teaching modeling, then follow-up questions were asked about the teaching approach, including the benefits and challenges of teaching mathematical modeling. If the teacher was not currently teaching modeling, then the teacher was asked why modeling was not being taught, and to describe any hypothetical benefits and challenges of teaching modeling.

To investigate how teachers’ conceptions of teaching mathematical modeling might influence their analyses of various “mathematical modeling” tasks, three tasks were selected. The tasks were chosen from EngageNY (n.d.), a widely used mathematics curriculum across the United States (Kaufman, 2017; Sahm, 2015). Algebra I tasks were chosen so that teachers could mentally analyze the tasks quickly without solving the tasks or spending significant time anticipating student responses. The tasks were purposefully selected to exploit possible misconceptions that the teachers may have about mathematical modeling. The first two tasks were used, as is, from the curriculum because, as purported by the authors of EngageNY, they were designed to scaffold students’ opportunities to engage in mathematical modeling. Lastly, the third task was modified by the researcher in a way that would allow students to engage in several aspects of mathematical modeling. Table 3.2 includes the three tasks and possible opportunities for mathematical modeling.

The first two tasks were not as open-ended as mathematical modeling problems could and, perhaps, should be. Hence, as written, students would be limited to focusing on computations with some analysis of the mathematical functions represented in the tasks. Moreover, students would be limited in opportunities to consider multiple variables, discover multiple solutions, or validate and revise a model. Investigating the teachers' analyses of the tasks could indicate their conceptions about mathematical modeling tasks and reflect potential curricular decisions.

Post-Lesson Study interviews were conducted after the lesson study process was completed. The Post-Lesson Study interview protocol contained slightly modified questions from the initial interview about teaching mathematical modeling but was relatively similar to the Pre-Lesson Study protocol. For instance, questions were revised to prompt teachers to reflect on their experiences of teaching mathematical modeling. These questions were designed to elicit teachers' conceptions of teaching mathematical modeling after participating in lesson study on mathematical modeling.

Table 3.2 Interview Protocol Tasks

Tasks	**Alignment to Modeling Definition (Cirillo, Pelesko et al., 2016)**	**Alignment to Modeling Cycle (GAIMME, 2016)**
Task A[7] Students are given late return penalty fees for two equipment rental companies. There are data sets for each company representing days 1 to 15. One data set represents a linear function, and one data set represents an exponential function. Students are asked: • Which company has a greater 15-day late charge? • Describe how the amount of the late charge changes from any given day to the next successive day in both Companies 1 and 2. • How much would the late charge have been after 20 days under Company 2? (*Engage New York*: *Algebra I, Module 3, Lesson 5*, pp. 51 – 52)	• Represents a contrived real-world situation • Make predictions	• Do the math: computations • Analyze the model
Task B Margie got $1,000 from her grandmother to start her college fund. She is opening a new savings account and finds out that her bank offers a 2% annual interest rate, compounded monthly. What type of function would best represent the amount of money in Margie's account? Justify your answer mathematically. **(***Engage New York***:** *Algebra I, Module 5, Lesson 3*, p. 42*)*	• Represents a contrived real-world situation	• Do the math: computations
Task C Noam and Athena are having an argument about whether it would take longer to get from New York City to Boston and back by car or train. Noam says it is faster to drive. Athena prefers to take Amtrak. Who do you agree with? Develop a mathematical model to justify your response. (Adapted from *Engage New York: Algebra I, Module 5, Lesson 2*, pp. 31 - 32*)*	• Represents a relatable real-world situation • Explain real-world situation • Can involve creativity • Can involve decisions • Can involve assumptions • Can involve multiple approaches	• Define variables • Do the math: computations • Analyze the model • Iterate to refine • Report the results

[7] Because Task A requires an entire page, the beginning is paraphrased. The entire task is available in Appendix D.

Data Analysis

Data for the study consisted of audio-recordings and transcripts of two interviews per teacher for a total of six interviews. Once the audio data were transcribed using oTranscribe (Bently, 2017), the transcripts were uploaded to Dedoose (2016) which is “a cross-platform app used to analyze qualitative and mixed-methods research.” The transcripts were analyzed using a constant comparative approach (e.g., Strauss, 1987; Hatch, 2002). Initial codes were developed using themes that emerged in the literature. Those codes became deductive codes to further code the data. As seen in Table 3.3, the deductive codes were taken primarily from the working definition of mathematical modeling and the GAIMME (2016) mathematical modeling cycle (i.e., Figure 3.2). Then the framework was revised further as inductive codes emerged from the data. The codes were refined through an iterative process for further analysis. Table 3.3 includes the final coding categories that were used to analyze the interview transcripts. Themes and findings were organized into analytic memos. The analysis of the data revealed how each teacher represented a case of teachers’ conceptions of teaching mathematical modeling.

Table 3.3 Conceptions of Teaching Mathematical Modeling Coding Categories

Conceptual Categories	**Deductive Codes**	**Inductive Codes**
Keywords from the description of mathematical modeling (Cirillo, Pelesko, et al., 2016)	• Real-World • Explain Phenomena • Predict • Creativity • Making Choices/Decisions • Iterative • Multiple Solution paths	• Purposes of Mathematical Modeling (e.g., who uses modeling)
Keywords from mathematical modeling cycle (GAIMME, 2016)	• Making Assumptions and Defining Variables • Developing a Formula • Using Formula to Compute and Calculate • Analyzing and Assessing the Model • Implementing and Reporting	N/A
Benefits of Teaching Mathematical Modeling (e.g., Stillman, Brown, & Galbraith, 2013; Galbraith & Clatworthy, 1990; Cirillo, Bartell, & Wager., 2016)	• Student interest • Gain Content Knowledge • Equity	• Group Work/Collaboration • Understanding the Problem-Solving Process
Challenges of Teaching Mathematical Modeling (Chan, 2013; Huson, 2016; Kuntze et al., 2013)	• Time • Curriculum Mandates • Student Knowledge • Teacher knowledge	N/A
Teaching Approach (developed through the data)	N/A	• Engaging with Student Thinking • Using Group Roles • Using a Lesson Plan • Incorporating More Mathematical Modeling

Findings

In this section, details for the three teachers' cases are provided through descriptions of each teacher's Pre-Lesson Study Conceptions and Post-Lesson Study Conceptions. As a preview of the findings, Loren learned pedagogical strategies for teaching mathematical modeling; Anne realized the benefits of teaching mathematical modeling; and Karen focused on shifting her classroom culture. Following the description of each case, cross-case reflections are discussed.

Case 1 – Loren: Learned Pedagogical Strategies for Teaching Mathematical Modeling

Pre-Lesson Study Conceptions

Expressed Benefits of Teaching Mathematical Modeling. In Loren's Pre-Lesson study interview, she indicated that teaching mathematical modeling could appeal to students' interest and provide opportunities for students to be creative through multiple approaches. Below, Loren shared her thoughts about teaching mathematical modeling:

> I don't know if it's just because I like it, but I feel that modeling makes mathematics more interesting to students…or, like I said, looking at how students can look at one model and see it from different perspectives and explain it in different ways, and I think that's something that is really valuable about mathematics because a lot of people think that math is, well there's one right answer…(Interview, July 25, 2018)

Although Loren had not yet taught mathematical modeling in an authentic way, she indicated that she thought mathematical modeling would increase student interest and give students opportunities to explore multiple approaches to modeling activities. As Loren reflected on her previous experience with modeling, she recognized that implementing mathematical modeling could provide various opportunities for students.

Expressed Challenges of Teaching Mathematical Modeling. When Loren described the challenges of teaching mathematical modeling, she was concerned that students would not appreciate the complexity of mathematical modeling activities, and they would give up easily. Here concerns are evident in her response below.

> Well, I think it can be complicated. I think for a lot of modeling tasks I could see where students would get easily frustrated or like if someone else gets it, and they don't feel like it's important to ask their peer why and try to understand it. They might just be like…'whatever.' I think a lot of modeling tasks are challenging, so I guess that's the only [challenge] that I see is just that students aren't interested in it and would just get frustrated easily. (Interview, July 25, 2018)

Before participating in lesson study, Loren seemed to believe that the complexity of modeling would be an obstacle for students, and she did not think students would persevere to complete modeling activities.

Expressed Teaching Approach to Mathematical Modeling. In the Pre-Lesson Study interview, Loren indicated that she had not taught authentic modeling. She had implemented "word problems," but she did not consider those to be modeling tasks. Loren reflected on her experience with teaching mathematical modeling thus far:

> The only experience that I've really had, and I don't even know if you would classify it as modeling is just like drawing pictures and setting up scenarios with word problems…I'm thinking of when I student taught geometry, and we would do those drawings with the ladder leaning on the house, and I've taught students ways to show mathematical situations, but I don't know if it's truly modeling. (Interview, July 25, 2018)

In the above example, Loren acknowledged that her use of word-problems that resulted in multiple representations were not actual modeling activities. As she conveyed, Loren did not really have experience teaching mathematical modeling, but she indicated that she was looking forward to the lesson study so that she could better understand how to teach mathematical modeling in a more authentic way.

Expressed Opportunities of Particular "Mathematical Modeling" Tasks. Prior to participating in lesson study, through Loren's analyses of the tasks, she demonstrated an understanding of the types of opportunities textbook tasks might provide for students. She indicated that the tasks labeled as "mathematical modeling" tasks were limited in the opportunities they might provide students to engage in mathematical modeling. For Task A, Loren said the task represented a real-world situation, however, she also expressed surprise that the task was not more open-ended. Below is her response when she was informed that the data for the task was provided to the students:

> Really? Wow? But even though they're given what the data are, they could still use that to give an actual equation for each one. But if it were me, and I really wanted the student to engage in modeling, I don't think I would give them that. I would want them to come up with [the data] on their own. (Interview, July 25, 2018)

Loren communicated a view that Task A was not open-ended enough for students to explore the real-world situation. Thus, she did not think that students would need to engage much in the modeling process.

For the second task, Loren also indicated that students would not need to do much in terms of modeling. She said that students could do "some type of modeling," but she still thought the task was similar to Task A in its limited opportunities for modeling. Finally, for Task C, Loren acknowledged that "it gives them the opportunity to dig into a whole lot, like traffic patterns and back roads, and just that [students] would have to get a lot of information to figure this out.” Loren’s responses indicate that she recognized that for Task C students would need to collect information. She also stated that students could “look into the differences” for Noam’s argument and Athena’s argument. In her Pre-Lesson Study interview, Loren’s descriptions of the tasks were indicative of her conceptions about how different types of tasks may or may not provide opportunities for students to engage in mathematical modeling. Given her responses, Loren could make purposeful choices about the types of modeling opportunities she wanted to provide for students through her task selection.

Post-Lesson Study Conceptions

Expressed Benefits of Teaching Mathematical Modeling. After participating in lesson study on mathematical modeling, Loren’s observations of students in her own and her two colleagues’ classrooms, allowed her to recognize additional benefits for students. As shown in Table 3.4, Loren initially identified student interest, creativity, and multiple approaches as benefits of teaching mathematical modeling. In her Post-Lesson Study interview, she described teaching mathematical modeling as a way to engage students in mathematics that “is valuable in the real world” and “really relevant.” Loren also reported that she wanted to teach modeling for its rigor and to “challenge the students because asking them to model something that they're just kind of given without any real specifics is a lot more rigorous for them than just teaching them how to do procedures.” Loren’s response provides evidence that she saw benefits in the implementation of ill-structured modeling activities. She also valued the collaborative nature that was required of the State Apportionment modeling task as indicated below.

> I think another reason that [mathematical modeling] should be taught in schools is that it's a really good way to engage [students] in group work...I think that if I introduced more modeling tasks, it would be a good way for students to have to work together because they're really trying to persevere through figuring this out. Like I said, it's challenging for them, so it helps them learn to work together as a team. (Interview, November 16, 2018)

Here Loren conveyed how engaging in mathematical modeling supported her students in collaborating with their classmates and provided them with opportunities to persevere through rigorous mathematics tasks. While Loren recognized possible benefits of teaching mathematical modeling in her initial interview, she had also expressed concern about the students struggling with the complexity of mathematical modeling activities. However, in her Post-Lesson Study interview, Loren's beliefs about her students' engagement with modeling had been expanded to include more benefits. Her responses may also show how participating in the lesson study influenced additional justification for her stated intentions to implement more modeling activities. To compare Loren's Pre-Lesson Study and Post-Lesson Study Conceptions, Table 3.4 includes Loren's conceptions of teaching mathematical modeling before and after participating in lesson study on mathematical modeling.

Table 3.4 Loren’s Conceptions of Teaching Mathematical Modeling

	Pre-Lesson Study Conceptions		**Post-Lesson Study Conceptions**	
	Themes	Teacher’s Expressions	Themes	Teacher’s Expressions
Benefits of Teaching Modeling	• Appeals to student interest • Provides opportunities for creativity • Can have multiple approaches	Modeling makes mathematics more interesting to students Students can look at one model and see it from different perspectives… I love when you have students explain the different ways that they think about the same mathematical model.	• Shows math is valuable and relevant • Provides rigor for students • Encourages collaboration	Students understand that mathematics is valuable in the real world and the modeling tasks…were really relevant. Asking them to model something without any real specifics is more rigorous. It's a really good way to engage them in group work…
Challenges of Teaching Modeling	• Students will struggle with the complexity of mathematical modeling tasks	Well I think it can be complicated…I could see where students would get easily frustrated.	• Requires time • Aligning to curriculum • Fostering productive group work	I guess the only drawback is the time that it takes. I think one challenge was, if it can be made relevant to the curriculum… I'm hesitant to do group work … students can kind of sink into the background…
Teaching Approach to Modeling	• Implementing Word-Problems	The only experience that I've really had, and I don't even know you would classify it as modeling is setting up scenarios with word problems…	• Using an annotated lesson plan • Incorporating Group Roles	…the lesson plan that we went in depth with was great. So, I think it'll change the way I teach. I was the only one of the three of us that didn't assign roles in the groups and I really wished I had.
Modeling Tasks	Tasks A & B are not open-ended enough Task C has opportunities for student research	If I really wanted the student to engage in modeling, I don't think I would give them [the data]. It's good cause it gives them the opportunity to dig into a whole lot.	Tasks A & B have Missed opportunities Task C: Opportunities to define variables and validate results	I don't think it provides much opportunity to engage in modeling… They can determine whether what they come up with actually makes sense…They have to think about all these variables.

Expressed Challenges of Teaching Mathematical Modeling. While Loren recognized many benefits to teaching mathematical modeling, she also realized, through experience, that teaching mathematical modeling could present some worthwhile challenges. As indicated in Table 3.4, after participating in the lesson study, Loren's conceptions about the challenges of teaching mathematical modeling included an increase in her concerns about how to incorporate modeling. Loren mentioned concerns about pedagogical strategies. For instance, she saw, during the classroom observations, that if students are not assigned specific roles in group work, they may not stay fully engaged in the process. Loren also discussed the challenge of "not tutoring [students] too much:"

> My challenge was like, I really tried to follow the lesson plan that we developed and highlight the questions for the teacher support, and only asked those things and tried to limit how much I wanted to like help them and kind of hint at the right way. But I think that's a great challenge too - something to be challenged about as a teacher, to really let them go forward with it and not tutoring them too much. (Interview, November 16, 2018)

Here Loren expressed the dichotomy of a benefit and a challenge. She mentioned, it was beneficial to plan ahead to support the students by asking targeted questions and not "help them," but it was also "a great challenge" to decide how to "hint at the right way." During the lesson study, the teachers developed an annotated lesson plan with instructor support, and they focused on facilitating and guiding students rather than telling. Loren's response displays how she reflected on her implementation of that lesson plan.

Other concerns that Loren mentioned involved the availability of resources. She discussed concerns about the time it takes to implement mathematical modeling and the alignment of modeling activities to her mandated curriculum. She articulated concerns that if modeling activities were not well-aligned to her curriculum, the students might not see the relevance. Although she was the first to suggest the “State Apportionment” task for the lesson study, Loren did not know if all of her students saw the purpose in the task. She said that using modeling tasks regularly would eliminate that problem: “the hope is that you could come up with modeling tasks that would be so readily ingrained in what you were doing that you could just turn it into an eventual, like actual, assessment.” Despite, the challenges that teaching mathematical modeling can present, Loren indicated that she would review the modeling activities in the curriculum materials from the lesson study to choose tasks for implementation that would align to her curriculum.

Expressed Teaching Approach. Although Loren had not taught mathematical modeling prior to the study, she indicated that she would continue to implement mathematical modeling. Loren indicated through the process of lesson study she learned pedagogical strategies for teaching mathematical modeling. In addition to being provided with curriculum materials for teaching mathematical modeling, Loren indicated that her participation in the project would support her future teaching of mathematical modeling through learned pedagogical strategies. When asked how she would change her teaching of mathematical modeling in the future, she said that she would change by incorporating group “roles.” She said that she “wished” she had incorporated group roles as her colleagues did. Participating in lesson study provided an opportunity for her to observe how her colleagues facilitated group work. Loren observed that when students were assigned group roles, they were more engaged throughout the lesson. For instance, Table 3.4 includes an example of how Loren communicated her intention to "incorporate role assignments" when she implements mathematical modeling. She also indicated that the lesson plan template and the process of anticipating student responses during lesson study was especially useful to her. When asked about how she would change her teaching of mathematical modeling in the future, Loren indicated that she would spend more time on the lesson planning. When referring to the "lesson planning part" of the lesson study, she said:

> It really showed me how valuable that is, and I like so enjoyed like going into [the lesson] knowing, what I wanted to say, what I didn't want to say, and what I thought the students would say. I planned timing and things like that, but the lesson plan that we did went in depth which was great. So, I think it'll change the way I teach. Not only teaching modeling but just teaching in general because it showed me the importance of lesson planning. (Interview, November 16, 2018)

Loren's interview response indicates that planning ahead, to decide how to use student thinking productively, supported her in teaching the mathematical modeling tasks. This transcript excerpt is also evidence for how lesson study provided opportunities for her to purposefully attend to student thinking and develop a robust lesson plan. In this way, participating in lesson study seemingly provided benefits related to improved pedagogy that went beyond teaching mathematical modeling.

Expressed Opportunities of Particular "Mathematical Modeling" Tasks. After the lesson study, Loren's descriptions of the tasks became more refined, with respect to modeling, and she recognized additional missed opportunities in Tasks A and B. In the Pre-Lesson Study interview, Loren recognized the tasks' limitations and strengths, but in her Post-Lesson Study interview, she communicated, in more specific ways, that Tasks A and B were not authentic modeling tasks (see Table 3.4). For Task A Loren said, “Yeah. So, I would say that I don't think it provides much opportunity to engage in modeling because it's kind of like the situations are already modeled for them. And they're just observing like what's already there.” For Task B, she still saw potential in the task, but she specified that “it's just one question with one answer, but it could be turned into like an actual entire task.” Loren's response indicates that she recognized how Tasks A and B could be limited in providing modeling opportunities, and she indicated that she would need to modify the tasks for them to be actual modeling tasks. In her Post-Lesson Study interview, Loren's conceptions about the mathematical modeling tasks provide evidence that she could be better prepared to select and adapt conventional textbook tasks to teach mathematical modeling in the future.

When Loren reviewed Task C, again, after participating in the lesson study, she described, in more detail, how the task would provide opportunities for students to engage in various aspects of mathematical modeling:

> Well, they would probably be like, um, “I don't have enough information.” I liked this one because I feel like they would almost have to do their own research maybe which is going to further show them that what they're doing is very real-world. You know, and they can determine whether what they come up with actually makes sense. So yeah, I like that they might have to do their own work to figure out how many miles, and are they going to take what kind

> of car? Like they have to think about all these variables. (Interview, November 16, 2018)

In this description, Loren described, in more detail, how students would need to verify their approaches and that students would need to be purposeful in choosing variables, whereas, in the Pre-Lesson Study interview, Loren focused mostly on the open-ended nature of the task and the opportunity for students to collect information. Based on this transcript excerpt, one could further conclude that Loren's conceptions about mathematical modeling tasks were more refined after participating in lesson study focused on mathematical modeling.

Case 2—Anne: Realized Benefits of Teaching Mathematical Modeling

Pre-Lesson Study Conceptions

Expressed Benefits of Teaching Mathematical Modeling. In Anne's Pre-Lesson Study interview, she mentioned that she had not quite taught mathematical modeling before, but she recognized how students could benefit from engaging in mathematical modeling. She communicated that the implementation of mathematical modeling could provide students with opportunities to engage in problem solving. Anne said that teaching mathematical modeling involves, "teaching them more the approach to solving problems rather than just rote memorization of solving problems, different representations and ways to solve things, but mostly just to get them thinking more analytically rather than plugging things into a formula." Anne conveyed how mathematical modeling could advance students' mathematical thinking and support problem-solving skills.

Expressed Challenges of Teaching Mathematical Modeling. Anne recognized that teaching mathematical modeling would be positive for her students, but she also acknowledged that it might not be the easiest endeavor. From her experience with implementing open-ended tasks, she was concerned that students would find modeling difficult as students were not used to solving problems with multiple solution approaches. When Anne spoke about her previous experiences, she said:

> Well, it was hard for them to figure out how to get started. ‘Cause they really want to be like spoon fed step-by-step. This is what you do. This is what yours should look like. And they struggled a lot with the idea that everyone should be different. (Interview, July 25, 2018)

Anne also said that, as a teacher, it was challenging for her to “not help.”

> Not helping them. That was the hardest part because I had to sit, and I would be like "You can figure this out." And they'd be like, "But we don't understand…” But I didn't want to tell them. I wanted them to come up with it. That was the biggest challenge. Letting them struggle. ‘Cause it's counter-intuitive to what you want to do as a teacher. You feel like you should help them, but struggling is super important. (Interview, July 25, 2018)

Anne recalled that her students were not comfortable with open-ended tasks, and she was challenged to not “tell them” too much. Her comments also suggest that Anne believed that struggling with mathematics was beneficial for students. As an additional challenge, Anne mentioned that her curriculum demands were an obstacle to teaching mathematical modeling, but she wanted to incorporate more open-ended tasks that were related to mathematical modeling. She said, “It's just sometimes it's hard with so much to get through. There's definitely room for more.” Anne’s beliefs about her students and her lack of curriculum resources with respect to modeling seemed to influence her ideas about perceived challenges of teaching mathematical modeling.

Expressed Teaching Approach to Mathematical Modeling. As mentioned earlier, Anne reported that she had really not taught mathematical modeling before. When asked about her teaching approach, she articulated that she had not formally taught mathematical modeling. She reported that her teaching approach to mathematical modeling had mainly consisted of application problems:

> I would say…I do try to pull in a lot of real-world application. We did a unit on polynomials, and we followed similar formats, but I pulled in some stuff. Like I had them model a roller coaster. They had to come up with their equation for the polynomial of the roller coaster and they drew it, and they had to talk about the domain and the range and what it meant in a real-world context. I try to do that once in a while. (Interview, July 25, 2018)

Anne indicated that one of her challenges was including modeling within her mandated curriculum and unit plans. Also, while Anne had some experience with teaching open-ended tasks that included aspects of modeling, she wanted to learn more about how to implement authentic mathematical modeling tasks.

Expressed Opportunities of Particular "Mathematical Modeling" Tasks. In her initial interview, when describing Task A, Anne said it needed some improvements, but she saw some opportunities for modeling. In particular, she said "I think it would have been better if they didn't give them the data… But I think that's definitely something that would be an opportunity to model just the two different types [i.e., linear, exponential]." She also thought that Task B could provide opportunities for students to explore exponential functions graphically as well as interpret some of the data produced by the function. Anne saw Task C as a task that could provide opportunities for student discussion. She said:

> That's just interesting, the way it's worded cause he says it's faster to drive, and she prefers Amtrak, so she's not necessarily saying it's faster to take Amtrak. So, you could agree with both of them in theory. It would be faster to drive, but you don't have to drive when you take Amtrak, so I think it would be interesting to see if they would pick up on that or if they would automatically jump to which one is faster, right?…I think that would be a good opportunity to see if they would even have that discussion. (Interview, July 25, 2018)

Anne saw that Task C was more open-ended than Tasks A and B, and she noted that for Task C students could produce more than one argument and have a mathematical discussion.

Post-Lesson Study Conceptions

Expressed Benefits of Teaching Mathematical Modeling. During Anne's Post-Lesson Study interview, she communicated several additional benefits that she identified after enacting the lesson and observing her colleagues' lesson enactments. As indicated in Table 3.5, Anne still indicated that one benefit was the opportunity for students to "get better at problem solving." She thought problem-solving skills would "translate to all other areas besides just math…just being able to grapple with problems and solve problems." Anne also added that engaging in mathematical modeling gives students opportunities to "collaborate with other students and things like that rather than just being instructed directly." After the study, Anne realized that engaging in mathematical modeling supported her students in productively struggling with mathematics and collaborating with other students. Anne, again, claimed that engaging with mathematical modeling could also support students in developing skills beyond mathematics.

In addition to expanding on her conceptions that teaching mathematical modeling can provide students with opportunities to gain problem-solving skills, Anne also realized how teaching mathematical modeling can provide a wider range of students with opportunities to display their thinking. For example, Anne seemed to be impressed by how she "was able to see skills in students that [she] wouldn't have seen in a traditional way of teaching them." For instance, Anne discussed ways in which engaging in mathematical modeling can position students as competent learners and afforded her a chance to engage with student thinking (also see Table 2.5):

> One of my students, like I got more out of him from this activity than I have the whole semester. I was able to see his thought process and things like that, that I had never seen before because I guess really, I'm always looking at a paper and what he's writing, and he's not a big sharer in class. So, it was really hard for me to see. But then when I saw what he was doing I was like, wow, he's really, really working on this. Like really his, the way his mind was working, was very much different than what I had thought. (Interview, November 17, 2018)

Anne further emphasized how she thought the group work contributed to this student's sharing of his ideas:

> I think it was being in the group and just everybody was kind of doing their own thing, and so he didn't feel like he was being singled out. Like he just felt like he was more part of a whole process, you know, I think because he's like, he's not a student that likes to call attention to himself and share it in that, in that environment. (Interview, November 17, 2018)

After participating in lesson study, Anne introduced additional benefits for her students that differed from those she spoke about before the lesson study. Anne's discussion about her observations of this particular student indicated that her initial beliefs, regarding students' abilities to engage with mathematical modeling, had evolved to seeing that students could persevere through modeling activities.

Another way that Anne realized benefits of teaching mathematical modeling was evident in the way that she spoke about how her participation in lesson study and the ways in which teaching mathematical modeling supported her engagement with student thinking. Anne reported:

> I saw students come up with ideas and, you know, different ways of coming up with the correct answer or even the wrong answer. But…it was easier for me to see how they were thinking and that thought process…I mean I definitely got a lot more insight into my students in their thinking than I would've if I did a traditional lesson for sure. (Interview, November 17, 2018)

In the above example, Anne indicated how her implementation of the modeling task provided her with greater access to student thinking. She attended to both correct and incorrect student approaches. Anne also benefited from observing other lesson enactments, and she adapted the use of Desmos (n.d.), an online tool for teachers, from Karen's class. Using Desmos as a knowledge-sharing platform supported the sharing of student thinking during whole-class discussions about the mathematical modeling tasks. To compare Anne's Pre-Lesson study and Post-Lesson study Conceptions, Table 3.5 includes Anne's conceptions of teaching mathematical modeling before and after participating in lesson study on mathematical modeling.

Table 3.5 Anne's Conceptions of Teaching Mathematical Modeling

	Pre-Lesson Study Conceptions		**Post-Lesson Study Conceptions**	
	Themes	Teacher's Expressions	Themes	Teacher's Expressions
Benefits of Teaching Modeling	• Promotes problem-solving	It's just teaching students how to model ways of solving problems like you know teaching them more the approach to solving problems rather than just rote memorization of solving problems.	• Promotes Problem-solving • Promotes student collaboration • Provides all students access • Focus on student thinking	they get better at problem solving and that'll translate to all other areas besides just math …collaborate with other students and things like that rather than just being instructed directly… for instance, one of my students, I got more out of him from this activity than I have the whole semester I definitely got a lot more insight into my students thinking than…I did a traditional lesson.
Challenges of Teaching Modeling	• Can be an unproductive struggle for students	Well, it was hard for them to figure out how to get started… they struggled a lot with the idea that everyone should be different.	• Requires a culture shift	For the kids… they're not used to struggling with problems…they've picked up on it pretty quickly…in the beginning, they weren't comfortable with it.
Teaching Approach to Modeling	• Teaching word-problems	I mean I do try to pull in a lot of real-world application. We did a unit on polynomials …Like I had them model a roller coaster. They had to come up with their equation for a polynomial…	• Increasing the use of modeling tasks • Using open-ended questions to foster collaboration	It's only been a couple of weeks, but I have… tried to incorporate… more mathematical modeling tasks in my lessons for sure. I definitely don't jump as fast to give them answers… when I'm planning my lesson I come up with more open-ended questions, so they'll have to do a little bit of grappling and working together…
Modeling Tasks	Tasks A & B are not open-ended enough Task C can have multiple approaches	… I think it would have been better if they didn't give them the data… … he says it's faster to drive, and she prefers Amtrak, … You could agree with both.	Tasks A & B are not quite modeling Task C promotes collaboration, defining variables, verifying results, and multiple approaches	There is just like one right answer. They'd have to justify their response…you have these other factors and things like that. I think it does definitely give the kids, more opportunity to engage with each other and explain their thinking. …There's not necessarily one right answer.

Expressed Challenges of Teaching Mathematical Modeling. While Anne still had concerns about teaching mathematical modeling after the study, she now realized that it was possible to overcome some of the obstacles. For instance, she noticed that teaching mathematical modeling required changes for teachers and students. The changes Anne referred to indicate that a shift in culture would be necessary for implementing modeling. Anne reported that students struggled at first. She also noted that teaching mathematical modeling required more work from teachers:

> It definitely requires more work as a teacher and, I think, for the kids. They're not used to struggling with problems and things like that. But I mean, for the most part, they picked up on it pretty quickly. So, I was pretty pleased about that. But maybe you could tell in the beginning they weren't comfortable with it. So, I think that would be a challenge for the students, you know, just getting comfortable with the process. (Interview, November 17, 2018)

In addition to the culture shift for students and teachers, Anne was still concerned about the resources required for teaching modeling and choosing modeling activities that "fit with what you're doing in your classroom."

Expressed Teaching Approach to Teaching Mathematical Modeling. In her Post-Lesson Study Interview, Anne remarked, "Before I really didn't know what it was. I think I would come up with tasks that I thought maybe were mathematical modeling, but now that we did the lesson study, um, you know, it's completely different" (Interview, November 17, 2018). Anne noted that her views about teaching mathematical modeling were "completely different" after participating in the lesson study (cf. Table 3.5). She also referred to the collaborative nature of lesson study as instrumental in preparing her to teach mathematical modeling. For instance, Anne said that she and her colleagues “don’t get a chance to collaborate like that as often as we should…it was great to work through our task as a team…” She mentioned that anticipating the student responses ahead of time through the process of lesson study allowed her to “guide” students and “let them struggle a little bit.” Another aspect of lesson study that Anne mentioned was that observing her colleagues’ lesson enactments and students during the implementation of the mathematical modeling lesson was “really helpful.” Although the interview was conducted soon after she taught the mathematical modeling lessons, Anne reported that she had already started to make changes to her teaching approach:

> I definitely don't jump as fast to give them answers to things, and I try when I'm planning my lesson to come up with more open-ended type questions for them. So, they'll have to …do a little bit of grappling and working together like they did during this task. So, I have, it has only been a couple of weeks, but I have kind of tried to incorporate some more tasks, more mathematical modeling tasks in my lessons for sure. (Interview, November 17, 2018)

Despite experiencing some challenges with teaching mathematical modeling, Anne wanted her students to benefit from the incorporation of more mathematical modeling within her teaching. Anne also reported that she was incorporating additional pedagogical strategies by being mindful to not “jump as fast to give them answers” when teaching mathematical modeling.

Expressed Opportunities of Particular “Mathematical Modeling” Tasks. Anne’s descriptions of the tasks after participating in lesson study are an indication that moving forward, she may be more prepared to select authentic modeling activities to implement in her classroom. Before participating in lesson study, Anne recognized that Tasks A and B were not very open-ended. Then in her Post-Lesson Study interview, Anne indicated that Tasks A and B were not quite modeling tasks. She used specific language about why they would not provide much opportunity to engage in mathematical modeling. For example, for Task A, she indicated that “it's almost too structured as a task… it's not like there's a lot of opportunity for them to create their own ideas or whatever. There is just one right answer.” Then for Task B, she pointed out, “Well, if they have the formula, then I mean they would just know that it's exponential rates. So, I don't know, it's less of an opportunity for modeling.” For Task C, Anne recognized that the students would need to engage in discussions, determine factors, and create multiple responses. She remarked:

> It's pretty open-ended where they'd have to justify their response. Obviously, a train travels faster than a car, but, you know, you have these other factors and traffic. And I think it does definitely give the kids more opportunity to engage with each other and explain their thinking. You know, and there's not necessarily one right answer. (Interview, November 17, 2018)

Here, Anne used more explicit terms to describe task C, such as “factors” and “not necessarily one right answer.” Anne’s descriptions of the tasks after the lesson study suggest that she had different expectations for mathematical modeling tasks than she did before the lesson study.

Case 3—Karen: Focused on Shifting Classroom Culture

Pre-Lesson Study Conceptions

Expressed Benefits of Teaching Mathematical Modeling. As the only teacher with experience teaching mathematical modeling, Karen referenced her early struggles with teaching modeling, but she also acknowledged how her students' engagement with modeling had evolved. Karen reflected on how implementing mathematical modeling had promoted collaboration and discourse. She said, **"**Eventually they started to deeply think about it and contribute to each other's ideas and bounce ideas off of each other and reference each other's input as a class discussion." Karen also conveyed an interest in supporting students to be agents of their own learning. Unlike in the previous two cases, because of her prior experience with teaching modeling, it seemed as though Karen had different beliefs about her students' experiences with modeling, and she had a clearer vision for her implementation of modeling.

Expressed Challenges of Teaching Mathematical Modeling. In her Pre-Lesson Study interview, Karen displayed a determination to teach modeling despite challenges. Karen's primary concerns about teaching mathematical modeling were with regard to her available resources. She indicated that when it comes to teaching mathematical modeling, "the primary constraint is the lack of great tasks and the lack of time." However, she also indicated that she would find time to include mathematical modeling tasks despite timelines and due dates, even if she has to include an activity after a test has already been given.

Expressed Teaching Approach to Mathematical Modeling. Karen's expressions about her approach and pedagogical strategies for teaching modeling prior to participating in the lesson study provided evidence of her determination to teach mathematical modeling as often as possible. Karen attempted to incorporate mathematical modeling activities within each unit. She mentioned that teaching modeling within a unit would sometimes "spoil the authenticity" of the task because students knew they were going to use the mathematics they had been studying in that particular unit. However, she said that, at times, rather than implement a complex modeling task, she will modify textbook tasks so that they could be more open-ended and connect to modeling in some way. She indicated that it was sometimes difficult to teach all the unit goals and attend to the goals of mathematical modeling at the same time.

Expressed Opportunities of Particular "Mathematical Modeling" Tasks. When Karen reviewed Task A, she did not see it as an authentic modeling task. Her reaction was as follows:

> So obviously we're comparing linear versus exponential and using the information given…Even though, ok so, this would be an example of something that would be called a modeling task, but I feel like there's only one correct answer… I don't see it as being without merit, but I don't necessarily see it as you taught students how to model. (Interview, November 16, 2018)

Karen acknowledged that Task A was like many tasks that textbooks identify as "modeling" tasks, but they were limited in the aspects of mathematical modeling that students might explore. Then, when she saw Task B, she said, "I don't think that's a modeling problem. I think that's a just a 'do you know the formula for compound interest'." Because of her prior experience with modeling, Karen described the "mathematical modeling" tasks as not really modeling tasks at all.

However, when Karen shared her thoughts on Task C, she indicated that students would have opportunities to engage in various aspects of modeling. Here is what she said:

> This is obviously open-ended. There's not one correct answer. Multiple entry points. Each student could defend their own reasoning using a variety of tools. There are all these questions that need to be answered before they even start. Variables that aren't given that aren't provided that they need to determine for themselves like what's relevant, what's not relevant, “what would I need to find out first.” Like they could actually do research ahead of time. Traffic patterns and average wait time at a red light and all kinds of variables that enter into the picture… (Interview, November 16, 2018)

Karen recognized that engaging with Task C, an open-ended modeling task, would require students to use creativity to define variables and make assumptions. Additionally, Task C could reveal more than one valid approach.

Post-Lesson Study Conceptions

Expressed Benefits of Teaching Mathematical Modeling. As shown in Table 3.6, in Karen’s Post-Lesson Study interview, her responses indicated that she saw extended benefits of teaching mathematical modeling. She communicated benefits beyond appealing to student interest and promoting collaboration. Karen also mentioned that teaching mathematical modeling can provide mathematical rigor and make mathematics relevant. She also focused on aspects related to the culture of engaging mathematical modeling. The transcript excerpt below demonstrates how Karen elaborated on the benefits of teaching mathematical modeling:

> [Mathematical Modeling] allows for students to have thoughts about what they're doing, why, or why they're doing what they're doing. It's not just procedural for the sake of doing a math problem…. Some of the benefits are that the students get to assume the role of a mathematician like mathematizing

> the world. I feel like they're more actively engaged in the problem-solving process and making connections. (Interview, November 16, 2018)

Karen mentioned how engaging in mathematical modeling can support students to act like mathematicians. The attributes for teaching mathematical modeling, as discussed by Karen, suggest that a culture shift is needed for students to truly appreciate mathematical modeling.

Karen also mentioned the collaborative aspect of mathematical modeling. She discussed how students could “build on each other’s ideas” and incorporate various mathematical practices such as “persevere in problem solving and critique the reasoning of others.” Again, Karen focused more on the cultural aspects of doing mathematical modeling, and how it provides more opportunities for a wide variety of students to become involved in the problem-solving process. To compare Karen’s Pre-Lesson Study and Post-Lesson Study Conceptions, Table 3.6 includes Karen’s conceptions of teaching mathematical modeling before and after the lesson study.

Table 3.6 Karen's Conceptions of Teaching Mathematical Modeling

	Pre-Lesson Study Conceptions		Post-Lesson Study Conceptions	
	Themes	Teacher's Expressions	Themes	Teacher's Expressions
Benefits of Teaching Modeling	• Appeals to student interest and agency • Promotes collaboration and discourse	I'm hoping that I can get them to enjoy it…and take ownership... Eventually, they started to deeply think and contribute to each other's ideas…and reference each other's input as a class discussion.	• Provides rigor for and relevancy • Students as mathematicians • Promotes collaboration	…students to have thoughts about what they're doing and why. It's not just procedural for the sake of doing a math problem…the outcomes are relevant and useful… … students get to assume the role of a mathematician like mathematizing the world. Collaborating, building on each other's ideas, you know rough draft thinking…
Challenges of Teaching Modeling	• Lack of authentic tasks and time	The primary constraint is the lack of the great tasks and the lack of time.	• Establishing appropriate culture	Like I am becoming more and more aware of the importance of establishing appropriate culture and it doesn't matter what the task is…
Teaching Approach to Modeling	• Incorporating modeling tasks into unit plans • Adapting tasks	… to make it as authentic as possible, but, because it's embedded in a certain unit, students automatically know that they're going to model with a specific…function. I use textbook tasks as my inspiration to create my own tasks….	• Interested in collaborating with colleagues • Intent to shift culture of classroom	…Just the ability to plan with someone else and anticipate things… I know my modeling tasks would be improved if I could do that. … an open-ended modeling problem. We do that four or five times a semester…I try to set it up like…" tomorrow is the day we're going to do, this cool task…It's going to be awesome."
Modeling Tasks	Tasks A & B are modeling tasks in disguise Task C is an open-ended modeling task	This would be an example of something that would be called a modeling task, but I feel like there's only one correct answer… There's not one correct answer. …There are all these questions…Variables that aren't provided…	Tasks A & B are modeling tasks in disguise Task C is an open-ended modeling task	I'm pretty sure said this before I received this more like an application task… It's not what I call doing mathematics. So, this to me is more of a modeling task. It's group worthy there's all kinds of things you could do with this task, talk about speed and you can talk about the traffic patterns…

Expressed Challenges of Teaching Mathematical Modeling. In her final interview, Karen again noted that she was challenged with finding good tasks, but she conveyed her biggest challenge to be shifting the culture in her classroom (see Table 3.6). Here, Karen discussed the challenge of fostering a creative classroom environment:

> Creating that environment where they want to do this type of math is an ongoing challenge of mine…just getting [students] to be comfortable with being uncomfortable is what I try and get to tell them. I am becoming more and more aware of the importance of establishing appropriate culture, and it doesn't matter what the task is or how wonderful my task is. If I can't get the students to buy what I'm selling, it's not, it's just not going to have the impact that I wanted to have. I need to have these kids believing that they can model. (Interview, November 16, 2018)

Karen was willing to implement modeling despite her limited resources (i.e., time and tasks), but she was still experimenting with developing a culture in her classroom where students embraced mathematical modeling.

Expressed Teaching Approach to Mathematical Modeling. After participating in the lesson study, just as she did beforehand, Karen indicated that she perceived mathematical modeling to be an important practice that should be embedded throughout mathematics courses. She also found that the collaborative nature of lesson study supported her to anticipate student thinking for the implemented lessons. Consequently, she would like to continue to plan with other teachers in her school to improve her teaching of mathematical modeling. More specifically, Karen said:

> …just the ability to plan with someone else and anticipate [student thinking], regardless if it's modeling or not modeling, but I wish I had that with my modeling tasks. I know my modeling tasks would be improved if I could do that. (Interview, November 16, 2018)

Karen indicated that during the lesson study her teaching of mathematical modeling was supported because she had the opportunity to collaborate with her colleagues, and she would like more opportunities to do so in order to continue to improve her teaching of modeling.

Karen also said that she incorporates many “application problems” within each unit, and she encourages group work. However, she said that despite time constraints, she teaches:

> an honest to goodness open-ended, take the whole class period, modeling problem four or five times a semester. That's all the time I have, but I try to set it up like, “okay guys, tomorrow is the day we're going to do, you know, we're going to do this cool task. I need you all to bring your A game. It's going to be awesome. (Interview, November 16, 2018)

For Karen, a whole class period lasts for 80 minutes, so this is a significant block of time to engage her students in one modeling activity. Karen discussed the importance of making time to incorporate mathematical modeling, and she aimed to shift the culture so that students also find the activities to be meaningful.

Expressed Opportunities of Particular “Mathematical Modeling” Tasks. As shown in Table 3.6, Karen’s descriptions of the tasks after the study did not vary much from her initial reactions. It seemed as though Karen saw Tasks A and B as modeling in disguise. She said that Tasks A and B were application tasks rather than modeling tasks and described them as “closed questions.” She mentioned that there were no opportunities for “interpretation,” and students would not be able to engage deeply in the modeling process. As before, Karen described Task C as an open-ended task with multiple solution paths. She also noted that it would be a good task for student collaboration. Given Karen's experience with teaching mathematical modeling combined with her already reasonable assessment of the modeling tasks prior to the study, the lack of difference in her descriptions of the tasks was to be expected.

Reflections across the Three Cases

Expressed Benefits of Teaching Mathematical Modeling. As indicated in the individual cases above, all three teachers acknowledged multiple benefits of teaching mathematical modeling before and after participating in the lesson study. When looking across the cases, the teachers' conceptions about the benefits of this important mathematical process showed the most commonality. As shown by the circled themes in Table 3.7, in the Pre-Lesson Study interviews, both Loren and Karen mentioned that mathematical modeling can appeal to student interest. Then, after participating in the study, both Loren and Karen discussed how they observed their students engage in rigorous and relevant mathematics during the mathematical modeling lesson on State Apportionment (see themes in the rectangles in Table 3.7). A common theme across all three teachers' Post-Lesson Study interviews was that teaching mathematical modeling promotes student collaboration. To compare the teachers' Pre-Lesson Study and Post-Lesson Study Conceptions, Table 3.7 includes the themes for each teachers' conceptions of teaching mathematical modeling before and after the lesson study.

Table 3.7 Three Secondary Teachers' Conceptions of Teaching Mathematical Modeling

	Pre-Lesson Study Conceptions			**Post-Lesson Study Conceptions**		
	Loren	Anne	Karen	Loren	Anne	Karen
Benefits of Teaching Modeling	• Appeals to student interest • Provides opportunities for creativity • Can have multiple approaches	• Promotes problem-solving	• Appeals to student interest and agency • Promotes collaboration and discourse	• Shows math is valuable and relevant • Provides rigor for students • Encourages collaboration	• Promotes Problem-solving • Promotes student collaboration • Provides all students access • Focus on student thinking	• Provides rigor and relevancy • Students as mathematicians • Promotes collaboration
Challenges of Teaching Modeling	• Students will struggle with the complexity of modeling	• Students will struggle with the complexity of modeling	• Lack of authentic tasks and time	• Requires time • Aligning to curriculum • Fostering group work	• Requires a culture shift	• Establishing appropriate culture
Teaching Approach to Modeling	• Implementing Word-Problems	• Teaching word-problems	• Incorporating modeling into unit plans • Adapting tasks	• Using annotated lesson plan • Incorporating Group Roles	• Increasing the use of modeling tasks • Using open-ended questions to foster collaboration	• Interested in collaborating with colleagues • Intent to shift culture of
Modeling Tasks	Tasks A & B are not open-ended enough Task C has opportunities to collect information	Tasks A & B are not open-ended enough Task C can have multiple approaches	Tasks A & B are disguised as modeling tasks Task C is an open-ended modeling task	Tasks A & B have Missed opportunities Task C has Opportunities to define variables and validate results	Tasks A & B are not quite modeling Task C promotes collaboration, defining variables, verification, multiple approaches	Tasks A & B are disguised as modeling tasks Task C is an open-ended modeling task

One final observation is that some of Loren's and Anne's themes *after* the study were more closely aligned with Karen's Pre-Lesson Study themes (see themes in the rectangles with dashed lines in Table 3.7). For instance, before the study, Karen indicated that student collaboration was a benefit of teaching mathematical modeling. However, Loren and Anne did not mention this benefit until after they observed their students exploring mathematical modeling. Additionally, Karen indicated that teaching mathematical modeling was challenging due to the lack of authentic tasks and required time. Yet, Loren and Anne did not mention those challenges until after the study. This observation is important because Karen had some experience teaching modeling, and Loren and Anne had not implemented mathematical modeling before the study. These cases provide empirical evidence about how these teachers' conceptions evolved through teaching mathematical modeling that was supported by lesson study. More importantly, one might hypothesize that Loren's and Anne's conceptions of teaching mathematical modeling progressed more expediently due to their participation in lesson study.

Expressed Challenges of Teaching Mathematical Modeling. In addition to noticing common themes across teachers' conceptions of benefits, the teachers also articulated similar ideas about the challenges of teaching mathematical modeling. In these cases, the teachers with similar experiences in teaching mathematical modeling (i.e., no previous experience) had more commonality. Prior to their participation in lesson study, both Loren and Anne were concerned about how students might struggle with the complexity of mathematical modeling. However, after observing how their students persevered when exploring a mathematical modeling activity, they did not mention that modeling activities would be too complex for their students, as a major concern, in their Post-Lesson Study interviews. Loren’s and Anne’s evolved conceptions about the benefits for students indicate a shift related to how students can engage with mathematical modeling. Also, both Anne and Karen spoke about aspects of teaching mathematical modeling that indicated that they wanted to see a culture shift for students that these teachers believed needs to be established when implementing mathematical modeling. Additionally, the challenges of teaching modeling described by Loren in the Post-Lesson Study interview had some alignment to Karen’s Pre-Lesson Study interview. This could be an indication that the lesson study supported Loren’s progression as a teacher of mathematical modeling. Although there was not as much commonality across teachers in their conceptions about the challenges of teaching mathematical modeling, all of the teachers’ conceptions were expanded in some way after participating in the lesson study.

Expressed Teaching Approach to Teaching Mathematical Modeling. The patterns observed, with respect to teachers’ experience and their commonalities of conceptions, were also observable when they discussed their teaching approach to modeling. Again, both Loren and Anne had only taught word problems or application problems prior to the study. However, all of the teachers discussed how their approaches had evolved after participating in the study. Additionally, it is possible that lesson study expedited Loren’s and Anne’s learning. Although Anne was interviewed only weeks after the lesson study, she communicated that she had already started incorporating more mathematical modeling into her classroom routines. It seems as though Anne's previously established classroom norms and routines (e.g., use of group roles) combined with her participation in the lesson study may have supported an evolution of her teaching approach to mathematical modeling.

Expressed Opportunities of Particular "Mathematical Modeling" Tasks. When the teachers described the opportunities that might be provided through particular "mathematical modeling" tasks, their teaching experience seemed to separate them again. In the Pre-Lesson Study interviews, all of the teachers indicated that Tasks A and B provided fewer opportunities to engage in the mathematical modeling process than Task C. However, Karen was the only teacher who pointedly did not classify Tasks A and B as authentic modeling tasks. In the Post-Lesson Study interviews, all three teachers indicated that Tasks A and B did not provide much, if any, opportunities to engage in the mathematical modeling process. In the Post-Lesson Study interview, when describing Task C, both Loren and Anne used terminology that was more specific, such as "defining variables" and "validating or verifying results." In the Post-Lesson Study interview, Karen classified Task C as a modeling task, and she focused on the characteristics that students could exhibit in a successful approach to the task.

Discussion

Teachers around the world have faced various challenges with teaching mathematical modeling. In the global context, teachers reported a lack of self-efficacy in teaching modeling, and they struggled with managing the implementation of open-ended mathematical modeling activities (e.g., Kuntze, Siller, & Vogl, 2013; Pereira de Oliveira, & Barbosa, 2013). In the United States, the sparse research indicates that teachers have had limited preparation for teaching mathematical modeling (e.g., Newton, Madea, Alexander, & Senk, 2014). Also, teachers may not believe that students can successfully engage with complex modeling activities (Anhalt, Cortez, & Bennet, 2018). Furthermore, U.S. teachers seems to lack resources such as time and high-quality curriculum materials for teaching mathematical modeling (e.g., Gould, 2013; Huson, 2016).

This study sought to understand more about how secondary teachers in the United States conceptualized teaching mathematical modeling through participation in lesson study. The outcomes for the teachers in this study provided insight into how teachers' conceptions about teaching mathematical modeling can evolve through such an experience. The teachers' conceptions in the study emerged into cases that could be indicative of how teachers with similar backgrounds and experiences could be influenced by lesson study on mathematical modeling.

Even the most veteran teacher, Karen, despite having prior experience with teaching mathematical modeling, still found benefits to participating in the lesson study. In her Pre-Lesson Study interview, Karen indicated that teaching mathematical modeling could increase student interest and provide learning opportunities through collaboration and discourse. These benefits were similar to those observed by other researchers (e.g., see Rubel, Lim, Hall-Wieckert, & Sullivan, 2016). However, Karen was also well aware of the challenges of teaching mathematical modeling. Similar to teachers in other U.S. studies (e.g., Gould, 2013; Huson, 2016), Karen had limited time and access to authentic modeling activities. Despite, those challenges, Karen had found ways to adapt and create her own modeling activities, and she found ways to incorporate those activities within her courses despite having a district-mandated curriculum. Then after the lesson study, Karen shared her appreciation for the collaborative nature of lesson study. She also indicated that her self-created modeling tasks could be improved through future collaborations with her colleagues. It is notable that through lesson study, Karen viewed her less-experienced colleagues as valuable resources for supporting the implementation of mathematical modeling. Also, after participating in the lesson study, a challenge that was different from those expressed in the Pre-Lesson Study interview became foregrounded for Karen. She discussed the challenge of motivating her students to participate in more meaningful ways and appreciate mathematical modeling. Although the influence of lesson study for Karen was not as dramatic as it was for Loren and Anne, Karen conveyed an interest in continued collaboration with colleagues and further developing the culture in her classroom so that students also appreciated the opportunities afforded by mathematical modeling.

As the novice teacher, Loren had some experiences with mathematical modeling as a student, but, thus far, she had not engaged her own students in mathematical modeling. At the beginning of the study, Loren’s concerns about teaching modeling were not unlike concerns reported by others in previous studies (e.g., Anhalt, Cortez, & Bennet, 2018). Initially, Loren believed that mathematical modeling tasks might be too complex for her students. Then after participating in the study, Loren reported observing several benefits of learning mathematical modeling for her students. For instance, she saw her students engage in rigorous and relevant mathematics. She also saw her students engage in “group work” in ways they had not done prior to the study. Loren’s experience seemed to change her beliefs about her students as she conveyed that she observed that modeling provided multiple benefits for her students. The benefits for students that Loren observed mirror the findings in other U.S. studies (e.g., Rubel, Lim, Hall-Wieckert, and Sullivan, 2016). Additionally, Loren was concerned about limited resources for teaching modeling such as the time required and the lack of available modeling activities, especially while using a mandated curriculum. Loren’s concern about a lack of resources was similar to U.S. teachers’ concerns in other studies (Gould, 2013; Huson, 2016). More specifically, Loren was unsure about how she might teach mathematical modeling or where she would find appropriate modeling activities that would fit within her curriculum. After participating in lesson study on mathematical modeling, Loren communicated that she had learned additional pedagogical strategies for teaching mathematical modeling. For instance, Loren said she intended to incorporate group roles and be more thoughtful and thorough in her lesson planning. Loren’s conceptions after the lesson study seemed to indicate that she was now better prepared to teach mathematical modeling.

While Anne, a former engineer, already saw the importance of teaching mathematical modeling, she had not yet taught an authentic mathematical modeling lesson. Like Loren and teachers in previous studies (e.g., see Anhalt, Cortez, &

Bennet, 2018), in her initial interview, Anne was concerned that mathematical modeling might be too complex for some students. After participating in the lesson study, she conveyed that she had the opportunity to observe student thinking in ways that she had not observed in her previous lessons. She also saw how students could become more adept at the problem-solving process through engaging in mathematical modeling. Anne's reflections on student thinking were similar to teachers observed by Widjaja et al. (2017) whose beliefs about students' abilities to solve problems were influenced by their participation in lesson study. Anne was able to see how modeling can provide opportunities for *all* of her students to participate in group discussions and engage in deeper mathematics. Also, soon after the lesson study was completed, Anne indicated that she had already started to implement more open-ended tasks into her regular classroom routines. She was also focused on not "telling" too much when supporting students in modeling activities.

Overall the three teachers' conceptions about teaching mathematical modeling evolved through participation in lesson study on mathematical modeling. In the Pre-Lesson Study interviews, the teachers in this study were concerned about a lack of resources for teaching modeling. Loren and Anne also had concerns about the complexity of modeling tasks for students. After participating in lesson study, the teachers' conceptions of teaching modeling were focused more on the benefits that modeling could provide for their students. Additionally, the teachers reflected on how lesson study supported their intentions to improve their implementation of mathematical modeling in the future.

It is also worth noting that while, internationally, teachers have reported a lack of self-efficacy (Kuntze, Siller, & Vogl, 2013) and struggled with the open-ended nature of modeling (Borromeo Ferri & Blum, 2013), the teachers in this study did not articulate concerns about how they would implement ill-structured tasks. Karen had experience teaching mathematical modeling, so she had likely overcome some of the challenges that can be present when implementing modeling. However, in the Pre-Lesson Study interviews, Loren and Anne were mostly concerned about their students struggling. Then after participating in lesson study, rather than focus on the challenges of teaching ill-structured tasks, both Loren and Anne focused on the benefits for students. Loren and Anne also mentioned how participating in the lesson study supported them in focusing on student thinking, so they were better prepared to not "tell" too much and "guide" student thinking in the moment.

Just as the teachers in Gould's (2013) study still had misconceptions of mathematical modeling after participating in professional development on mathematical modeling, the teachers in this study were not experts in teaching mathematical modeling after participating in two cycles of lesson study. However, these teachers' conceptions of teaching mathematical modeling were more sophisticated after the study. While the teachers did not always explicitly mention aspects of mathematical modeling such as its iterative process, they emphasized how mathematical modeling included working with real-world situations unlike the contrived situations found in many textbook tasks. They also indicated that authentic modeling tasks can have multiple valid solutions and extend beyond procedural steps.

These findings are important because each of these teachers had a variety of background experience, and they all reported evolved conceptions about teaching mathematical modeling through participation in lesson study.

Conclusion

The cases in this study provide evidence that lesson study is a promising strategy for supporting the implementation of mathematical modeling in school mathematics. Based on the teachers' responses in the interviews, it seems that the activities of lesson study influenced their conceptions of teaching mathematical modeling. One way the teachers were supported was through the provision of resources. More specifically, the teachers received mathematical modeling curriculum materials that they did not have prior to the study. To support the teachers with respect to limited time, lesson study provided structured time that allowed the teachers to collaboratively plan a lesson on mathematical modeling. All of the teachers in the study reported that they found value in collaborating with their colleagues. Loren and Anne specifically, reported that they learned pedagogical strategies through the iterative process of planning, teaching and observing, and debriefing the lesson on mathematical modeling. Loren and Anne also discussed how the focus on student thinking impacted their beliefs about students and improvement of instructional strategies. While Loren, the novice teacher was mostly focused on developing pedagogical strategies, the two experienced teachers, Anne and Karen, had shifted their focus to cultural aspects of teaching mathematical modeling. In summary, all of the teachers shifted their conceptions about teaching mathematical modeling in ways that illustrate progress.

Limitations and Future Research. While this study included a small sample size of three teachers, the qualitative methods contributed to nuanced understandings about how secondary teachers can think about teaching mathematical modeling, an important mathematical process. Because the methods relied on interview data, and the sample size is small, it is not possible to conclude cause and effect or correlational relationships. However, this study contributes to areas in mathematics education which still have few empirical studies. From this study, hypotheses can be developed about how to support secondary teachers in teaching mathematical modeling. These hypotheses can be tested through larger studies. For example, a study which incorporates multiple groups of teachers participating in long-term lesson study may provide further insight into how lesson study influences teachers' conceptions of teaching mathematical modeling. Additionally, assessments that measure teachers' learning about teaching mathematical modeling could provide more information regarding teachers' learning.

Chapter 4

CLOSING THOUGHTS

> "*In qualitative work, it is understood that the act of studying a social phenomenon influences the enactment of that phenomenon. Researchers are a part of the world they study; the knower and the known are taken to be inseparable.*" (Hatch, 2002, p. 10)

Throughout each phase of this study, I did my best to be purposeful in my choices and reflective on the outcomes. In doing so, I had to consider my own "conceptions" about research, lesson study, and teaching mathematical modeling. As a result, I realized how I would have made some different choices. I also reflected on further observations or phenomena within my data that went beyond the scope of answering my research questions. Additionally, one of the most exciting and rewarding parts of this journey was developing new hypotheses and ideas for future research. The goal of this chapter is to share some of these reflections.

Alternate Decisions

While I recognize that there are multiple ways that I could have conducted this study, many of my methodological decisions were influenced heavily by my lack of access to resources, namely time and funding. Thus, I will not focus on how I would have changed the broader research methodology. Instead, I discuss a few nuanced decisions that I made in the moment, as a participant observer, that likely influenced the findings of each sub-study. Throughout the study, I aimed to balance the roles of researcher, knowledgeable other, and facilitator of lesson study. Like the teachers in my study, I was also influenced by my own academic and professional background. I personally, had nine years of secondary teaching experience concurrent with seven years of leading various teacher teams and school-level PD meetings. Then, I had two years of experience as an education administrator where my primary responsibilities involved leading PD on a variety of topics for school leaders and teachers (e.g., school organization, academic policy, data management, and mathematics instruction). Finally, as a researcher, I had participated in lesson study through my graduate research assistantship. Consequently, like the teachers who had more experience in teaching with conventional mathematics tasks versus open-ended modeling activities, I had more experience with leading traditional workshop PD rather than the open-ended and iterative process of lesson study. Finally, just as it was for the teacher participants, teaching mathematical modeling is also a relatively new idea for me as a practitioner and researcher.

Given my professional background and the balancing of roles, I often found myself making key decisions which would have unpredictable outcomes. For instance, once the lesson study planning meeting was underway, I decided to emphasize the focus on student thinking over explicitly attending to a mathematical modeling cycle. I primarily did this because the teachers were learning about lesson study, focusing on student thinking, and implementing mathematical modeling. I realized that it would be overly ambitious to expect the teachers to become experts in all three areas after only two cycles of lesson study. I hypothesized that focusing on student thinking would naturally support the teaching of mathematical modeling, so I foregrounded the attention to student thinking. As a result, the teachers focused explicitly on student thinking throughout the lesson study process, so the findings in regard to engagement with student thinking were more fruitful than I had initially expected. However, it was unclear whether or not mathematical modeling had been emphasized enough. Within the lessons, there were some aspects of mathematical modeling that fell short.

Because I realized that modeling may not have been emphasized enough, once the formal interviews were conducted, I discussed, with each teacher, how modeling could have been enhanced within the lessons. In short, the teachers and I agreed that the students engaged in mathematical modeling with respect to various aspects of our chosen modeling definition and cycle, but other elements of mathematical modeling were absent. For instance, the students had opportunities to make choices and assumptions, explore various models of state apportionment, interpret the models in terms of fairness, and make comparisons of a simplified model to real-world outcomes. However, the students were not asked to validate their models or interpret how the models corresponded to real-world data. The students were also not explicitly told that they were engaging in mathematical modeling or given any information about mathematical modeling. Informally, each of the teachers agreed that being more explicit about the modeling process would have enhanced the lesson. We also recognized that as a first modeling experience for students, it may not be necessary to be explicit about the actual process right away, but that it would also be effective to be more explicit about mathematical modeling in subsequent modeling activities (see, e.g., Borromeo Ferri, 2018).

In considering how I may have better emphasized mathematical modeling, I discovered a conundrum in my research methodology. Part of why I chose the focus on student thinking over the focus on modeling was because I had not anticipated the teachers' responses to the PD activities. On the one hand, I wanted to leave the process open-ended and see how the teachers progressed naturally. I did not want to over-influence the teachers' engagement with student thinking or their conceptions about teaching mathematical modeling. On the other hand, I needed to make use of judicious telling (Lobato et al., 2005), so that I guided the teachers in ways that could be beneficial to themselves and students. Because I did not have time in between the teachers' selection of the modeling task and the first planning meeting because these events occurred on the same day, I was not able to plan ahead by familiarizing myself with the selected task. As a result, I did not plan when I might need to interject to guide the teachers to focus on specific modeling aspects. Thus, in future studies, I will plan more time for the curriculum study so that teachers select a modeling activity prior to the first planning meeting. In doing so, I will have time to engage with the activity before the planning meeting in order to better facilitate the development of the lesson plan. This will allow me to annotate the teachers' selected modeling activity with respect to specific components of mathematical modeling. For example, I will annotate when students might need support in making assumptions or validating their models. Then during the planning meetings, using my annotations of mathematical modeling competencies, I can interject and explicitly guide teachers to incorporate important features of mathematical modeling into the lesson. I will encourage teachers to include learning goals and teacher support in the lesson plan that supports them to specifically attend to aspects of mathematical modeling while implementing the lessons. In doing so, the teachers could be better prepared to support students to engage with various aspects of mathematical modeling.

Finally, another change I would make involves how I conducted the interviews. My limitations here could be attributed to my lack of experience as a researcher. While I intended to conduct semi-structured interviews, and I had written possible probing questions, I missed several opportunities to ask follow-up questions. This was especially true during the Post-Lesson Study interviews. I believe that I would have collected richer interview data if I had asked the teachers to expand on some of their answers. For instance, in her Post-Lesson Study interview, Anne mentioned that she had started to incorporate more modeling tasks. While I was analyzing the data, I realized that I would like to know more about those tasks. I should have asked her to give me an example of a task that she had implemented. In the future, I will work on being more responsive to teachers' interview responses.

Additional Observations from the Data

As indicated in both manuscripts, there were several data collected during the studies that have not been formally analyzed. Thus, in the near future, I would like to explore some of that data further. One observation that stood out to me, through reflections on the data and my experience, were various cultural factors throughout this lesson study. Because, previous researchers have acknowledged that lesson study may need to be adapted to cultural aspects of the United States (e.g., Lewis, C., 2016;

Stigler & Hiebert, 2016; Takahashi & McDougal, 2016), some of my observations were expected. As I mentioned previously in Chapter 1 and Paper 1 (i.e., Chapter 2), the teachers and I had to adapt to various structural constraints, such as their school schedules. For instance, we chose to meet over the summer to plan the lessons. Because not all of the teachers were available to watch the lesson enactments live, we relied on video-recordings of the lessons. Since other researchers had documented similar challenges and the adaptations made during their implementations of lesson study in the United States, I was able to quickly make adaptations. For this reason, I think it would be important to share my experiences as confirmation that there are indeed cultural factors that influence how lesson study can be implemented in U.S. schools.

Furthermore, I explicitly asked the teachers about the lesson study process during the Post-Lesson Study interviews. All of the teachers mentioned that they felt very supported when collaborating with their colleagues. Loren and Anne also appreciated the focus on student thinking as they had not attended to student thinking through other modes of PD. Each of the teachers expressed an interest in continuing lesson study at their school, but they all had concerns about how lesson study would fit within their school structure. Despite their concerns, the teachers had ideas about how lesson study could work in their setting. Exploring my data further could fill in some research gaps with respect to how to adapt lesson study within the United States.

With respect to mathematical modeling, there are places where the teachers developed their own solution approaches and discussed possible student approaches to the tasks in the lesson. Their primary goal in doing so was to anticipate student responses. However, as the teachers anticipated student responses to the modeling tasks, they also experienced the tasks as learners. Tools from research could support analyses of this activity. For example, Borromeo Ferri (2018) used frameworks to analyze modeling routes and modeling competencies of teachers. It would be illuminating to see how such frameworks might apply to some of my data. Those findings could have further implications for how the lesson study influenced the teachers' conceptions of mathematical modeling.

Future Research

Lastly, I would like to share ideas for future studies. In the manuscripts, I indicated how I might conduct future research that builds on my dissertation study. In addition to those studies, this research experience has provoked other related interests. For instance, through conducting this study, I realized that teaching mathematical modeling can promote equity in two distinct ways. First, teaching mathematical modeling can promote access by providing opportunities for all students to deeply engage with mathematics. Second, mathematical modeling can provide opportunities for students to explore issues of fairness in their own communities and throughout the world. In the future, I would like to conduct research that investigates how mathematical modeling can promote access and provide students with opportunities to explore equity.

With respect to issues of equity, there are other tangential topics that influence equitable opportunities for teachers and students. In future studies, I would be interested in learning more about how lesson study influences teachers' mathematical modeling content knowledge. I would also like to investigate the use of technology for teaching mathematical modeling, how mathematical modeling appeals to students' motivation and interest, and how teachers can provide opportunities for students to productively struggle with challenging mathematical modeling activities.

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Appendix A

MEETING AGENDAS

Lesson Study on Mathematical Modeling

- Warm Up: Describe your favorite Professional Development experience? Why was this your favorite? (5 minutes)
- Overview of Study (5 minutes)
- Introduction to Lesson Study (15 minutes)
- Videos About Lesson Study and Discussion (35 minutes)
- Anticipating Student Responses to *Traffic Jam* Task (Illustrative Mathematics, n.d.) (30 minutes)
- The Five Practices (5 minutes)
- Our Lesson Study Process (5 minutes)
- Break (5 minutes)

Curriculum Study: Mathematical Modeling

- Review Example Tasks (i.e., GAIMME, 2016; Gould, Murray, & Sanfratello, 2012) (15 minutes)
  - Break (15 minutes)
  - Discuss the features of the Tasks (10 minutes)
- What is mathematical modeling and why do we teach it? (5 minutes)
- Mathematical Modeling in the CCSSM (5 minutes)
- Mathematical Modeling Cycle (10 minutes)
- Explore Mathematical Modeling Curriculum Materials (60 minutes)
  - Review Materials
  - Discuss Potential Tasks for Lesson Plans
- Closing Discussion and Next Steps (10 minutes)

Table A.1 Binder Contents: Curriculum Study Materials

Resource	**Description**
Mathematical Modeling Tasks from EngageNY Workshop (EngageNY, n.d.)	Open-ended mathematical modeling tasks that were distributed during an EngageNY workshop to address the CCSSM modeling requirements. The materials for the workshop were made available online.
Chapter 1 from *5 Practices for Orchestrating Productive Mathematics Discussions* (Smith & Stein, 2018)	This chapter introduces the 5Ps but does not give much information on how to use the 5Ps. We did not read this chapter as a group.
Articles on Discourse Moves (i.e., Cirillo, 2013; Jacobs & Ambrose, 2008; Jacobs & Phillip, 2004)	Articles that provide discourse moves and types of questions to ask in order to elicit and build on students' thinking.
Lesson Plan Template	See Appendix B
Low Inference Observation Form	See Appendix C
The Modeling Process, Proportionality, and Geometric Similarity (Giordano, Fox, & Horton, 2013)	A chapter from a college textbook that introduces the process of mathematical modeling. The modeling cycle from an earlier edition of this textbook was provided to the teachers in the PowerPoint presentation. The modeling cycle in this edition varies slightly. We had a brief discussion of the differences in the cycle.
CCSSM Standards which include the Practice of Modeling (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)	The CCSSM Standards have a section on mathematical modeling and content standards that could include modeling have an asterisk (or star). The modeling section and starred standards were provided to the teachers.
Next Generation Science Standards on Science and Engineering Practice (NGSS, 2013)	The NGSS Standards on Science and Engineering Practice were provided to the teachers as they have many aspects of mathematical modeling.
GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education, Chapters 1 and 3 (2016)	GAIMME provides guidance on how to teach mathematical modeling across the grades. There are also several modeling tasks within the guide. The teachers were provided the entire book electronically.
Mathematical Modeling Handbook (Gould, Murray, Sanfratello, 2012)	This handbook consists of mathematical modeling tasks that are mostly appropriate for high school and college students. The teachers were also provided a CD of the book.

Appendix B

LESSON PLAN TEMPLATE

<table>
<tr><td colspan="3">Title of Lesson

Knowledge of Students:

Learning Goals:

Associated Standards:

Equipment:

Associated Files:

Associated Text:</td></tr>
<tr><td>Students’ Learning Activities, Teacher’s Questions and Anticipated Student Responses</td><td>Teacher’s Support</td><td>Notes/Reflection Include hypothesis to try out in the future.</td></tr>
<tr><td>Launch/Warm Up (Time)</td><td rowspan="2"></td><td rowspan="2"></td></tr>
<tr><td>Rationale (relationship to learning goal(s)):</td></tr>
<tr><td>Exploration (Time)</td><td rowspan="2"></td><td rowspan="2"></td></tr>
<tr><td>Rationale (relationship to learning goal(s)):</td></tr>
<tr><td>Group Work (Time)</td><td rowspan="2"></td><td rowspan="2"></td></tr>
<tr><td>Rationale (relationship to learning goal(s)):</td></tr>
<tr><td>Summary of Lesson (Time)</td><td></td><td></td></tr>
</table>

Rationale (relationship to learning goal(s)):		
Homework (Time)		
Rationale (relationship to learning goal(s)):		

Adapted from UD Math 25X Lesson Plans, Sample lesson plans by Lewis and Hurd (2011), and International Math-teacher Professionalization Using Lesson Study (IMPULS) example lesson plans

Appendix C

LESSON PLANS AND TASKS

State Apportionment Day 1
Knowledge of Students: Students will be heterogeneously grouped. Students should be familiar with percentages and ratios. Students should be familiar with arithmetic mean and geometric mean. **Learning Goals:** By the end of the lesson, the students will be able to… • Make sense of the task and persevere in solving. • Explain their mathematical thinking about their selected proportions or state apportionment methods. • Apply weighted averages to state apportionment. • Compare and contrast various methods of apportionment. **Associated Standards:** **A-SSE: Interpret the structure of expressions** 1. Interpret expressions that represent a quantity in terms of its context. SMP's: 1, 3, 4, 6, 8 **Equipment/Materials:** • Chrome book

- State Apportionment Worksheet 1
- State Apportionment Worksheet 2

Associated Files & Websites:

- Lesson Plan State Apportionment
- State Apportionment Maps in Google Drive Folder

https://www.census.gov/library/video/census_apportionment_machine.html (apportionment machine)
http://www.cut-the-knot.org/ctk/Democracy.shtml
http://www.census.gov/history/www/programs/demographic/methods_of_apportionment.html
http://www.census.gov/population/apportionment/about/index.html
https://www.census.gov/library/visualizations/2010/dec/2010-map.html (Maps)
http://www.ctl.ua.edu/math103/apportionment/appmeth.htm
http://www.math.colostate.edu/~spriggs/m130/apportionment2.pdf
https://epubs.siam.org/doi/abs/10.1137/1020040
https://www.youtube.com/watch?v=YWfEqWLz9pc (Hamilton Method)
https://www.youtube.com/watch?v=weGGVmy9yLc (Jefferson Method)

Associated Text:
Mathematical Modeling Handbook

Students' Learning Activities, Teacher's Questions and Anticipated Student Responses	**Teacher's Support**	**Notes/Reflection Include hypothesis to try out in the future.**
Launch/Warm Up (10 minutes) If you were in charge of determining how many representatives each state in the United States should have, what information would you need. How would you use that information? What obstacles do you think might be present? <u>Valid Responses</u> Students will tie the population to the number of representatives. I will calculate the percent population of each state. I will find the total number of congressional seats.	Once students have two or three valid responses, choose those students to share out. Can you provide an example from the map?	Students only said they would use the population. They did not give these

I will use the average. I would make sure the states with a higher population have higher seats. Smaller states have fewer, larger states have more (with regards to population). 1 representative per 100,000 people <u>Emerging Responses</u> I will divide by the total number of states. States with a larger area should have the highest number of representatives. Students think they need to know political parties, or the conversation turns into a discussion about who to elect. **Warm Up Share Out** After students share their thoughts: How would you define apportionment? Write down your rough draft thoughts. Watch video as a whole class https://www.census.gov/library/video/census_apportionment_machine.html	How do you know how many seats should be allocated? Why do you think that method is fair? For example, should Delaware have the same number of representatives as California? Why or why not? Students share definitions of apportionment on an online discussion board.	responses or specific responses. We might need to make the question more specific.
Rationale (relationship to learning goal(s)): This question sets the stage and gives students a visual of the varying sizes of states and populations. Students may begin thinking about population density.		
Exploration/Group Work (25 minutes) Part 1 1. A. If the new country plans on having 25 representatives in its House of Representatives, how many should each state receive? B. What if they plan to have only 17 representatives? How did you calculate how many representatives each state should receive? Did you use the same method for both 25 and 17 representatives? A. <u>Valid Responses</u> Students use percentages to calculate the number of seats to be allocated. Students round up or down to decide on the number of seats per state. Students divided each state population by the divisor (4,000) instead of using percentages.	Explain your calculation.	

With 25 representatives, the states should receive the following apportionment: A = 4; B = 4; C = 7; D = 10.

State	Population	Calculation	Number of Seats
A	15,000	0.15 x 25 = 3.74 or 15,000/4,000	4
B	17,000	0.17 x 25 = 4.25 or 17,000/4,000	4
C	28,000	0.28 x 25 = 7 or 28,000/4,000	7
D	40,000	0.4 x 25 = 10 or 40,000/4,000	10

Emerging Responses

Students may divide by four and round.
Students may not use percentages. They might just guess and check.

Use a simple example (e.g., if we share a pizza and I pay \$15 and you pay \$5 how much of the pizza should you get to eat?)

Students do not round correctly, or students' allocations may not sum to 25.

How would you round the number of people?

B. Valid Responses

Same as above.
With 17 representatives, the states should receive the following apportionment: A = 2; B = 3; C = 5; D = 7.

Why are we not able to round all of the allocations appropriately?

Initially, students follow rounding rules and round up to find the total of seats is 18 (see table below). Then students round A down since it has a smaller population.

Is this a fair allocation? Why or why not?

Can you elaborate on that?

State	Population	Calculation	Number of Seats
A	15,000	0.15 x 17 = 2.55 or 15,000/5882	3
B	17,000	0.17 x 17 = 2.89 or 17,000/5882	3
C	28,000	0.28 x 17 = 4.76 or 28,000/5882	5
D	40,000	0.4 x 17 = 6.8 or 40,000/5882	7

Emerging Responses

Students do not round correctly, or students' allocations may not sum to 17. Students may round the divisor to 6,000.

2. Which states (if any) would disagree with the apportionment that you have created in each of these cases? Do both scenarios create the same problems? Can you create a method that is fair to all states in both cases? Describe how your method works and why you believe it to be fair.

Valid Responses

For 1A: A and B get the same number of representatives even though B has more people. Students should question the fairness of this allocation.

For 1B: Students will notice that for state A, you have to round down even though the number is 2.55.

Emerging Responses

Students explanations provide a weak explanation for why their method works.

States A and B might complain because they don't get as many representatives as the states with higher populations. No states will disagree because it's based on population. **Share Out of Exploration Part 1 (10 minutes)** <u>Student Response Sequencing</u> First: Guess and check weighting method Second: Percentage method & rounded to get too many or too few representatives Final: Rounded and ended up with the right number of representatives (compare two groups that rounded differently)	Monitor students and ask students to share out their responses using the document camera. What do you notice about each method? What are the pros and cons of each method? What are your revisions for your initial responses?	
Exploration/Group Work (30 minutes) Part 2 3. The Hamilton Method was devised by Alexander Hamilton as a technique for fair apportionment. Investigate what the Hamilton Method was and if you agree or disagree with its fairness. Do either of your methods share any similarities with the Hamilton Method? Watch the video on the Hamilton Method: https://tinyurl.com/SGHamilton <u>Valid Responses</u> Students will research the Hamilton Method on their laptops. The Hamilton Method always gives the states with the highest remainder the extra seat(s). For 17 total seats the remainders are: A = 2; B = 3; C = 5; D = 7 Since A has the lowest remainder, the number of seats for State A is rounded down and the other states are rounded up. <u>Emerging Responses</u> Students will round using traditional rounding rules.	How does this method make sense? Can you justify why Hamilton is rounding everyone down in the beginning? How does he decide who gets the extra representatives? After the explanation of the method you can pause the video and start to use the Hamilton method. You may	

<table>
<tr><td>Students will not be able to decide on the extra person.</td><td>use software to build in pauses and interject questions.

Refer to the video: Why did they add the extra person in this one? How did they decide?</td><td></td></tr>
<tr><td>Exploration Share Out Part 2 (10 – 15 minutes)
Choose various organization techniques to share.
Rationale (relationship to learning goal(s)):
This task provides opportunities for students to think about how state representatives can be allocated fairly. Students use percentages and proportions to decide how states can be represented fairly.</td><td>Does anyone have anything else to add?</td><td></td></tr>
<tr><td>Exploration/Group Work (10 - 15 minutes) Part 3
4. Suppose that 1000 people move from state B to state A. How would this affect the Hamilton Method with both 25 and 17 representatives? Is this fair? Why or why not? Does it make sense that moving 1000 would change the results?
<u>Valid Responses</u>
Under the new population, with 25 representatives, the states receive the following apportionments: A = 4; B = 4; C = 7; D = 10.
<table>
<tr><th>State</th><th>Populatio
n</th><th>Calculation</th><th>Number of Seats</th></tr>
<tr><td>A</td><td>16,000</td><td>.16 x 25 = 4</td><td>4</td></tr>
<tr><td>B</td><td>16,000</td><td>.16 x 25 = 4</td><td>4</td></tr>
<tr><td>C</td><td>28,000</td><td>.28 x 25 = 7</td><td>7</td></tr>
</table></td><td>Split up calculating 25 and 17 within groups (e.g., two students calculate 25 and each calculate 17).</td><td></td></tr>
</table>

D	40,000	.40 x 25 = 10	10

With 17 representatives, the states should receive the following apportionments: A = 2; B = 3; C = 5; D = 7.

State	Population	Calculation	Number of Seats
A	16,000	.16 x 17 = 2.72	3
B	16,000	.16 x 17 = 2.72	3
C	28,000	.28 x 17 = 4.76	5
D	40,000	.40 x 17 = 6.8	7

This method creates 18 seats, so A or B would need to be rounded down.
In the latter apportionment, states A and B have the same population but do not receive equal representation.

Emerging Responses

Students will not find the correct allocations.
Students will not find the total number or representatives.
Students choose arbitrary reason for which state (A or B) to assign more representatives. For example, a student was observed, allocated an extra seat to State B, based the order of her calculations.

Do you think your reasoning is fair? Why?

Share Out of Exploration Part 3 (10 – 15 minutes)
With your partner discuss what you notice and wonder.

Does it make sense that moving 1000 would change the results?

Have a whole-class discussion about student responses.	You may want to use an online sharing platform to display student responses.	
Summary of Lesson (5 minutes) What are the advantages of the Hamilton method? What areas for improvement do you see?		
Homework (Time) Watch the video on the geometric mean		
Rationale (relationship to learning goal(s)): Students watch a new method that will be discussed the following day.		

STATE APPORTIONMENT DAY 1[8]

Student Name:____ Date:______________________

In the United States House of Representatives, the number of seats that each state receives is based on the population of the state. Each state is guaranteed at least one representative, but after that, it is determined solely by the number of people living in the state according to the census taken every ten years. There have been many different ways that the US state apportionment has been determined in the past.

Warm Up

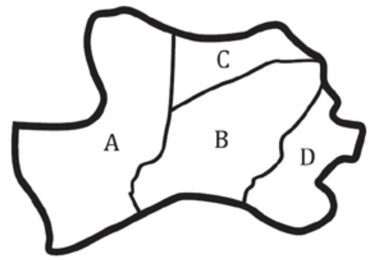
If you were in charge of determining how many representatives each state in the United States should have, what information would you need. How would you use that information? What obstacles do you think might be present?

How would you define apportionment? Write down your rough draft thoughts.

[8] Task adapted from Sanfratello, A. (2012). State apportionment. In H. Gould, D.R. Murray, & A. Sanfratello (Eds.), *Mathematical modeling handbook* (pp. 133 – 140). Bedford, MA: The Consortium for Mathematics and Its Applications (COMAP).

For simplicity, imagine that a newly formed country wishes to copy the US House of Representatives. This new country has just 100,000 people split up into only four different states, listed in the table below.

State	Population
A	15,000
B	17,000
C	28,000
D	40,000



2. A. If the new country plans on having 25 representatives in its House of Representatives, how many should each state receive?

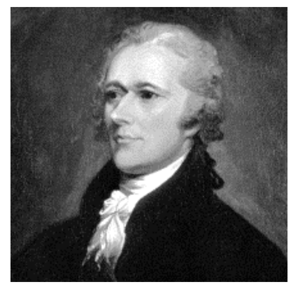
State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

B. What if they plan to have only 17 representatives? How did you calculate how many representatives each state should receive? Did you use the same method for both 25 and 17 representatives?

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

2. Which states (if any) would disagree with the apportionment that you have created in each of these cases? Do both scenarios create the same problems? Can you create a method that is fair to all states in both cases? Describe how your method works and why you believe it to be fair.

3. The Hamilton Method was devised by Alexander Hamilton as a technique for fair apportionment. Investigate what the Hamilton Method was and if you agree or disagree with its fairness. Do either of your methods share any similarities with the Hamilton Method?
Watch the video on the Hamilton Method:
https://tinyurl.com/SGHamilton



Hamilton's Apportionment for 25 Seats

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

Hamilton's Apportionment for 17 Seats

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

4. Suppose that 1000 people move from state B to state A. How would this affect the Hamilton Method with both 25 and 17 representatives?

Hamilton's Apportionment for 25 Seats

State	Population	Calculation	Number of Seats
A			
B			
C			
D			

Hamilton’s Apportionment for 17 Seats

State	Population	Calculation	Number of Seats
A			
B			
C			
D			

Is this fair? Why or why not? Does it make sense that moving 1000 people would change the results?

State Apportionment Day 2

Knowledge of Students:
Students will be heterogeneously grouped.
Students should be familiar with percentages and ratios.

Learning Goals:
By the end of the lesson, the students will be able to…

- Make sense of the task and persevere in solving.
- Apply weighted averages to state apportionment.
  Recognize that none of the methods work perfectly.
- Use data about the actual state apportionment results to reinforce that apportionment is based on population.

Associated Standards:

A-SSE: Interpret the structure of expressions
1. Interpret expressions that represent a quantity in terms of its context.

SMP's: 1, 3, 4, 6, 8

Equipment/Materials:

- Chrome book
- State Apportionment Worksheet 1
- State Apportionment Worksheet 2

Associated Files & Websites:

- Lesson Plan State Apportionment
- Mathematical Modeling Handbook State Apportionment Task Handouts

http://www.cut-the-knot.org/ctk/Democracy.shtml
http://www.census.gov/history/www/programs/demographic/methods_of_apportionment.html
http://www.census.gov/population/apportionment/about/index.html
http://www.ctl.ua.edu/math103/apportionment/appmeth.htm
http://www.math.colostate.edu/~spriggs/m130/apportionment2.pdf

https://epubs.siam.org/doi/abs/10.1137/1020040
https://www.youtube.com/watch?v=YWfEqWLz9pc (Hamilton Method)
https://www.youtube.com/watch?v=weGGVmy9yLc (Jefferson Method)
https://www.youtube.com/watch?v=l74j-auLjZE (Huntington-Hill Method)
https://www.youtube.com/watch?v=_UdGUULKN-E (Geometric Mean)

Associated Text:
Mathematical Modeling Handbook

Students' Learning Activities, Teacher's Questions and Anticipated Student Responses	**Teacher's Support**	**Notes/Reflection Include hypothesis to try out in the future.**

Launch/Warm Up (15 minutes)

Find the arithmetic mean and the geometric mean of each pair of numbers. Round to 2 decimal places when needed.

	Arithmetic Mean	Geometric Mean
6 and 12	6 + 12 = 18/2 = 9	$\sqrt{(6\ x\ 12)}$= 8.49
5 and 9	5 + 9 = 14/2 =7	$\sqrt{(5\ x\ 9)}$= 6.71
4 and 7	4 + 7 = 11/2 = 5.5	$\sqrt{(4\ x\ 7)}$= 5.29

What do you notice about how the two values compare to each other?

Valid Response

The geometric mean is always slightly smaller than the arithmetic mean.

Exploration/Group Work (30 - 35 minutes) Part 1

1. Watch the video on the Jefferson method, apply the Jefferson method for the 25 representatives.

Link: https://tinyurl.com/SGJefferson

Use the tables to apply the Jefferson method (students are provided three tables on the worksheet).

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

Why did Jefferson use this method?

What are the differences and similarities between the Jefferson Method and the Hamilton Method?

<u>Valid Responses</u>

The Jefferson Method involves modifying the divisor, d, which is calculated by taking the quotient of the total population and the number of seats. d is then decreased until the quotient of each state's population and the new d add up to the exact number of seats needed.

Let d = 4000 (100,000/25)

State	Population	Calculation	Number of Seats
A	15,000	15,000/4,000 = 3.75	3
B	17,000	17,000/4,000 = 4.25	4
C	28,000	28,000/4,000 = 7	7
D	40,000	40,000/4,000 = 10	10

Using the same method, 3,900 still does not work.

Let d = 3,700

State	Population	Calculation	Number of Seats
A	15,000	15,000/3,700 = 4.05	4
B	17,000	17,000/3,700 = 4.59	4
C	28,000	28,000/3,700 = 7.56	7
D	40,000	40,000/3,700 = 10.81	10

Students explain why some columns worked and others did not work.

<u>Emerging Responses</u>

If students struggle, assign each group a different denominator to try.
Discourage students from erasing any trials that do not work.

What's the purpose of this method? What do you hope to accomplish? How do you know when to stop guessing?

Have each person in your group choose a different divisor. It does not matter what divisor you choose. The end goal is to find the number of seats.

Students focus on the procedure without attaching any meaning to the output. Students struggle with finding the denominator to use. Students will use a denominator that is not helpful such as 4,001 or 3,999. **Share Out** First: Have one group share that chose a divisor that did not work Second: Have a different group share a divisor that works. Third: Have another group share a divisor that works.	This is a really complicated method. This is why people are still arguing over state apportionment. There is no right or wrong way.	
Rationale (relationship to learning goal(s)): Students use a different method for state apportionment.		
Exploration/Group Work (30 - 35 minutes) Part 2 2. The current method, Huntington-Hill, used by the US uses the geometric mean as the denominator and the state's population in the numerator. Does this method work well? What the video in the link below and complete the table. https://tinyurl.com/SGHHmethod Valid Responses If the quotient is greater than the geometric mean, give the number of seats equal to the upper quota.	Pause the video periodically and ask students what they notice about the video. What questions do you have right now? Discuss your thoughts and questions with your group.	We may want to remove the formula

For 25 seats

State	Population	Calculation				Number of Seats
		Quota	Lower Quota (n)	Upper Quota (n + 1)	Geometric Mean $\sqrt{(n(n+1))}$	
A	15,000	.15 x 25 = 3.75	3	4	$\sqrt{(3\ x\ 4)}$= 3.46	4
B	17,000	.17 x 25 = 4.25	4	5	$\sqrt{(4\ x\ 5)}$= 4.47	4
C	28,000	.28 x 25 = 7				7
D	40,000	.4 x 25 = 10				10

for the students. Let's see how students in the second teaching handle it.

For 17 seats

State	Population	Calculation				Number of Seats
		Quota	Lower Quota (n)	Upper Quota (n + 1)	Geometric Mean $\sqrt{(n(n+1))}$	
A	15,000	.15 x 17 = 2.55	2	3	$\sqrt{(2\ x\ 3)}$= 2.45	3
B	17,000	.17 x 17 = 2.89	2	3	$\sqrt{(2\ x\ 3)}$= 2.45	3
C	28,000	.28 x 17 = 4.76	4	5	$\sqrt{(4\ x\ 5)}$= 4.47	5
D	40,000	.4 x 17 = 6.8	6	7	$\sqrt{(6\ x\ 7)}$= 6.48	7

The H-H method does not work well with 17 seats. It leads to 18 total seats.

Emerging Responses

Student groups will get stuck at different points along the way.
Students will complete the calculations, but not make sense of what the numbers mean.
Students will struggle with applying the formula correctly.

How many seats are you trying to fill? Remind students to review the goal of the task.

Share Out of Exploration Part 2 (10 minutes) **<u>Share Out 1 (Mid-summary)</u>** Choose groups that did not complete the chart. Have them share sticking points and receive feedback from the rest of the class. **<u>Whole-Class Share Out</u>** Whole-class discussion about the groups’ final conclusions.	Discuss the outcomes with your groups. What do you notice or wonder?	
Final Exploration—Connecting to the Real World 2. Look up the actual state populations and their apportionments. https://en.wikipedia.org/wiki/United_States_congressional_apportionment How many representatives does Delaware have? How does Delaware compare to other states? What are your thoughts about the current state apportionments? What else jumps out at you? What do you notice? Explore the Wikipedia and pick out something that speaks to you. <u>Valid Responses</u> Students will notice some state have many more representatives. For example, California has 53 representatives and Delaware only has 1. Students will recognize other states that also only have one representative but are much larger in size.	You are going to explore a website. Be prepared to share out anything that speaks to you. Remind students that they can sort the chart towards the middle of the page.	

There was one time in history that Delaware had 2 representatives. New York had 45 representatives. Then it went down to 16. Students reference population density or other reasons for varying numbers representatives. <u>Emerging Responses</u> Big states only have one representative, they should have more.	Washington, D.C. does not have any representatives because it is not a state. What are your thoughts about that?	
Rationale (relationship to learning goal(s)): These tasks support students in their mathematical discourse and productive struggle. Students have the opportunity to engage in an authentic mathematical modeling task.		
Summary of Lesson (Time) Go back to your rough-draft thoughts about apportionment.	How did your ideas about apportionment change?	
Homework (Time) TBD		

STATE APPORTIONMENT DAY 2

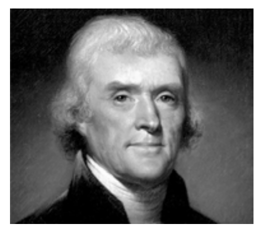
Student Name:____ Date:______________________

Warm Up

Find the arithmetic mean and the geometric mean of each pair of numbers. Round to 2 decimal places when needed.

	Arithmetic Mean	Geometric Mean
6 and 12		
5 and 9		
4 and 7		

What do you notice about how the two values compare to each other?



1. Watch the video on the Jefferson method, apply the Jefferson method for 25 representatives. Link: https://tinyurl.com/SGJefferson

Use the tables to apply the Jefferson method.

Jefferson's Apportionment for 25 Seats

State	**Population**	**Calculation**	**Number of Seats**
A	15,000		
B	17,000		
C	28,000		
D	40,000		

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

Why did Jefferson use this method?

What are the differences and similarities between the Jefferson Method and the Hamilton Method?

2. The current method, Huntington-Hill, used by the US uses the geometric mean as the denominator and the state's population in the numerator. Does this method work well?

Watch the video in the link below and complete the table.
https://tinyurl.com/SGHHmethod

Huntington-Hill's Apportionment for 25 Seats

State	**Population**	**Calculation**				**Number of Seats**
		Quota	Lower Quota (n)	Upper Quota (n + 1)	Geometric Mean $\sqrt{(n(n+1))}$	
A	15,000					
B	17,000					
C	28,000					
D	40,000					

Huntington-Hill's Apportionment for 17 Seats

State	**Population**	**Calculation**				**Number of Seats**
		Quota	Lower Quota (n)	Upper Quota (n + 1)	Geometric Mean $\sqrt{(n(n+1))}$	
A	15,000					
B	17,000					
C	28,000					
D	40,000					

2. Look up the actual state populations and their apportionments.
https://en.wikipedia.org/wiki/United_States_congressional_apportionment
How many representatives does Delaware have? How does Delaware compare to other states? What are your thoughts about the current state apportionments?

What else jumps out at you? What do you notice? Explore the Wikipedia and pick out something that speaks to you.

Appendix D

OBSERVATION FORM FOR TEACHERS

Date:
Lesson Title:
Learning Goals: By the end of the lesson, the students will be able to…

-

Teacher Says/Does	Student Says/Does
Inferences/Other Notes	

Appendix E

DEBRIEF SESSION PROTOCOL

- Commentary from teacher who enacted the lesson
- Commentary from the teachers who observed the lesson
- Discussion of lesson study team (Questions from Gorman et al., 2010, p. 101))
    - What evidence is there that lesson goals were met?
    - What insights and conclusions can we draw from our observation of students engaging with the research lesson?
    - What have we learned about student thinking, mathematics, and the lesson?
    - What new questions do we have?
    - What improvements or revisions to the lesson do the data suggest?

Appendix F

STUDY TIMELINE

Table F.1 Research Project Timeline

Event	Timeline
Segment 1: Before the Lesson study	
Initial Teacher Interviews: Loren Anne Karen	 July 25, 2018 July 25, 2018 July 26, 2018
Segment 2: Lesson study	
Phase A: Introduction to Lesson study Meeting (2 hours) Curriculum Study Meeting (2 hours)	 July 26, 2018 July 26, 2018
Phase B: Lesson Planning Meeting 1 (4 hours) Lesson Planning Meeting 2 (3 hours)	Tuesday, July 31, 2018 Friday, July 10, 2018
Phase C: Lesson Enactments and Observations Loren Enacted the Lesson Karen Enacted the Lesson Anne Enacted the Lesson	 Tuesday, September 25, 2018 Wednesday, September 26, 2018 Monday, October 15, 2018 Tuesday, October 16, 2018 Thursday, November 8, 2018 Friday, November 9, 2018
Phase D: Debrief Lesson Enactments and Observations Debrief Loren's Lesson Enactment Debrief Karen's and Anne's Lesson Enactments	 Wednesday, September 26, 2018 Friday, November 9, 2018
Segment 3: After the Lesson study	
Final Interviews Karen Loren Anne	 November 16, 2018 November 16, 2018 November 17, 2018

Appendix G

INTERVIEW PROTOCOLS

Pre-Lesson Study Interview Protocol

Questions about Mathematical Modeling

1. From your perspective, what is mathematical modeling? What are some reasons to engage in mathematical modeling?
    a. If the teacher only mentions one context for engaging in mathematical modeling, ask: How might scientists or other professionals engage in mathematical modeling? How about K-12 students?

    [If the teacher asks what is meant by mathematical modeling say the following: "For the purpose of this interview mathematical modeling is defined as 'a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena'" (GAIMME, 2016, p. 8).]

2. Do you believe that mathematical modeling should be taught in schools? Why? Why not?

3. Are you currently teaching mathematical modeling, or have you ever taught mathematical modeling? Describe your teaching approach to teaching mathematical modeling (e.g., frequency, resources for tasks, aspects of the modeling cycle addressed).

If the teacher has taught or is currently teaching mathematical modeling, ask the following.

- Why have you approached mathematical modeling in this way?
- What benefits, if any, do you see in teaching mathematical modeling?
- What drawbacks, if any, do you see in teaching mathematical modeling?
- What challenges do you and your students experience when you're teaching math modeling?
- Has your teaching of math modeling changed with time and why? In what ways?

If the teacher is not currently teaching mathematical modeling but has taught mathematical modeling in the past, ask:

- Why are you not currently teaching modeling?

If the teacher is not currently teaching or has never taught mathematical modeling, ask the following.

- What benefits could you see to teaching mathematical modeling?
- What challenges would you expect you and your students to experience if you were teaching math modeling?

Note: Each question and sub-question may be subjected to the following questions:

- "What do you mean by…?"
- "Why do you think that is important"
- "Could you tell me more about that?"
- "I'm not sure I understand what you mean. Can you explain that again? Or can you explain that in a different way?"
- "Can you give me an example from your experiences?"

4. The following tasks were adapted from the mathematics curriculum, *Engage New York*. The authors of *Engage New York* identified the tasks as mathematical modeling tasks. From your perspective, what types of opportunities do the following tasks provide for students to engage in mathematical modeling, if at all? [Teachers will be provided, one at a time, with a copy of the tasks.]

"Mathematical Modeling" Tasks

Task A

Two equipment rental companies have different penalty policies for returning a piece of equipment late.

Company 1: On day $\mathbf{1}$, the penalty is \$$\mathbf{5}$. On day $\mathbf{2}$, the penalty is \$$\mathbf{10}$. On day $\mathbf{3}$, the penalty is \$$\mathbf{15}$, and so on, increasing by \$$\mathbf{5}$ each day the equipment is late.

Company 2: On day $\mathbf{1}$, the penalty is \$$\mathbf{0.01}$. On day $\mathbf{2}$, the penalty is \$$\mathbf{0.02}$. On day $\mathbf{3}$, the penalty is \$$\mathbf{0.04}$, and so on, doubling in amount each additional day late.

Jim rented a digger from Company 2 because he thought it had the better late return policy. The job he was doing with the digger took longer than he expected, but it did not concern him because the late penalty seemed so reasonable. When he returned the digger $\mathbf{15}$ days late, the penalty fee shocked him. What did he pay, and what would he have paid if he had used Company 1 instead?

Company 1	
Day	**Penalty**
1	**\$5**
2	**\$10**
3	**\$15**

Company 2	
Day	**Penalty**
1	**\$0.01**
2	**\$0.02**
3	**\$0.04**

4	$\$20$
5	$\$25$
6	$\$30$
7	$\$35$
8	$\$40$
9	$\$45$
10	$\$50$
11	$\$55$
12	$\$60$
13	$\$65$
14	$\$70$
15	$\$75$

4	$\$0.08$
5	$\$0.16$
6	$\$0.32$
7	$\$0.64$
8	$\$1.28$
9	$\$2.56$
10	$\$5.12$
11	$\$10.24$
12	$\$20.48$
13	$\$40.96$
14	$\$81.92$
15	$\$163.84$

a. Which company has a greater $\mathbf{15}$-day late charge?

b. Describe how the amount of the late charge changes from any given day to the next successive day in both Companies 1 and 2.

c. How much would the late charge have been after $\mathbf{20}$ days under Company 2?

(*Engage New York*: *Algebra I*, *Module 3*, *Lesson 5*, pp. 51 – 52)

Task B

Margie got $\$\mathbf{1,000}$ from her grandmother to start her college fund. She is opening a new savings account and finds out that her bank offers a $\mathbf{2}\%$ annual interest rate, compounded monthly. What type of function would best represent the amount of money in Margie's account? Justify your answer mathematically.

(*Engage New York: Algebra I, Module 5, Lesson 3,* p. 42)

Task C

Noam and Athena are having an argument about whether it would take longer to get from New York City to Boston and back by car or train. Noam says it is faster to drive. Athena prefers to take Amtrak. Who do you agree with? Develop a mathematical model to justify your response.

(Adapted from *Engage New York: Algebra I, Module 5, Lesson 2,* pp. 31 - 32)

Post-Lesson Study Interview Protocol

Questions about Mathematical Modeling

1. From your perspective, what is mathematical modeling? What are some reasons to engage in mathematical modeling?
   a. If the teacher only mentions one context for engaging in mathematical modeling, ask: How might scientists or other professionals engage in mathematical modeling? How about K-12 students?

   [If the teacher asks what is meant by mathematical modeling say the following: "For the purpose of this interview mathematical modeling is defined as 'a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena'" (GAIMME, 2016, p. 8).]

2. Do you believe that mathematical modeling should be taught in schools? Why? Why not?

3. After teachers have taught the lesson developed through Lesson study ask:

   - Based on your previous experience and now your experience in teaching the State Apportionment lesson:
     a. What benefits, if any, do you see in teaching mathematical modeling?
     b. What drawbacks, if any, do you see in teaching mathematical modeling?
     c. What challenges do you and your students experience when you're teaching math modeling?
     d. How would you compare your teaching of modeling before our Lesson study to now?
     e. How do you anticipate your teaching of math modeling will change with time and why?

   - One of the things I wondered is whether the students realized they were doing modeling?
     a. Do you think the task we implemented was modeling and in what ways?
     b. Do you think the students understood that they were doing modeling? How? Why? Did they understand more about modeling by doing the task?
     c. Is important that students understood that what they were doing was modeling??

Note: Each question and sub-question may be subjected to the following questions:

- "What do you mean by…?"
- "Why do you think that is important"
- "Could you tell me more about that?"
- "I'm not sure I understand what you mean. Can you explain that again? Or can you explain that in a different way?"
- "Can you give me an example from your experiences?"

4. The following tasks were adapted from the mathematics curriculum, *Engage New York*. The authors of *Engage New York* identified the tasks as mathematical modeling tasks. From your perspective, what types of opportunities do the following tasks provide for students to engage in mathematical modeling, if at all? [Teachers will be provided, one at a time, with a copy of the tasks.] (See Pre-Lesson Study Interview Protocol for tasks.)

Appendix H

IRB Approval Letter

UNIVERSITY OF DELAWARE

RESEARCH OFFICE

210 Hullihen Hall
University of Delaware
Newark, Delaware 19716-1551
Ph: 302/831-2136
Fax: 302/831-2828

DATE: July 23, 2018

TO: Jenifer Hummer
FROM: University of Delaware IRB

STUDY TITLE: [1254341-1] Lesson Study on Mathematical Modeling for

Secondary Teachers SUBMISSION TYPE: New Project

ACTION: APPROVED
APPROVAL DATE: July 23, 2018
EXPIRATION DATE: July 22, 2019
REVIEW TYPE: Expedited Review

REVIEW CATEGORY: Expedited review category # *(6,7)*

Thank you for your submission of New Project materials for this research study. The University of Delaware IRB has APPROVED your submission. This approval is based on an appropriate risk/benefit ratio and a study design wherein the risks have been minimized. All research must be conducted in accordance with this approved submission.

This submission has received Expedited Review based on the applicable federal regulation.

Please remember that informed consent is a process beginning with a description of the study and insurance of participant understanding followed by a signed consent form. Informed consent must continue throughout the study via a dialogue between the researcher and research participant. Federal regulations require each participant receive a copy of the signed consent document.

Please note that any revision to previously approved materials must be approved by this office prior to initiation. Please use the appropriate revision forms for this procedure.

All SERIOUS and UNEXPECTED adverse events must be reported to this office. Please use the appropriate adverse event forms for this procedure. All sponsor reporting requirements should also be followed.

Please report all NON-COMPLIANCE issues or COMPLAINTS regarding this study to

this office. Please note that all research records must be retained for a minimum of

three years.

Based on the risks, this project requires Continuing Review by this office on an annual basis. Please use the appropriate renewal forms for this procedure.

If you have any questions, please contact Nicole Farnese-McFarlane at (302) 831-1119 or nicolefm@udel.edu. Please include your study title and reference number in all correspondence with this office.